Classical motivation. (Why study perverse sheaves?)

Reference: Kirwan & Woolf 'An introduction to intersection homology theory'. (2006)

§1. The role of the homology of a manifold

• Intersection theory.

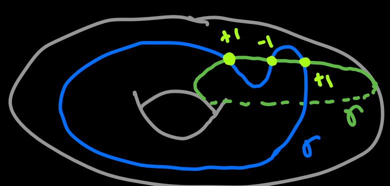
Let $n \in \mathbb{Z}_{\geq 0}$, M a compact oriented manifold w/ $\dim_{\mathbb{R}} M = n$.
For every $0 \leq i \leq n$ there is a bilinear pairing

$$H_i(M) \otimes_{\mathbb{C}} H_{n-i}(X) \longrightarrow \mathbb{C}$$

$$([\sigma] \otimes [\gamma]) \longmapsto \sum_{t \in \sigma \cap \gamma} (-1)^{f(t)}$$

a function on t

2 representatives in "generic" position



$\rightsquigarrow 1 + -1 + 1 = 1$

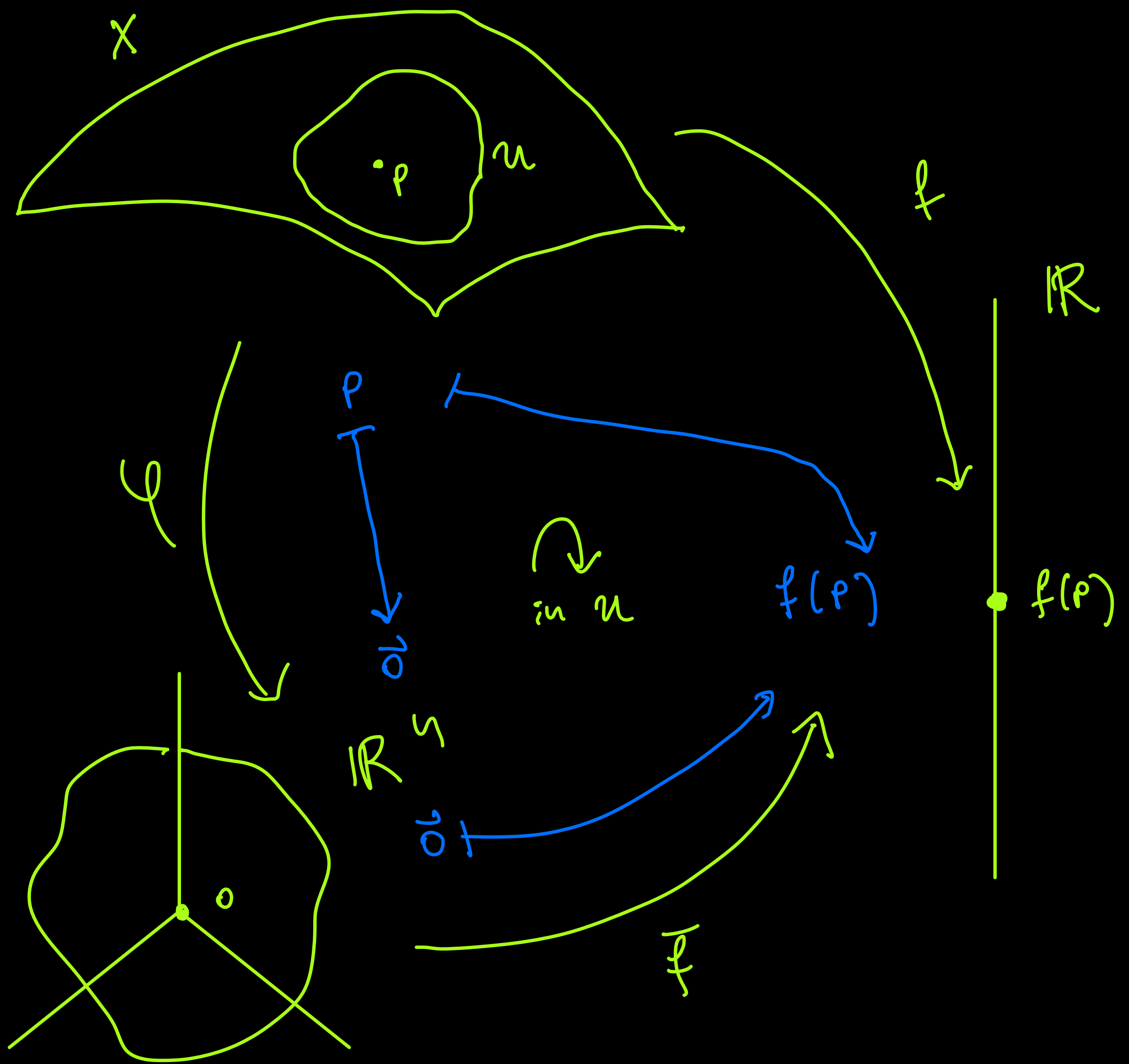
$[\sigma] \otimes [\gamma] \longmapsto 1$

Poincaré duality states such bilinear pairings are non-degenerate in both variables, then for each i-dim submanifold Y the there is another submanifold A of dimension i and a submanifold B of dimension $n-i$ which are in "general position" which respect each other. In particular,

$$\dim_{\mathbb{C}} H_i(M) = \dim_{\mathbb{C}} H_{n-i}(M).$$

• Morse theory

Let X a compact smooth manifold. Let $f: X \longrightarrow \mathbb{R}$ be a smooth "generic" function (more precisely a "Morse function"). By Morse Lemma such a function near a critical point $p \in X$ can be expressed as a diagonal quadratic form for some suitable chart.



$$\bar{f}(x_1, \dots, x_n) = -x_1^2 - x_2^2 - \dots - x_\alpha^2 + x_{\alpha+1}^2 + \dots + x_n^2$$

index of f at p.

where the index of f at p is the dimension of the largest subspace of T_pX on which the Hessian $H_p(f)$ is negative definite.

Corollary. The set of critical points of f are isolated and hence finite.

Let p critical point of f.

$$H_p(f) : T_pX \times T_pX \longrightarrow \mathbb{R}$$

Suppose f "separate" critical points. We have

Prop.

- If $y \neq f(x)$ $\forall x$ critical there is $\varepsilon > 0$ s.t. the inclusion

$$X_{y-\varepsilon} \xhookrightarrow{\iota} X_{y+\varepsilon} \qquad \left(X_z := f^{-1}\big((-\infty, z)\big)\right)$$

induces an isomorphism

$$H_i(X_{y-\varepsilon}) \xrightarrow{\cong} H_i(X_{y+\varepsilon}) \qquad \forall i, \text{ or}$$

- If $y = f(x)$ for some unique x critical

$$H_k(X_{y+\varepsilon}, X_{y-\varepsilon}) = \begin{cases} 0 & \text{if } k \neq \alpha(x) \\ \mathbb{C} & \text{if } k = \alpha(x). \end{cases}$$

NB. $\alpha(x) = \#\{\text{negative eigenvalues of } H_x(f)\}$

This leads to

$$\sum_{x \in C(f)} t^{\alpha(x)} - \sum_{i \geq 0} t^i \dim_{\mathbb{C}} H_i(X) = (1+t)\underbrace{R(t)}_{\in \mathbb{Z}_{\geq 0}[t]}$$

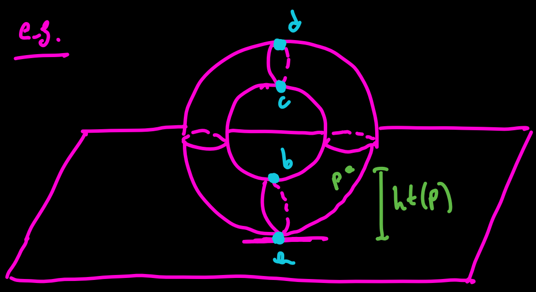
$C(f)$: set of critical points of f

which leads to

$$\dim_{\mathbb{C}} H_i(X) \leq \# \{ x \in C(f) : \alpha(x) = i \}.$$

Morse inequalities

e.g.



$f : \bigcirc \longrightarrow \mathbb{R}$ given by the height

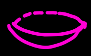
$p \longmapsto ht(p)$ w.r.t a tangent plane

By looking at the Hessian of f at a, b, c, d:

$$\alpha(a) = 0, \quad \alpha(b) = 1, \quad \alpha(c) = 1, \quad \alpha(d) = 2$$

$$X_0 = f^{-1}(\{<0\}) = \phi \quad ,$$

For $\varepsilon > 0$ small enough

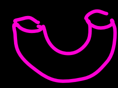
$X_1 = f^{-1}(\{<\varepsilon\}) =$  $\sim pt$ by the proposition above we have

$$H_i(X_1, \phi) = \begin{cases} 0 & \text{if } i \neq 1 \\ \mathbb{C} & \text{if } i = 1 \end{cases}$$

$$\Rightarrow \quad 0 \to H_0(\phi) \to H_0(X_1) \to \mathbb{C} \to H_1(\phi) \to H_1(X_1) \to 0$$

$$\Rightarrow H_i(X_1) = \begin{cases} \mathbb{C} & i = 0. \\ 0 & i \neq 0. \end{cases}$$

$f^{-1}(\{< f(b) - \varepsilon\}) \sim X_1 \sim \bullet$

$X_2 = f^{-1}(\{< f(b) + \varepsilon\}) \sim$  $\sim S^1$

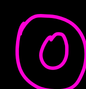
By our proposition

$H_i\Big($  ,  $\Big) = \begin{cases} \mathbb{C} & i = 1 \\ 0 & o/w \end{cases}$

$$0 \to \underbrace{H_1(X_1)}_{0} \to H_1(X_2) \to \mathbb{C} \to \underbrace{H_2(X_1)}_{0} \to H_2(X_2) \to 0$$

$$\to H_i(X_2) = \begin{cases} \mathbb{C} & \text{if } i = 0 \text{ or } 1 \\ 0 & o/w \end{cases} \quad \text{as expected}$$

In the same way we can compute the homology of

$X_3 =$  and $X_4 =$  .

- other interpretations.
  - $H_*(M, \mathbb{R}) \longrightarrow$ differential forms M compact mfld
    or harmonic forms
  - Lefschetz hyperplane theorems

relates $H^i(X)$ and $H^i(X \cap H)$.

§2. The situation with singular spaces

Dealing with real singular manifolds is a huge task. In general, it is not possible to have a rich interpretation for those spaces. However, one can restrict the attention to singular projective varieties and try to solve this problem. This is the context where perverse sheaves, the derived category of constructible sheaves and the intersection homology play an important role.

e.g.

Consider, $X \subset \mathbb{CP}^2$ the complex projective variety:

$$X = \{ [x:y:z] \in \mathbb{CP}^2 \mid yz = 0 \}.$$

$\dim_{\mathbb{C}} X = 1$. e.g. on the big open set $U_0 = \{x \neq 0\}$.

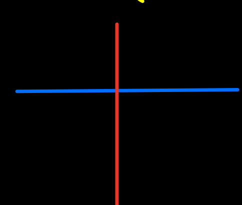
$$X \cap U_0 = \{ [1:y:z] \in \mathbb{CP}^2 \mid yz = 0 \}$$

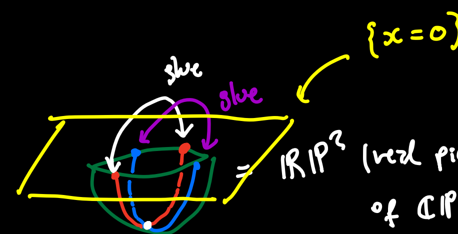
$\cong \{ (y,z) \in \mathbb{C}^2 \mid yz = 0 \}$ affine subvar of $\mathbb{A}^2_{\mathbb{C}}$.

$$\cong \underset{\{y=0\}}{\underset{\shortparallel}{\mathbb{C}}} \cup \underset{\{z=0\}}{\underset{\shortparallel}{\mathbb{C}}}$$

$\mathbb{C} \cap \mathbb{C} = \bullet$

$= [1:0:0]$



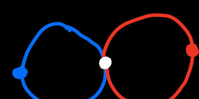


$= \mathbb{RP}^2$ (real picture of $\mathbb{CP}^2$)

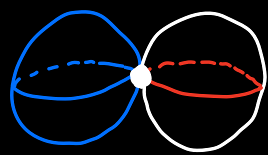
we have to glue $\bullet\ \bullet$

and $\bullet\ \bullet$

Therefore the real picture is

 $= \mathbb{RP}^1 \vee \mathbb{RP}^1$.

Therefore $X = \mathbb{CP}^1 \vee \mathbb{CP}^1$. The complex picture is

$X \simeq$  $\simeq S^2 \vee S^2$.

$$H_i(S^2 \vee S^2) = \left\{ \begin{array}{ll} \mathbb{C} & , i = 0 ; \\ 0 & , i = 1 ; \\ \mathbb{C} \oplus \mathbb{C} & , i = 2 \end{array} \right\}$$ can be computed by "eye".

The other way to see this is to use the fact that $S^2 \vee S^2$ is connected

then $H_0(S^2 \vee S^2) = \mathbb{C}$ and

$$H_i(S^2 \vee S^2) = H_i(S^2) \oplus H_i(S^2) \quad \text{for } i > 0.$$

(Can be deduced from Mayer-Vietoris.)

We see that

$$X = \{[x:y:z] \in \mathbb{CP}^2 \mid y = 0\} \cup \{[x:y:z] \in \mathbb{CP}^2 \mid z = 0\}.$$

These two sets are homeo to $\mathbb{CP}^1$ and intersect in $[1:0:0]$.

Therefore Poincaré duality fails since

$$\dim_{\mathbb{R}} X = 2, \dim_{\mathbb{C}} X = 1. \text{ So } n = 2.$$

$$\dim_{\mathbb{C}} H_0(X) = 1 \neq \dim_{\mathbb{C}} H_2(X) = 2.$$

One solution is introducing the intersection homology groups $IH_*(X)$ of X.

These are cx. vector spaces which are topological invariants of X. They have the property that for any cx. proj. variety of complex dimension n (i.e. real dimension $2n$) whether singular or not, there are non-degenerate pairings (in both variables):

$$IH_i(X) \otimes IH_{2n-i}(X) \longrightarrow \mathbb{C}$$

for $0 \le i \le 2n$. For non-singular X there is a natural isomorphism of functors

$$IH_*(X) \simeq H_*(X)$$

and this pairing is identified with the pairing in (1.3).

The existence of the pairings for intersection homology is a purely topological fact; it does not rely on the complex geometry of X.

This is a huge restriction in the topology, for example, if X is a cx proj variety we have $\dim_{\mathbb{R}} X$ is even. However, the theory allow us to have the same interpretations as before (Morse theory, de Rham isomorphisms, etc...) and new things without a classical analog appear.