MATH 595 Tuesday 23 January

Bertini's theorem, tangent sheaf, canonical sheaf, projective genus

(1) **Chapter II, Exercise 8.4 (c), (d), (e), (f), (g).** *Complete intersections in* $\mathbb{P}^n$.

A closed subscheme Y of $\mathbb{P}^n$ is called a *complete intersection* if the homogeneous ideal I of Y in $S = k[x_0, \ldots, x_n]$ can be generated by $r = \operatorname{codim}(Y, \mathbb{P}^n)$ elements.

You may assume parts (a) and (b) without proof.

(c) Assume that Y is a complete intersection, and furthermore that Y is normal. Show that for all $\ell \geq 0$, the natural map

$$\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(\ell)) \to \Gamma(Y, \mathcal{O}_Y(\ell))$$

is surjective.
In particular, when $\ell = 0$, show that Y is connected.
Hint: Use part (b) and Exercise 5.14.

(d) Suppose given integers $d_1, \ldots, d_r$ (with $r < n$). Use Bertini's theorem to show that there exist non-singular hypersurfaces H_i of degree d_i in $\mathbb{P}^n$ such that the scheme $Y = H_1 \cap \ldots H_r$ is irreducible and non-singular of codimension r in $\mathbb{P}^n$.

(e) Given Y as in (d), prove that $\omega_y \simeq \mathcal{O}_Y(\sum d_i - n - 1)$.

(f) If Y is a non-singular hypersurface of degree d in $\mathbb{P}^n$, use (c) and (e) to show that $p_g(Y) = \binom{d-1}{n}$. (Note that this means that $p_g(Y) = p_a(Y)$ in this case.)
In particular, if Y is a non-singular plane curve of degree d, then $p_g(Y) = \frac{1}{2}(d-1)(d-2)$.

(g) If Y is a non-singular curve in $\mathbb{P}^3$ which is a complete intersection of non-singular surfaces of degree d and e, then prove that $p_g(Y) = \frac{1}{2}de(d+e-4)+1$.