Monday, 11 March, 2019

Last time integrating over rectangles.

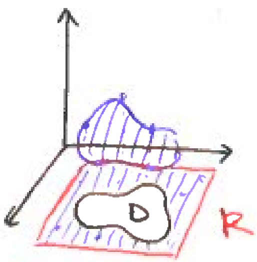
[1] • e.g. Find $\iint_D (3x^2 + 3y^2)\, dA$, where $D = [0,2]\times[-3,1]$

(see slides for solution)

§ DOUBLE INTEGRAL OVER GENERAL REGIONS (§ 15.3)

Let f be a continuous function on a bounded set $D \subset \mathbb{R}^2$.

We can choose a rectangle R that contains D.



Define a function $\hat{f}$ on R as follows:

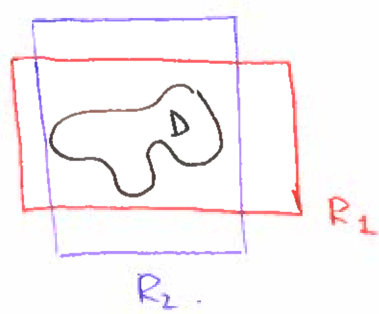
$$\hat{f}(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in D \\ 0 & \text{if } (x,y) \notin D \end{cases}$$

If $\hat{f}$ is integrable, define $\iint_D f\, dA = \iint_R \hat{f}\, dA$.

• Note: if $f \geq 0$, $\iint_R \hat{f}\, dA$ = volume of the solid under Γ_f and over D

↳ in particular, we get the same number even if we choose a different rectangle

$$\iint_{R_1} \hat{f}^1\, dA = \iint_{R_2} \hat{f}^2\, dA$$



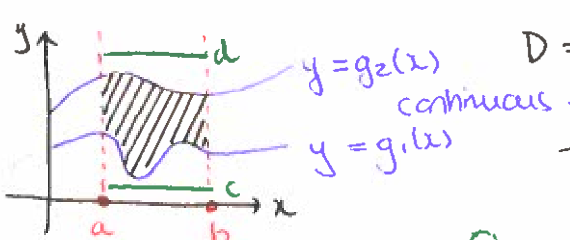
$\Rightarrow$ $\boxed{\iint_D f\, dA \text{ is well-defined.}}$

Example: $\iint_D 1\, dA = ?$

(area of D)(1) = area of D.

* How do we compute $\iint_D f\, dA$?

Example:



$D = \{(x,y) \mid a \leq x \leq b, \; g_1(x) \leq y \leq g_2(x)\}$

This is called a region of type I.

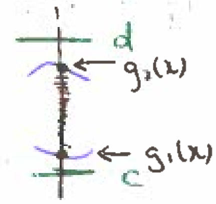
Choose a rectangle $R = [a,b]\times[c,d]$ containing D.

So $\iint_D f(x,y)\,dA = \iint_R \hat{f}(x,y)\,dA$

$$= \int_a^b \int_c^d \hat{f}(x,y)\,dy\,dx. \qquad \text{[Fubini]}.$$

But for fixed x, $\int_c^d \hat{f}(x,y)y = \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy$,

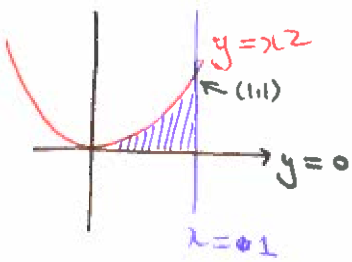
since $\hat{f}(x,y) = \begin{cases} 0 & \text{if } y \notin [g_1(x), g_2(x)] \\ f(x,y) & \text{if } y \in [g_1(x), g_2(x)] \end{cases}$.



So $\boxed{\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx.}$ for D a region of Type I.

Statement of theorem on slides.

Example: Let D be the region bounded by $y=x^2$, $x=1$, $y=0$.



- $[a,b]$ is the "shadow" of D on the x-axis.
  - All x_0 such that the line $\{x=x_0\}$ passes through D.

  $\Rightarrow [a,b] = [0,1]$.
- $g_1(x_0)$ is the smallest value of y such that $(x_0, g_1(x_0)) \in D$.

  $\Rightarrow g_1(x) = 0$.
- $g_2(x_0)$ - largest

  $\Rightarrow g_2(x) = x^2$.

$\Rightarrow D = \{(x,y) \mid 0 \le x \le 1, \quad 0 \le y \le x^2\}$ is of type I.

Now find $\iint_D 12xy\,dA$.

$$\iint_D 12xy\,dA = \int_0^1 \int_0^{x^2} 12xy\,dy\,dx$$

$$= \int_0^1 \left[6xy^2\right]_{y=0}^{x^2} dx = \int_0^1 6x^5\,dx$$

$$= \left[x^6\right]_0^1 = 1.$$

We also have regions of type II, where we swap the order of x & y.

[see slides]

Definition A region $D \subset \mathbb{R}^2$ is of Type II if it is of the form

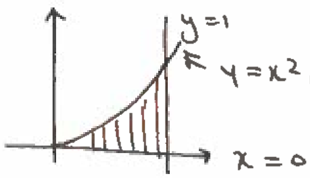
$$D = \{(x,y) \mid c \le y \le d, \quad h_1(y) \le x \le h_2(y)\}$$

where $h_1, h_2 : [c,d] \to \mathbb{R}$ are continuous.

Theorem: For f continuous on D of type II as above,

$$\iint_D f\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy.$$

Example: our D from before is also a region of type 2.



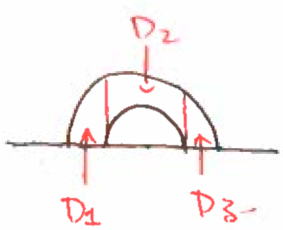
$y = x^2$, $x = \sqrt{y}$. $\boxed{1}$ Q: if $D = \{(x,y) \mid c \le y \le d, \ h_1(y) \le x \le h_2(y)\}$ what are $c, d, h_1(y)$ and $h_2(y)$?

$\boxed{2}$ Find the Area of D.

[see slides for solution].

Problem: Not all regions are of type I or II.

Example: $D = \{(x,y) \mid 1 \le x^2 + y^2 \le 4 \text{ and } y \ge 0\}$.

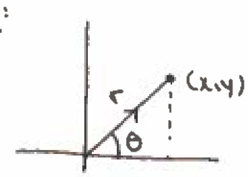


Method 1: Divide D into 3 regions which are of type I / II.

Method 2: Choose better coordinates.

§ POLAR COORDINATES (15.4)

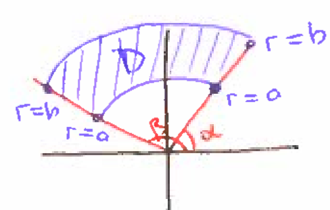
Recall:



- $x = r\cos\theta$, $y = r\sin\theta$, $r \ge 0$, $0 \le \theta \le 2\pi$.
- conversely, $r = \sqrt{x^2+y^2}$, $\theta = \arctan(y/x)$.

Definition: A polar rectangle is a region D given by

- $a \le r \le b$;
- $\alpha \le \theta \le \beta$.



Theorem: For D a polar rectangle as above and f a continuous function on D,

$$\iint_D f\,dA = \int_{\theta=\alpha}^{\beta} \int_{r=a}^{b} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta.$$

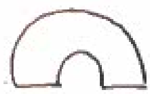
WARNINGS: $\cdot$ f is evaluated at $\cancel{f}(r\cos\theta, r\sin\theta)$

$\cdot$ there is an extra r.

Example: $D = \{(x,y) \mid 1 \leq x^2+y^2 \leq 4, \quad y \geq 0\}$.

is a ~~pla~~ polar rectangle.

- $x^2+y^2 = r^2 \rightarrow 1 \leq r \leq 2$.



- $\cancel{y}$ $\quad 0 \leq \theta \leq \pi$.

So $\displaystyle\iint_{\cancel{R}D} 8y^2\, dA = \int_0^{\pi}\int_1^2 8(r\sin\theta)^2\, r\, dr\, d\theta$.

Example. Find $\displaystyle\iint_{\cancel{R}D} y\, dA$.

$\boxed{2}$ (See slides for solution)