Last time: integrating vector fields on surfaces

- $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\ dS$.
- If $\mathbf{r}(u, v), (u, v) \in D$ is a parametrization of S such that $\mathbf{r}_u \times \mathbf{r}_v$ is positively oriented, then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\ dA.$$

Let $S = \{3x + 2y + z = 1, x \geq 0, y \geq 0, z \geq 0\}$, oriented upward. S is parametrized by $\mathbf{r}(u, v) = \langle u, v, 1 - 3u - 2v \rangle$, where $(u, v) \in D = \{u \geq 0, v \geq 0, 3u + 2v \leq 1\}$.

Sketch S and D. Find $\mathbf{r}_u \times \mathbf{r}_v$. Is it positively oriented?

(a) Yes, it is positively oriented.

(b) No, it is negatively oriented.

(c) I don't remember what that means.

Solution & example

We can calculate that $\mathbf{r}_u \times \mathbf{r}_v = \langle 3, 2, 1 \rangle$.

Since we are told that S is *oriented upward* and the $\mathbf{k}$ component of $\mathbf{r}_u \times \mathbf{r}_v$ is positive, we must have that $\mathbf{n}$ and $\mathbf{r}_u \times \mathbf{r}_v$ point in the same direction. So $\mathbf{r}_u \times \mathbf{r}_v$ is positively oriented.

Example: calculate $\iint_S \mathbf{F} \cdot d\mathbf{S}$ for S as above and $\mathbf{F}(x, y, z) = \langle 1, 0, 1 \rangle$.

We can use the formula

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\ dA.$$

$$\Rightarrow \iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \langle 1, 0, 1 \rangle \cdot \langle 3, 2, 1 \rangle dA$$

$$= \iint_D 3 + 0 + 1 dA$$

Example continued

So

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = 4 \iint_D dA = 4(\text{Area of } D).$$

Since D is a right triangle, we can calculate its area easily: $\frac{1}{12}$.
So

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = 4\frac{1}{12} = \frac{1}{3}.$$

Practice with Stokes' Theorem

Let $\mathbf{F} = \langle x \sin z, y \sin z, e^{x+y} \rangle$, and let $S = \{x^2 + y^2 + z^2 = 9\}$, oriented outwards.

Find $\iint_S \text{curl}\mathbf{F} \cdot d\mathbf{S}$.

(a) -12π

(b) -9

(c) 0

(d) 9

(e) 12π

Answer: (c)