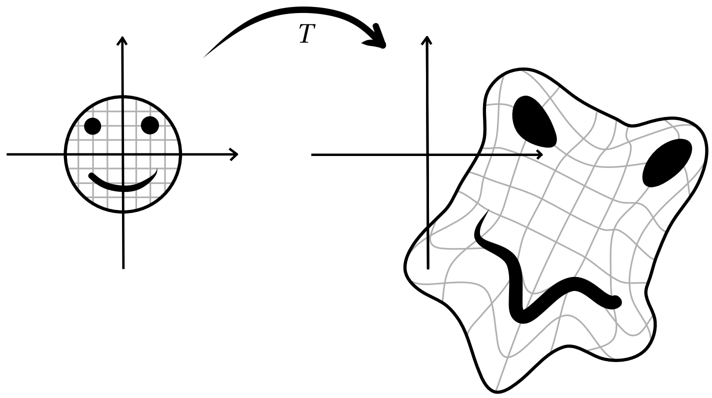
Tuesday, March 26 $*$ **Solutions** $*$ *Transformations of* $\mathbb{R}^2$.

Purpose: In class, we've seen several different coordinate systems on $\mathbb{R}^2$ and $\mathbb{R}^3$ beyond the usual rectangular ones: polar, cylindrical, and spherical. The lectures on Friday and Monday will cover the crucial technique of simplifying hard integrals using a change of coordinates (Section 15.9). The point of this worksheet is to familiarize you with some basic concepts and examples for this process.

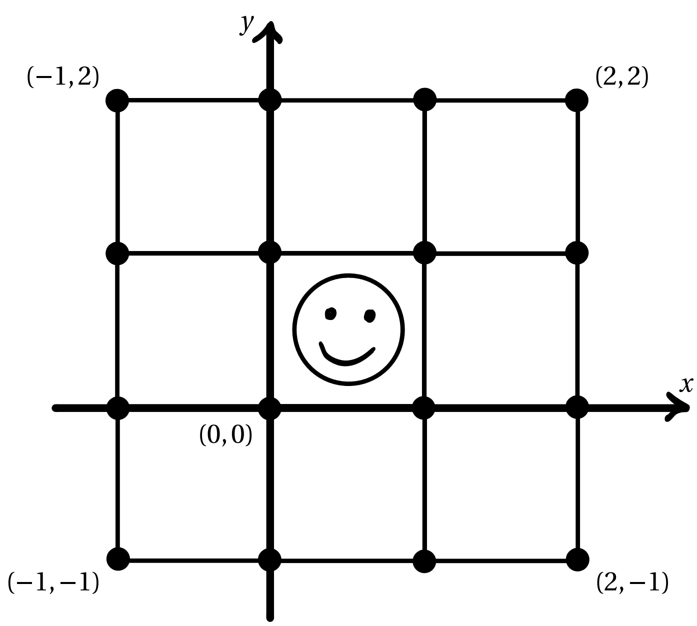
Starting point: Here we consider a variety of transformations $T\colon \mathbb{R}^2 \to \mathbb{R}^2$. Previously, we have used such functions to describe vector fields on the plane, but we can also use them to describe ways of distorting the plane:



1. Consider the transformation $T(x, y) = (x - 2y, x + 2y)$.

   (a) Compute the image under T of each vertex in the below grid and make a careful plot of them, which should be fairly large as you will add to it later.

   To speed this up, divide the task up among all members of the group.



SOLUTION:

See the image following part (f).

(b) For each pair A and B of vertices of the grid joined by a line, add the line segment joining $T(A)$ to $T(B)$ to your plot. This gives a rough picture of what T is doing.

SOLUTION:

See the image following part (f).

(c) What is the image of the x-axis under T? The y-axis?

SOLUTION:

The image of the x-axis is the line $y = x$. The image of the y-axis is the line $y = -x$. To see this, parametrize the x-axis as $\mathbf{r}(t) = (t, 0), -\infty < t < \infty$. Then $T(\mathbf{r}(t)) = (t, t), -\infty < t < \infty$, which traces out the line $y = x$. Do the same for the y-axis.

(d) Consider the line L given by $x + y = 1$. What is the image of L under T? Is it a circle, an ellipse, a hyperbole, or something else?

SOLUTION:

Parametrize L by $\mathbf{r}(t) = (t, 1-t), -\infty < t < \infty$. $T(L)$ is parametrized by $T(\mathbf{r}(t)) = (t - 2(1-t), t + 2(1-t)) = (3t-2, -t+2)$. These are the parametric equations of a line.

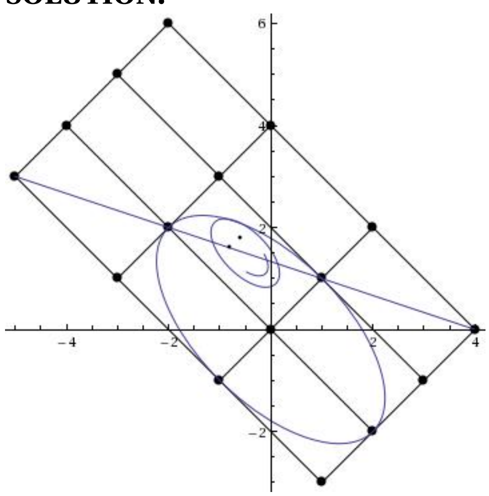
(e) Consider the circle C given by $x^2 + y^2 = 1$. What is the image of C under T?

SOLUTION:

Parametrize C by $\mathbf{r}(t) = (\cos t, \sin t), 0 \le t \le 2\pi$. Then $T(\mathbf{r}(t)) = (\cos t - 2\sin t, \cos t + 2\sin t), 0 \le t \le 2\pi$. Note that if we let $x = \cos t - 2\sin t, y = \cos t + 2\sin t$, then $y - x = 4\sin t$ and $y + x = 2\cos t$. So the curve $T(C)$ satisfies the equation $\left(\frac{y-x}{4}\right)^2 + \left(\frac{y+x}{2}\right)^2 = 1$. This is the equation of an ellipse.

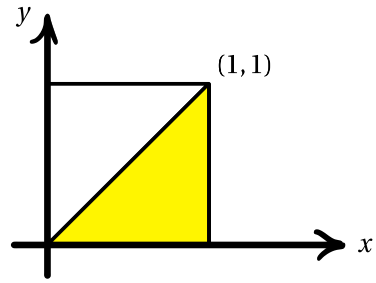
(f) Add $T(L)$, $T(C)$ and $T($☺$)$ to your picture. Check your answer with the instructor.

SOLUTION:



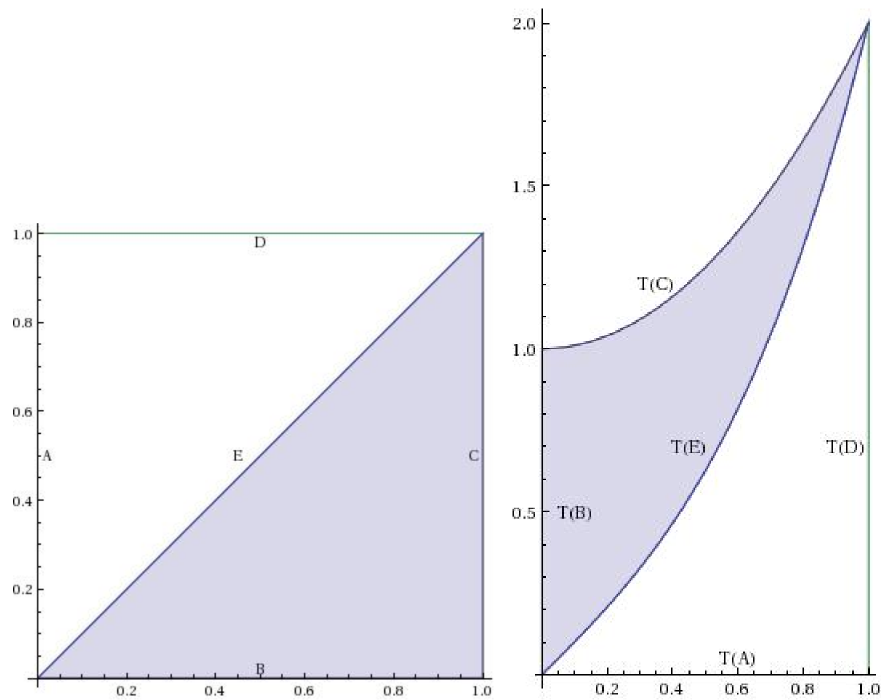
Note: The transformation T is a particularly simple sort called a *linear transformation.*

2. Consider the transformation $T(x, y) = \big(y, x(1 + y^2)\big)$. Draw the image of the picture below under T.



SOLUTION:

Label the 5 line segments as at left below. The image of the left hand picture is the right hand picture.



We can figure this out as follows. First parametrize the line segments:

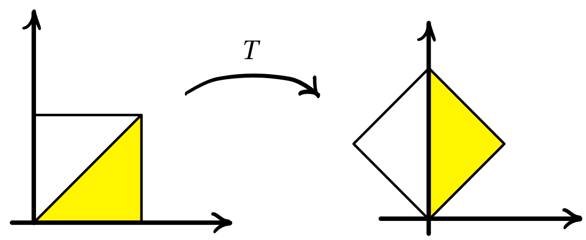
$$\begin{array}{ll} \mathbf{r}_A(t) = (0,t), 0 \le t \le 1 & \mathbf{r}_B(t) = (t,0), 0 \le t \le 1 \\ \mathbf{r}_C(t) = (1,t), 0 \le t \le 1 & \mathbf{r}_D(t) = (t,1), 0 \le t \le 1 \\ \mathbf{r}_E(t) = (t,t), 0 \le t \le 1 & \end{array}$$

Then compute the image under T of each of these:

$$\begin{array}{cc} T(\mathbf{r}_A(t)) = (t,0), 0 \le t \le 1 & T(\mathbf{r}_B(t)) = (0,t), 0 \le t \le 1 \\ T(\mathbf{r}_C(t)) = (t, 1+t^2), 0 \le t \le 1 & T(\mathbf{r}_D(t)) = (1,2t), 0 \le t \le 1 \\ T(\mathbf{r}_E(t)) = (t, t(1+t^2)), 0 \le t \le 1 & \end{array}$$

Graphing each of these gives the image above at left.

3. In this problem, you'll construct a transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ which rotates counter-clockwise about the origin by $\pi/4$, as shown below.



(a) Give a formula for T in terms of polar coordinates. That is, how does rotation affect r and θ?

SOLUTION:

$$T(r,\theta) = (r, \theta + \pi/4)$$

(b) Write down T in terms of the usual rectangular (x, y) coordinates. Hint: first convert into polar, apply part (a) and then convert back into rectangular coordinates.

SOLUTION:

First convert (x, y) into polar:

$$(r,\theta) = (\sqrt{x^2+y^2}, \arctan(y/x))$$

Then apply T in polar coordinates:

$$T(r,\theta) = (r, \theta + \pi/4)$$

Then convert the result to rectangular coordinates:
$T(x, y) = (r\cos(\theta+\pi/4), r\sin(\theta+\pi/4))$, where $r = \sqrt{x^2+y^2}, \theta = \arctan(y/x)$.
Recall the double angle formulas $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$ and $\sin(a+b) = \sin(a)\cos(b) + \sin(b)\cos(a)$. Using these we see that

$$\cos(\theta+\pi/4) = \cos\theta\cos(\pi/4) - \sin\theta\sin(\pi/4) = \sqrt{2}/2(\cos\theta - \sin\theta)$$

and

$$\sin(\theta+\pi/4) = \sin\theta\cos(\pi/4) + \sin(\pi/4)\cos\theta = \sqrt{2}/2(\sin\theta + \cos\theta).$$

Hence we have

$$r\cos(\theta+\pi/4) = \sqrt{2}/2(r\cos\theta - r\sin\theta) = \sqrt{2}/2(x-y)$$

and

$$r\sin(\theta+\pi/4) = \sqrt{2}/2(r\sin\theta + r\cos\theta) = \sqrt{2}/2(x-y).$$

So we have

$$T(x,y) = (\sqrt{2}/2(x-y), \sqrt{2}/2(x-y)).$$