# MATH 595 Thursday 8 February

## Cohomology and affine schemes; introduction to Čech cohomology

(1) **Chapter III, Exercise 3.1.**

Let $X$ be a Noetherian scheme. Prove that $X$ is affine if and only if $X_{\text{red}}$ is affine.

Hint: Let $\mathscr{N}$ be the sheaf of nilpotent elements in $X$. Show that for each coherent $\mathscr{F}$ there is a finite filtration

$$0 \subset \ldots \subset \mathscr{N}^2 \cdot \mathscr{F} \subset \mathscr{N} \cdot \mathscr{F} \subset \mathscr{F}.$$

Do induction on the length of this sequence to prove that the higher cohomologies of $\mathscr{F}$ vanish. Now show that the higher cohomologies of any quasi-coherent sheaf must vanish too. Apply Serre's criterion.

(2) **Chapter III, Exercise 3.2.**

Let $X$ be a reduced noetherian scheme. Show that $X$ is affine if and only if each irreducible component is affine.

Hint: Recall that to show that $X$ is noetherian, it's enough to show that for any coherent sheaf of ideals $\mathscr{I}$ on $X$, $H^1(X, \mathscr{I})$ vanishes. So let $\mathscr{I}$ be such a sheaf, corresponding to some closed subscheme $Y$, and let $X = \bigcup Y_i$ be the decomposition of $X$ into finitely many affine irreducible components.

Consider a short exact sequence of the form

$$0 \to \mathscr{I}_{Y \cup Y_1} \to \mathscr{I}_Y \to \mathscr{F} \to 0.$$

Prove that the quotient $\mathscr{F}$ comes from pushing forward some ideal sheaf on $Y_1$; conclude that $H^1(X, \mathscr{F}) = 0$. What does this tell you about $H^1(X, \mathscr{I}_Y)$? Repeat this procedure using $Y \cup Y_1$ and $Y_2$ instead of $Y$ and $Y_1$. Induct...

(3) **Chapter III, Exercise 3.8.**

In this exercise, we show that without the noetherian hypothesis, it is not the case that whenever $I$ is injective as a module, the associated sheaf $\widetilde{I}$ must be a flasque sheaf.

Let $A$ be the quotient of the polynomial ring $k[x_0, x_1, x_2, \ldots]$ by the relations $\{x_0^n x_n = 0\}_{n=1,2,\ldots}$. Let $I$ be an injective $A$-module containing $A$. Prove that the map $\phi : I \to I_{x_0}$ cannot be surjective.

Hint: if $\phi$ is surjective, there exists some $b \in I$ such that $\phi(b) = \frac{1}{x_0}$ in $I_{x_0}$. What does this mean?

(4) *The Čech differential*

Prove that $d^2 = 0$.