MATH 595 Thursday 22 February

Cohomology of projective space

(1) **Chapter III, Exercises 5.1, 5.2, 5.3.**

Let X be a projective scheme over a field k, and let $\mathscr{F}$ be a coherent sheaf on X. The *Euler characteristic* of $\mathscr{F}$ is defined by

$$\chi(\mathscr{F}) = \sum (-1)^i \dim_k H^i(X, \mathscr{F}).$$

- It is an exercise in homological algebra (you can do it later if you like) to show that if there is a short exact sequence of coherent sheaves

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0,$$

then $\chi(\mathscr{F}) = \chi(\mathscr{F}') + \chi(\mathscr{F}'')$.

Now let $\mathcal{O}_X(1)$ be a very ample invertible sheaf on X over k. Assume that the dimension of X is r. Show that there is a polynomial $P(z) \in \mathbb{Q}[z]$ such that $\chi(\mathscr{F}(n)) = P(n)$ for all $n \in \mathbb{Z}$, as follows:

(a) We will do induction on the dimension of the support of $\mathscr{F}$. When this dimension is 0, reduce to the case that $\mathscr{F}$ is a skyscraper sheaf, and prove that $\chi(\mathscr{F}(n))$ is constant.

(b) For the induction step, you will need the following fact: If $f : \mathbb{Z} \to \mathbb{Z}$ is a function such that the difference $\Delta f(n) = f(n+1) - f(n)$ is equal to $Q(n)$ for every integer n, for some polynomial $Q(z) \in \mathbb{Q}[z]$ of degree D, then $f(n) = P(n)$ for some polynomial of degree $D+1$. To use this fact to your advantage, choose a suitable element $s \in \Gamma(X, \mathcal{O}_X(1))$ and construct an exact sequence

$$0 \to \mathscr{R} \to \mathscr{F}(-1) \to \mathscr{F} \to \mathscr{Q} \to 0.$$

The polynomial $P(z) = P_{\mathscr{F}}(z)$ is called the *Hilbert polynomial* of $\mathscr{F}$.

(c) Let $X = \mathbb{P}^r_k$, and let $M = \Gamma_*(\mathscr{F})$, considered as a graded $S = k[x_0, \ldots, x_r]$-module. Prove that $P_{\mathscr{F}}(z)$ just defined agrees with the Hilbert polynomial of M defined in Chapter I, section 7.

We define the *arithmetic genus* of X by

$$p_a(X) = (-1)^r(\chi(\mathcal{O}_X) - 1).$$

If X is integral and k is algebraically closed, it is not hard to show that $H^0(X, \mathcal{O}_X) \cong k$; then the formula for $p_a(X)$ can be written as

$$p_a(X) = \sum_{i=0}^{r-1} (-1)^i \dim_k H^{r-i}(X, \mathcal{O}_X).$$

(d) Let X be a plane curve of degree d. What is $p_a(X)$? (Use our Čech cohomology computation from last time.)

(e) If X is a closed subvariety of $\mathbb{P}^r_k$, show that this definition of the arithmetic genus agrees with the definition given in Chapter I, Exercise 7.2, which appeared to depend on the choice of embedding.

(2) **Chapter III, Exercise 5.5.**

Let k be a field, let $X = \mathbb{P}^r_k$, and let Y be a closed subscheme of dimension $q \geq 1$ which is a complete intersection. The prove the following collection of statements, by induction on the codimension of Y:

(a) For any integer n, the natural map

$$H^0(X, \mathcal{O}_X(n)) \to H^0(Y, \mathcal{O}_Y(n))$$

is surjective.

(b) Y is connected.

(c) $H^i(Y, \mathcal{O}_Y(n)) = 0$ for $0 < i < q$ and for any $n \in \mathbb{Z}$.

(d) $p_a(Y) = \dim_k H^q(Y, \mathcal{O}_Y)$.

Hint: To carry out the induction step, write $Y = H_1 \cap H_k$, let Y_0 be the complete intersection $H_1 \cap H_{k-1}$, and write a short exact sequence giving $\mathcal{O}_Y$ as a quotient of $\mathcal{O}_{Y_0}$.