

Thursday, January 31 \*\* Functions of several variables; Limits.

1. For each of the following functions  $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ , draw a sketch of the graph together with pictures of some level sets.

(a)  $f(x, y) = xy$

(b)  $f(\mathbf{x}) = |\mathbf{x}|$ . Please note here that  $\mathbf{x}$  is a vector. In coordinates, this function is  $f(x, y) = \sqrt{x^2 + y^2}$ .

For (a), the result is one of the many quadric surfaces. What is the name for this type? Is the graph in (b) also a quadric surface?

2. Consider the function  $f: \mathbb{R}^2 \rightarrow \mathbb{R}$  given by

$$f(x, y) = \frac{2x^3y}{x^6 + y^2} \quad \text{for } (x, y) \neq \mathbf{0}$$

In this problem, you'll consider  $\lim_{(x,y) \rightarrow \mathbf{0}} f(x, y)$ .

- (a) Look at the values of  $f$  on the  $x$ - and  $y$ -axes. What do these values show the limit  $\lim_{(x,y) \rightarrow \mathbf{0}} f(x, y)$  must be **if it exists**?
- (b) Show that along each line in  $\mathbb{R}^2$  through the origin, the limit of  $f$  exists and is 0.
- (c) Despite this, show that the limit  $\lim_{(x,y) \rightarrow \mathbf{0}} f(x, y)$  does not exist by finding a curve over which  $f$  takes on the constant value 1.
3. Consider the function  $f: \mathbb{R}^2 \rightarrow \mathbb{R}$  given by

$$f(x, y) = \frac{xy^2}{\sqrt{x^2 + y^2}} \quad \text{for } (x, y) \neq \mathbf{0}$$

In this problem, you'll show  $\lim_{\mathbf{h} \rightarrow \mathbf{0}} f(\mathbf{h}) = 0$ .

- (a) For  $\epsilon = 1/2$ , find some  $\delta > 0$  so that when  $0 < |\mathbf{h}| < \delta$  we have  $|f(\mathbf{h})| < \epsilon$ . Hint: As with the example in class, the key is to relate  $|x|$  and  $|y|$  with  $|\mathbf{h}|$ .
- (b) Repeat with  $\epsilon = 1/10$ .
- (c) Now show that  $\lim_{\mathbf{h} \rightarrow \mathbf{0}} f(\mathbf{h}) = 0$ . That is, given an arbitrary  $\epsilon > 0$ , find a  $\delta > 0$  so that that when  $0 < |\mathbf{h}| < \delta$  we have  $|f(\mathbf{h})| < \epsilon$ .
- (d) Explain why the limit laws that you learned in class on Wednesday aren't enough to compute this particular limit.