Last time: cylindrical and spherical coordinates

Recall that (x, y, z) and (ρ, θ, ϕ) are related by

$$x = \rho \sin\phi \cos\theta, \qquad y = \rho \sin\phi \sin\theta, \qquad z = \rho \cos\phi.$$

Consider the solid C lying between the half-cone $z = \sqrt{x^2 + y^2}$ and the half-sphere $z = \sqrt{9 - x^2 - y^2}$. Sketch C and write it in spherical coordinates:

$$C = \{(\rho, \theta, \phi) \mid \square \leq \theta \leq \square,\ \square \leq \phi \leq \square,\ \square \leq \rho \leq \square\}.$$

What is the sum of the numbers in the boxes?

(a) $9 + \frac{5\pi}{4}$

(b) $3 + \frac{5\pi}{4}$

(c) $3 + 3\pi$

(d) $9 + 3\pi$

Announcements

The American Society of Mechanical Engineers at the University of Illinois wants you to sign up for their talent show. Here is a link to their poster:
Poster.

Recall: Integrating in spherical coordinates

Let B be a "spherical wedge":

$$B = \{(\rho, \theta, \phi) \mid \alpha \le \theta \le \beta,\ a \le \rho \le b,\ c \le \phi \le d\}.$$

Let $f : B \to \mathbb{R}$ be a continuous function. Then

$$\iiint_B f dV =$$

$$\int_c^d \int_\alpha^\beta \int_a^b f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi) \rho^2 \sin\phi \ d\rho d\theta d\phi.$$

Example

Let's find the volume of the solid C from the first question.

$$C = \{(\rho, \theta, \phi) \mid 0 \leq \theta \leq 2\pi,\ 0 \leq \phi \leq \frac{\pi}{4},\ 0 \leq \rho \leq 3\}$$

So we have

$$\begin{aligned}
V(C) &= \iiint_C dV \\
&= \int_0^{\frac{\pi}{4}} \int_0^{2\pi} \int_0^3 \rho^2 \sin\phi \ d\rho d\theta d\phi \\
&= \int_0^{\frac{\pi}{4}} \int_0^{2\pi} \left[\frac{1}{3}\rho^3 \sin\phi\right]_0^3 \ d\theta d\phi \\
&= \int_0^{\frac{\pi}{4}} \int_0^{2\pi} 9 \sin\phi \ d\theta d\phi \\
&= \int_0^{\frac{\pi}{4}} 18 \sin\phi \ d\phi = 9\pi(2 - \sqrt{2}).
\end{aligned}$$

Practice with Jacobians

Find the Jacobian for each of the examples (1), (2), (3). Which is the largest?

(a) (1)

(b) (2)

(c) (3)

(d) It's a tie.

(e) I don't know how to do this.

Change of variables — why does it work?

We calculate $\iint_{T(D)} f(x,y)dA$ by dividing D into small boxes $\square$ of area ΔA.

Then $T(D)$ is divided into small parallelograms $T(\square)$ of area $|\frac{\partial(x,y)}{\partial(u,v)}|\Delta A$.

We should choose test points (x^*, y^*) in each $T(\square)$. We do this by choosing points (u^*, v^*) in $\square$, and taking $(x^*, y^*) = T(u^*, v^*)$.

Then to compute the integral, we take the sum over all the parallelograms of the contribution

$$f(x^*, y^*) \cdot \text{Area}(T(\square)) = f(T(u^*, v^*)) \cdot |\frac{\partial(x,y)}{\partial(u,v)}|\Delta A,$$

and taking the limit as the number of boxes goes to infinity.

But this is just the integral $\iint_D f(T(u,v))|\frac{\partial(x,y)}{\partial(u,v)}|dA$.