

MATH 402 Worksheet 5

Monday 22 October, 2018

This worksheet is about working with isometries in matrix form. This will allow us to understand how to compose different isometries to get new isometries.

Recall that in order to represent translations using matrices, we had to introduce the *affine model* for Euclidean geometry, where points are represented by points in the $\{z = 1\}$ plane: rather than denoting a point by (x, y) , we write

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Exercise 1. Warm-up:

- Recall the definition of Euclidean distance between two points in \mathbb{R}^3 . Check that the distance between $(x, y, 1)$ and $(x', y', 1)$ in \mathbb{R}^3 is the same as the distance between the two points (x, y) and (x', y') in \mathbb{R}^2 . (This means that we can use distance in \mathbb{R}^3 to determine whether line segments in the affine model are congruent.)
- On the other hand, notice that if we do ordinary vector addition, $(x, y, 1) + (x', y', 1)$, we get $(x + x', y + y', 2)$, which is not even a point in the affine model! What should we define to be the new way of “adding” $(x, y, 1)$ and $(x', y', 1)$ to get something that agrees with our old way of adding vectors in \mathbb{R}^2 ?

Exercise 2. Matrix form of some isometries:

(Flip to the back page for a review of how we multiply matrices with column vectors and with each other, if it feels rusty.)

Consider the following matrices:

$$A = \begin{bmatrix} 1 & 0 & v_1 \\ 0 & 1 & v_2 \\ 0 & 0 & 1 \end{bmatrix}; \quad B = \begin{bmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad C = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad \mathbf{x} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- Calculate $A\mathbf{x}$, and show that it agrees with our earlier formulas for translation by $v = (v_1, v_2)$. (Recall that our formulas from last week were in ordinary Cartesian coordinates, not in affine form, so you’ll have to change into the correct form.)
- Calculate $B\mathbf{x}$, and show that it agrees with our earlier formulas for rotation about 0 by angle ϕ .
- Calculate $C\mathbf{x}$ and show that it agrees with our earlier formulas for reflection across the y -axis. (Actually, we didn’t write down the formulas for reflection across the y -axis, only reflection across the x -axis—what is the correct formula?)
- What is the matrix that will give reflection across the x -axis?

Exercise 3. Composing isometries

Recall that if f is a linear function that corresponds to multiplication by a matrix M , and g is a linear function that corresponds to multiplication by a matrix N , then $f \circ g$ corresponds to multiplication by the matrix $M \cdot N$.

Let w be the vector corresponding to $Rot_\phi(v)$. (So we can find w by multiplying the affine form of v by B .) Prove that $Rot_\phi \circ T_v = T_w \circ Rot_\phi$, by showing that the matrices corresponding to each side of the equation are equal.

You do not need to hand your work in, but you are expected to complete it. If you get stuck or are unsure about your answers, come to office hours. This material is examinable and will not be covered in ordinary lecture format, so you must make sure that you understand it as it is presented here.

Review about matrix multiplication

We'll focus on 3 by 3 matrices here. Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}; \quad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}; \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Recall that by the (i, j) th entry of a matrix, we mean the entry in the i th row and the j th column.

To calculate the product $A \cdot B$, we need to write down the (i, j) th entry for $i, j = 1, 2, 3$; we'll call it $(AB)_{ij}$. To calculate it, we look at the i th row of A and the j th column of B . In order for matrix multiplication to make sense, we need these vectors to have the same number of entries (in our example, both are of length 3):

$$\begin{bmatrix} a_{i1} & a_{i2} & a_{i3} \end{bmatrix}; \quad \begin{bmatrix} b_{1j} \\ b_{2j} \\ b_{3j} \end{bmatrix}$$

Then $(AB)_{ij}$ is defined by taking the dot product of these two vectors, or in other words pairing up the elements in order, multiplying each pair together, and adding this all up:

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j}.$$

In particular, we can compute $A \cdot \mathbf{x}$, because the rows of A all have length 3, and the (one!) column of \mathbf{x} has length 3 too. Note that $\mathbf{x} \cdot A$ doesn't make sense, because the rows of \mathbf{x} have length 1, so we can't match them up with the columns of A .

Let me know if you would like more practice questions to review this skill.