

t-structures, leavts, and the recollement situation

Reference. Adami's book section A.7.

§1. t-structures

Let us recall some facts about triangulated categories.

A category \mathcal{T} is a **triangulated category** if it is additive, possesses an autoequivalence $[1]: \mathcal{T} \rightarrow \mathcal{T}$ called the **shift functor**, and a collection of diagrams

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

called **distinguished triangles**. Satisfying certain properties.

Exercise. For a d.t.

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

we have $g \circ f = h \circ g = f[1] \circ h = 0$.

e.g. In section, $D(A)$. For a complex $A^\bullet = (A^i, d_A^i)_{i \in \mathbb{Z}}$

we have $(A^\bullet[1])^i = A^{i+1}$, $d_{A^\bullet[1]}^i = -d_A^{i+1}$.

And exact triangles

$$A^\bullet \xrightarrow{f} B^\bullet \longrightarrow \text{Cone}(f) \xrightarrow{f[1]} A^\bullet[1]$$

$$\text{Cone}(f)^i = (A^\bullet[1] \oplus B)^\bullet = A^{i+1} \oplus B.$$

$$d_{\text{Cone}(f)}^i = \begin{pmatrix} d_{A^\bullet[1]}^i & 0 \\ f[1] & d_B^i \end{pmatrix} = \begin{pmatrix} -d_A^{i+1} & 0 \\ f[1] & d_B^i \end{pmatrix}.$$

Notation: $[n] := [1]^n$ $n \in \mathbb{Z}$.

let \mathcal{T} be a triangulated category, and $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ be a pair of strictly full subcategories.

for $n \in \mathbb{Z}$, let

$$\mathcal{T}^{\leq n} := \mathcal{T}^{\leq 0}[-n] \quad \text{and} \quad \mathcal{T}^{\geq n} := \mathcal{T}^{\geq 0}[-n]$$

The pair $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ is called a **t-structure** on \mathcal{T} if:

(1) $\mathcal{T}^{\leq -1} \subseteq \mathcal{T}^{\leq 0}$ and $\mathcal{T}^{\geq -1} \supseteq \mathcal{T}^{\geq 0}$.

(2) If $X \in \mathcal{T}^{\leq -1}$ and $Y \in \mathcal{T}^{\geq 0}$, then $\text{Hom}(X, Y) = 0$

(3) for any $X \in \mathcal{T}$, there exists a d.t.

$$A \longrightarrow X \longrightarrow B \longrightarrow A[1]$$

with $A \in \mathcal{T}^{\leq -1}$, $B \in \mathcal{T}^{\geq 0}$.

let $\mathcal{H} := \mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq 0}$. This is called the **heart** of $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$.

A t-structure is said to be **bounded below** if $\exists n$ s.t

$$\mathcal{T}^{\geq n} = \mathcal{T}$$

" " " " " **above** " " " "

$$\mathcal{T}^{\leq n} = \mathcal{T}$$

e.g. let \mathcal{A} be an abelian category. Consider $D(\mathcal{A})$ and

$$D(\mathcal{A})_{\leq n}^{\leq 0} = \{ X \mid H^i(X) = 0 \text{ for } i > n \}$$

$$D(\mathcal{A})_{\geq n}^{\geq 0} = \{ X \mid H^i(X) = 0 \text{ for } i < n \}$$

this forms a t-structure on $D(\mathcal{A})$ called the **natural t-structure**.

Idea: For $n \in \mathbb{Z}$ consider the truncation functors

$$\tau^{\leq n} : \mathcal{C}(k) \longrightarrow \mathcal{C}(k)$$

$$A^\bullet \longmapsto \tau^{\leq n} A^\bullet$$

$$(\tau^{\leq n} A^\bullet)^i = \begin{cases} A^i & i < n \\ \ker(d^n) & i = n \\ 0 & i > n \end{cases}$$

$$\tau^{\geq n} : \mathcal{C}(k) \longrightarrow \mathcal{C}(k)$$

$$A^\bullet \longmapsto \tau^{\geq n} A^\bullet$$

$$(\tau^{\geq n} A^\bullet)^i = \begin{cases} 0 & i < n \\ \operatorname{coker}(d^{n-1}) & i = n \\ A^n & i > n \end{cases}$$

Why these strange definitions? • These functors lift to $\operatorname{Fun}(\mathcal{D}(k), \mathcal{D}(k))$

- If $A^\bullet \in \mathcal{D}(k)^{\leq n}$ i.e. $H^i(A^\bullet) = 0$ for $i > n$ then we can replace A^\bullet by $\tau^{\leq n}(A^\bullet)$, i.e. we can work with a complex X^\bullet such that $X^i = 0 \forall i > n$ instead since the map:

$$i : \tau^{\leq n}(A^\bullet) \longrightarrow A^\bullet \text{ given by}$$

$$\begin{array}{ccccccc} \tau^{\leq n}(A^\bullet) : & \dots & \longrightarrow & A^{n-1} & \longrightarrow & \ker d^n & \longrightarrow & 0 & \longrightarrow & \dots \\ & & & \downarrow & & \downarrow & & \downarrow & & \\ i \downarrow & & & & & & & & & \\ A^\bullet : & \dots & \longrightarrow & A^{n-1} & \longrightarrow & A^n & \longrightarrow & A^{n+1} & \longrightarrow & \dots \end{array}$$

is a quasi-isomorphism.

Similarly, for $A^\bullet \in \mathcal{D}(A)^{\geq n}$

$$\tau: A^\bullet \longrightarrow \tau^{\geq n}(A^\bullet) \text{ given by}$$

$$\begin{array}{ccccccc} A^\bullet & : \dots & \longrightarrow & A^{n-1} & \longrightarrow & A^n & \longrightarrow & A^{n+1} & \longrightarrow & \dots \\ \tau \downarrow & & & \downarrow & & \downarrow & & \downarrow & & \\ \tau^{\leq n}(A^\bullet) & : \dots & \longrightarrow & 0 & \longrightarrow & \text{coker } d^n & \longrightarrow & A^{n+1} & \longrightarrow & \dots \end{array}$$

is a quasi-isomorphism.

The axiom (1) is obvious. Let us prove (2) and leave (3) as an exercise.

We will prove the stronger lemma:

Lemma. If $n < m$, $X^\bullet \in \mathcal{D}(A)^{\leq n}$ and $Y^\bullet \in \mathcal{D}(A)^{\geq m}$

$$\text{then } \text{Hom}_{\mathcal{D}(A)}(X^\bullet, Y^\bullet) = 0.$$

Proof. Since $Y^\bullet \in \mathcal{D}(A)^{\geq m}$, $Y^\bullet \simeq \tau^{\geq m}(Y^\bullet)$ so we can
i.i.

assume $Y^i = 0$ for $i < m$.

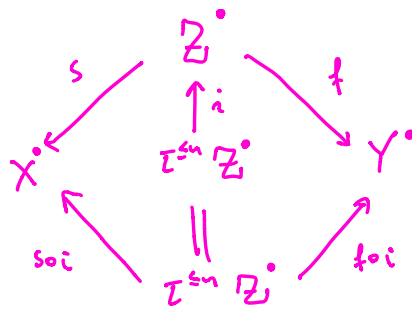
Suppose we have a roof

$$\begin{array}{ccc} & Z^\bullet & \\ s \swarrow & & \searrow f \\ X^\bullet & & Y^\bullet \end{array}$$

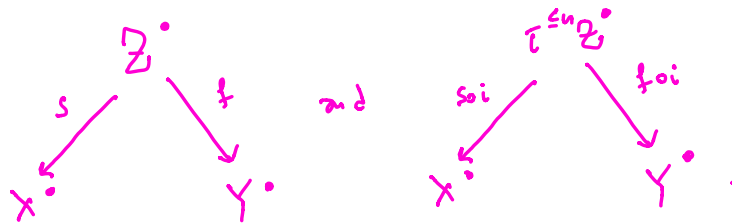
where s is a p.isom. Then $Z^\bullet \simeq X^\bullet \simeq \tau^{\leq n}(X^\bullet)$
i.i.

Therefore $Z^\bullet \in \mathcal{D}(A)^{\leq n}$, so we can assume $Z^\bullet = \tau^{\leq n} Z^\bullet$

since



is an equivalence b/w the roofs



foi is the homotopy class of the map

$$\begin{array}{ccccccc}
 \tau^{\leq n} \mathbb{Z}^{\bullet} & : & \mathbb{Z}^m & \rightarrow & \ker(d^n) & \rightarrow & 0 \rightarrow \dots \rightarrow 0 \rightarrow \dots \\
 \downarrow \text{foi} & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 Y & : & 0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & \dots & 0 & \rightarrow Y^m \rightarrow \dots
 \end{array}$$

Therefore $\text{foi} = 0 \in \pi_1$.

Exercise. $A \xrightarrow{D} D(A)$ is fully faithful

$$A^{\bullet} \longmapsto (\dots \rightarrow 0 \rightarrow A \rightarrow 0 \rightarrow \dots)$$

degree 0

Then $A \cong \text{essential image of } D \cong D^{\leq 0}(A) \cap D^{\geq 0}(A)$.

In particular, the heart of $D(A)$ is abelian!

Warning. Let \mathcal{T} be a triangulated category with

$(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ a t -structure, let $\mathcal{E} = \mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq 0}$ be its heart. In general, it is not true that

$$\mathcal{D}(\mathcal{E}) \simeq \mathcal{T}.$$

Lemma. (A.7.8 and A.7.9.) Let \mathcal{T} and $\mathcal{E} = \mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq 0}$ as before.

The heart \mathcal{E} is a full abelian subcategory of \mathcal{T} . Furthermore, if

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

are two morphisms in \mathcal{E} , the sequence

$$0 \rightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \rightarrow 0$$

is exact iff there exists $h: Z \rightarrow X[1]$ in \mathcal{T} s.t.

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

is a dist. triangle.

Let $\mathcal{T}_1, \mathcal{T}_2$ two triangulated categories with t -structures

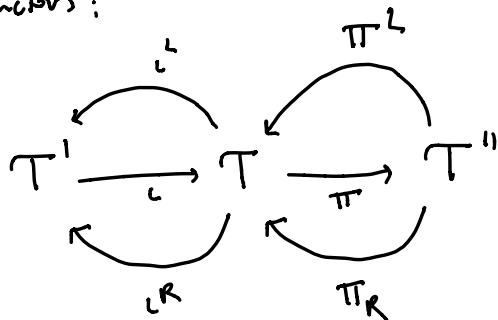
$(\mathcal{T}_1^{\leq 0}, \mathcal{T}_1^{\geq 0})$ and $(\mathcal{T}_2^{\leq 0}, \mathcal{T}_2^{\geq 0})$ resp. A triangulated

functor $F: \mathcal{T}_1 \rightarrow \mathcal{T}_2$ is left t -exact (resp. right)

if $F(\mathcal{T}_1^{\geq 0}) \subset \mathcal{T}_2^{\geq 0}$ (resp. $F(\mathcal{T}_1^{\leq 0}) \subset \mathcal{T}_2^{\leq 0}$).

§2. The recollement situation (Exercises A.7.4. and A.7.6)

A recollement diagram consists of three triangulated categories and six triangulated functors:



such that

- (a) (L^L, L) & (L, L^R) are adjoint pairs.
- (b) (Π^L, Π) & (Π, Π^R) are adjoint pairs.
- (c) $\Pi \circ L = 0$.
- (d) For any $X \in T$, there are d.f.

$$\begin{aligned} L^R(X) \longrightarrow X \longrightarrow \Pi^R \Pi(X) \longrightarrow L^R(X) [1] \\ \Pi^L \Pi(X) \longrightarrow X \longrightarrow L^L(X) \longrightarrow \Pi^L \Pi(X) [1] \end{aligned}$$

(e) L, Π^L, Π^R are fully faithful.

Suppose T' has a t-structure $(T'^{\leq 0}, T'^{\geq 0})$,

T'' has a t-structure $(T''^{\leq 0}, T''^{\geq 0})$.

There is a unique t-structure $(T^{\leq 0}, T^{\geq 0})$ in T s.t. L, Π are t-exact. More precisely,

$$\begin{aligned} T^{\leq 0} &= \{ X \in T \mid L^L(X) \in T'^{\leq 0} \text{ and } \Pi(X) \in T''^{\leq 0} \} \\ T^{\geq 0} &= \{ X \in T \mid L^R(X) \in T'^{\geq 0} \text{ and } \Pi(X) \in T''^{\geq 0} \} \end{aligned}$$

Furthermore, ι^R and π^R are left t -exact,
 ι^L and π^L are right t -exact.

This t -structure on \mathcal{T} is said to be obtained by **recollament**
or **gluing** from those on \mathcal{T}' and \mathcal{T}'' .