MATH 595 Tuesday 1 May

Ruled surfaces

(1) Let $X \to C$ be a ruled surface. Denote by K_X a canonical divisor on X, and by $\mathfrak{k}$ and canonical divisor on C. We proved that

$$K_X \sim -2C_0 + (\mathfrak{k} + \mathfrak{e})f.$$

Use this to deduce that in $\mathrm{Num}(X)$, we have $K_X = -2C_0 + (2g - 2 - e)f$. Conclude that $K_X^2 = 8(1 - g)$.

(2) Suppose that $X \cong \mathbb{P}(\mathscr{E})$ is a ruled surface over C, where $\mathscr{E}$ is a normalized rank 2 locally free sheaf on C. Suppose furthermore that $\mathscr{E}$ is decomposable (i.e. can be written as a direct sum of two invertible sheaves).

Prove that $\mathscr{E} \cong \mathcal{O}_C \oplus \mathscr{L}$, where $\mathscr{L}$ is some invertible sheaf of degree ≤ 0.

Prove that it is possible to produce a ruled surface X over C with invariant e for any $e \geq 0$.

(3) **Exercise V.2.2**

Let X be the ruled surface $\mathbb{P}(\mathscr{E})$ over a curve C. Prove that $\mathscr{E}$ is decomposable if and only if there exist two sections C' and C'' of X which do not intersect.