MATH 595 Thursday April 5

Hurwitz's Theorem

(1) **IV.2.3 (a)**

Let $X \subset \mathbb{P}^3$ be a curve of degree d. Choose a line L in $\mathbb{P}^2$ which is not tangent to X, and define a map

$$\begin{aligned} \phi : X &\to L \\ P &\mapsto T_P(X) \cap L. \end{aligned}$$

(a) Suppose that $P \in X \cap L$. Prove that ϕ is ramified at P.
(Hint: Choose coordinates on $\mathbb{P}^2$ so that $P = 0 \in \mathbb{A}^2$, $T_P(X) = \{x = 0\}$, and $L = \{y = 0\}$. If $X \cap \mathbb{A}^2$ is given by $\operatorname{Spec}(k[x, y]/(f))$, what do you know about f? Use this to write down an explicit formula for ϕ.)

(b) Now consier P on on L. Show that ϕ is ramified at P if and only if P is an inflection point of X.
(Hint: Choose coordinates on $\mathbb{P}^2$ so that $P = 0 \in \mathbb{A}^2$, $T_P(X) = \{x = 0\}$, and L is the line at infinity $\{z = 0\}$.)

(2) **IV.2.2** *Classification of curves of genus 2*

Fix a field k, algebraically close and of characteristic $\neq 2$.

(a) Let X be a curve of genus 2. Recall that the linear system $|K|$ has degree $2g - 2 = 2$, and dimension $g - 1 = 1$, and hence determines a degree 2 morphism $f : x \to \mathbb{P}^1$.
Use Hurwitz's theorem to show that f is ramified at exactly 6 points, with ramification index 2 at each of these points.
(Note that f is determined up to automorphism of $\mathbb{P}^2$, and so we have associated to X and unordered set of 6 points in BP^1, up to automorphism of $\mathbb{P}^1$.)

(b) Now let $\alpha_1, \dots, \alpha_6$ be 6 unordered points of k. Let K be the extension of $k(x)$ determined by the equation $z^2 = (x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_6)$. Let X be the projective curve with $K(X) = K$, and let $f : X \to \mathbb{P}^1$ be the morphism determined by the extension $k(x) \subset K$.
Prove that f is ramified over $Q \in \mathbb{P}^1$ if an only if $Q = \alpha_i$ for $i = 1, \dots 6$; prove also that for these points, f has ramification index 2. Use this to show that $g(X) = 2$.
(Hint: work over $\mathbb{A}^1 = \mathbb{P}^1 \setminus \infty$, so that X^0 can be written as $\operatorname{Spec}(k[x, z]/(z^2 - h(x))$. Because $h(x) = (x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_6)$ is square-free, we can prove that $k[x, z]/(z^2 - h)$ is integrally closed (see Ex. II.6.4), so that X^0 is normal. Write the map $f : X^0 \to \mathbb{A}^1$ explicitly, and check when $x - Q$ is a local parameter at $P \in f^{-1}(Q)$.)

(c) With X, f still as in part (b), check that f is the map determined by $|K|$.
(Hint: write $f^*\mathcal{O}_{\mathbb{P}^1} \cong \mathscr{L}(D)$ for some effective divisor D of degree d; then f is determined by a linear system $\mathfrak{d} \subset |D|$ of dimension 1. Prove that $D \sim K$ and $\mathfrak{d} = |K|$.)

Recall that the automorphism group of $\mathbb{P}^1$ acts freely and transitively on triples of distinct points in $\mathbb{P}^1$. Thus if we order the six points α_i, we can find a unique automorphism ϕ of $\mathbb{P}^1$ such that $\phi(\alpha_1) = 0$, $\phi(\alpha_2) = 1$, $\phi(\alpha_3) = \infty$. Then we are left with three distinct points in $k \setminus \{0, 1\}$. Now we can define an action of the symmetric group S_6 on such triples $(\beta_1, \beta_2, \beta_3)$: act on $(0, 1, \infty, \beta_1, \beta_2, \beta_3)$ by $\sigma \in S_6$, and then apply the unique automorphism ϕ to take the first three terms back to $(0, 1, \infty)$. The remaining three terms give $\sigma \cdot (\beta_1, \beta_2, \beta_3)$. You can check that this is an action if you like.

Summing up, we obtain a bijection between isomorphism classes of genus 2 curves X and the set

$$\{(\beta_1, \beta_2, \beta_3) \mid \beta_i \in k \setminus \{0, 1, \}\}/S_6.$$