t-structures, hearts, and the recollement situation

Reference. Achar's book section A.7.

§1. t-structures

Let us recall some facts about triangulated categories.

A category $\mathcal{T}$ is a triangulated category if it is additive, possesses an autoequivalence $[1]: \mathcal{T} \longrightarrow \mathcal{T}$ called the shift functor, and a collection of diagrams

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

called distinguished triangles. Satisfying certain properties.

Exercise. For a d.t.

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

we have $g \circ f = h \circ g = f[1] \circ h = 0$.

e.g. $\mathcal{A}$ abelian, $D(\mathcal{A})$. For a complex $A^\bullet = (A^i, d_A^i)_{i \in \mathbb{Z}}$

we have $(A^\bullet[1])^i = A^{i+1}$, $d^i_{A[1]} = -d_A^{i+1}$.

And exact triangles

$$A^\bullet \xrightarrow{f} B^\bullet \longrightarrow \mathrm{Cone}(f) \xrightarrow{f[1]} A^\bullet[1]$$

$$\mathrm{Cone}(f)^i = (A^\bullet[1] \oplus B)^i = A^{i+1} \oplus B.$$

$$d^i_{\mathrm{Cone}(f)} = \begin{pmatrix} d^i_{A[1]} & 0 \\ f[1] & d^i_B \end{pmatrix} = \begin{pmatrix} -d^{i+1} & 0 \\ f[1] & d^i_B \end{pmatrix}.$$

Notation: $[n] := [1]^n \quad n \in \mathbb{Z}$.

Let $\mathcal{T}$ be a triangulated category, and $(\mathcal{T}^{\le 0}, \mathcal{T}^{\ge 0})$ be a pair of strictly full subcategories.

For $n \in \mathbb{Z}$, let

$$\mathcal{T}^{\le n} := \mathcal{T}^{\le 0}[-n] \quad \text{and} \quad \mathcal{T}^{\ge n} := \mathcal{T}^{\ge 0}[-n]$$

The pair $(\mathcal{T}^{\le 0}, \mathcal{T}^{\ge 0})$ is called a t-structure on $\mathcal{T}$ if:

(1) $\mathcal{T}^{\le -1} \subseteq \mathcal{T}^{\le 0}$ and $\mathcal{T}^{\ge -1} \supseteq \mathcal{T}^{\ge 0}$.

(2) If $X \in \mathcal{T}^{\le -1}$ and $Y \in \mathcal{T}^{\ge 0}$, then $\mathrm{Hom}(X, Y) = 0$

(3) For any $X \in \mathcal{T}$, there exists a d.t.

$$A \longrightarrow X \longrightarrow B \longrightarrow A[1]$$

with $A \in \mathcal{T}^{\le -1}$, $B \in \mathcal{T}^{\ge 0}$.

Let $\mathcal{C} := \mathcal{T}^{\le 0} \cap \mathcal{T}^{\ge 0}$. This is called the heart of $(\mathcal{T}^{\le 0}, \mathcal{T}^{\ge 0})$.

A t-structure is said to be bounded below if $\exists n$ s.t.

$$\mathcal{T}^{\ge n} = \mathcal{T}$$

" " " " " above " " " "

$$\mathcal{T}^{\le n} = \mathcal{T}.$$

e.g. Let $\mathcal{A}$ be an abelian category. Consider $D(\mathcal{A})$ and

$$D(\mathcal{A})^{\le 0}_{\le n} = \{ X \mid H^i(X) = 0 \text{ for } \underset{>n}{i>0} \}$$

$$D(\mathcal{A})^{\ge 0}_{\ge n} = \{ X \mid H^i(X) = 0 \text{ for } \underset{<n}{i<0} \}$$

This forms a t-structure on $D(\mathcal{A})$ called the natural t-structure.

Idea: For $n \in \mathbb{Z}$ consider the truncation functors

$$\tau^{\le n} : C^\bullet(\mathcal{A}) \longrightarrow C^\bullet(\mathcal{A})$$

$$A^\bullet \longmapsto \tau^{\le n} A^\bullet$$

$$(\tau^{\le n} A^\bullet)^i = \begin{cases} A^i & i < n \\ \ker(d^n) & i = n \\ 0 & i > n \end{cases}$$

$$\tau^{\ge n} : C^\bullet(\mathcal{A}) \longrightarrow C^\bullet(\mathcal{A})$$

$$A^\bullet \longmapsto \tau^{\ge n} A^\bullet$$

$$(\tau^{\ge n} A^\bullet)^i = \begin{cases} 0 & i < n \\ \operatorname{coker}(d^{n-1}) & i = n \\ A^n & i > n \end{cases}$$

Why these strange definitions? • These functors lift to $\mathrm{Fun}(D(\mathcal{A}), D(\mathcal{A}))$

- If $A^\bullet \in D(\mathcal{A})^{\le n}$ i.e. $H^i(A^\bullet) = 0$ for $i > n$ then we can replace $A^\bullet$ by $\tau^{\le n}(A^\bullet)$, i.e. we can work with a complex $X^\bullet$ such that $X^i = 0$ $\forall i > n$ instead since the map:

$$i : \tau^{\le n}(A^\bullet) \longrightarrow A^\bullet \quad \text{given by}$$

$$\begin{tikzcd}
\tau^{\le n}(A^\bullet) : \arrow[d, "i"'] & \cdots \arrow[r] & A^{n-1} \arrow[r] \arrow[d] & \ker d^n \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
A^\bullet : & \cdots \arrow[r] & A^{n-1} \arrow[r] & A^n \arrow[r] & A^{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

is a quasi-isomorphism.

Similarly, for $A^\bullet \in D(\mathcal{A})^{\geq n}$

$$q : A^\bullet \longrightarrow \tau^{\geq n}(A^\bullet) \quad \text{given by}$$

$$\begin{tikzcd}
A^\bullet \arrow[d, "q"'] : \cdots \arrow[r] & A^{n-1} \arrow[r] \arrow[d] & A^n \arrow[r] \arrow[d] & A^{n+1} \arrow[r] \arrow[d] & \cdots \\
\tau^{\leq n}(A^\bullet) : \cdots \arrow[r] & 0 \arrow[r] & \operatorname{coker} d^n \arrow[r] & A^{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

is a quasi-isomorphism.

The axiom (1) is obvious. Let us prove (2) and leave (3) as an exercise.

We will prove the stronger lemma:

__Lemma.__ If $n < m$, $X^\bullet \in D(\mathcal{A})^{\leq n}$ and $Y^\bullet \in D^{\geq m}(\mathcal{A})$

then $\operatorname{Hom}_{D(\mathcal{A})}(X^\bullet, Y^\bullet) = 0$.

__Proof.__ Since $Y^\bullet \in D(\mathcal{A})^{\geq m}$, $Y^\bullet \underset{q.i.}{\simeq} \tau^{\geq m}(Y^\bullet)$ so we can

assume $Y^i = 0$ for $i < m$.

Suppose we have a roof

$$\begin{tikzcd}
& Z^\bullet \arrow[dl, "s"'] \arrow[dr, "f"] & \\
X^\bullet & & Y^\bullet
\end{tikzcd}$$

where s is a q.isom. Then $Z^\bullet \underset{q.i}{\simeq} X^\bullet \simeq \tau^{\leq n}(X^\bullet)$

Therefore $Z^\bullet \in D(\mathcal{A})^{\leq n}$, so we can assume $Z^\bullet = \tau^{\leq n} Z^\bullet$

Since

$$
\begin{tikzcd}
 & Z^\bullet \arrow[dl, "s"'] \arrow[dr, "f"] & \\
X^\bullet & \tau^{\leq n} Z^\bullet \arrow[u, "i"'] & Y^\bullet \\
 & \tau^{\leq n} Z^\bullet \arrow[ul, "s\circ i"] \arrow[ur, "f\circ i"'] \arrow[u, equal] &
\end{tikzcd}
$$

is an equivalence b/w the roofs

$$
\begin{tikzcd}
 & Z^\bullet \arrow[dl, "s"'] \arrow[dr, "f"] & \\
X^\bullet & & Y^\bullet
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
 & \tau^{\leq n} Z^\bullet \arrow[dl, "s\circ i"'] \arrow[dr, "f\circ i"] & \\
X^\bullet & & Y^\bullet
\end{tikzcd}
.
$$

$f\circ i$ is the homotopy class of the map

$$
\begin{tikzcd}
\tau^{\leq n} Z^\bullet \arrow[d, "f\circ i"] & : & Z^{n-1} \arrow[r] \arrow[d] & \ker(d^n) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
Y & : & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots & 0 \arrow[r] & Y^m \arrow[r] & \cdots
\end{tikzcd}
$$

Therefore $f\circ i = 0$ $\square$.

Exercise. $\mathcal{A} \xrightarrow{D} D(\mathcal{A})$ is fully faithful

$$A^\bullet \longmapsto (\cdots \to 0 \to \underset{\text{degree } 0}{A} \to 0 \cdots)$$

Then $\mathcal{A} \cong$ essential image of D $\cong D^{\leq 0}(\mathcal{A}) \cap D^{\geq 0}(\mathcal{A})$.

In particular, the heart of $D(\mathcal{A})$ is abelian!

⚠ Warning. Let $\mathcal{T}$ be a triangulated category with $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ a t-structure, let $\mathcal{E} = \mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq 0}$ be its heart. In general, it is not true that

$$D(\mathcal{E}) \simeq \mathcal{T}.$$

Theorem. (A.7.8 and A.7.9) Let $\mathcal{T}$ and $\mathcal{E} = \mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq 0}$ as before. The heart $\mathcal{E}$ is a full abelian subcategory of $\mathcal{T}$. Furthermore, if

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

are two morphisms in $\mathcal{E}$, the sequence

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

is exact iff there exists $h: Z \longrightarrow X[1]$ in $\mathcal{T}$ s.t.

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

is a dist. triangle.

Let $\mathcal{T}_1, \mathcal{T}_2$ two triangulated categories with t-structures $(\mathcal{T}_1^{\leq 0}, \mathcal{T}_1^{\geq 0})$ and $(\mathcal{T}_2^{\leq 0}, \mathcal{T}_2^{\geq 0})$ resp. A triangulated functor $F: \mathcal{T}_1 \longrightarrow \mathcal{T}_2$ is left t-exact (resp right) if $F(\mathcal{T}_1^{\geq 0}) \subset \mathcal{T}_2^{\geq 0}$ (resp. $F(\mathcal{T}_1^{\leq 0}) \subset \mathcal{T}_2^{\leq 0}$.

§2. The recollement situation (Exercises A.7.4. and A.7.6)

A **recollement diagram** consists of three triangulated categories and six triangulated functors:

$$\begin{tikzcd} T' \arrow[r, "\iota"'] & T \arrow[l, bend right, "\iota^L"'] \arrow[l, bend left, "\iota^R"] \arrow[r, "\pi"'] & T'' \arrow[l, bend right, "\pi^L"'] \arrow[l, bend left, "\pi_R"] \end{tikzcd}$$

such that

(a) (ι^L, ι) & (ι, ι^R) are adjoint pairs.

(b) (π^L, π) & (π, π^R) are adjoint pairs.

(c) $\pi \circ \iota = 0$.

(d) For any $X \in T$, there are d.t.

$$\iota\iota^R(X) \longrightarrow X \longrightarrow \pi^R\pi(X) \longrightarrow \iota\iota^R(X)[1]$$
$$\pi^L\pi(X) \longrightarrow X \longrightarrow \iota\iota^L(X) \longrightarrow \pi^L\pi(X)[1]$$

(e) ι, π^L, π^R are fully faithful.

Suppose T' has a t-structure $(T'^{\le 0}, T'^{\ge 0})$,
T'' has a t-structure $(T''^{\le 0}, T''^{\ge 0})$.

There is a unique t-structure $(T^{\le 0}, T^{\ge 0})$ in T s.t. ι, π are t-exact. More precisely,

$$T^{\le 0} = \{ X \in T \mid \iota^L(X) \in T'^{\le 0} \text{ and } \pi(X) \in T''^{\le 0} \}$$
$$T^{\ge 0} = \{ X \in T \mid \iota^R(X) \in T'^{\ge 0} \text{ and } \pi(X) \in T''^{\ge 0} \}$$

Furthermore, ι^R and π^R are left t-exact,

ι^L and π^L are right t-exact.

This t-structure on $\mathcal{T}$ is said to be obtained by recollement or gluing from those on $\mathcal{T}'$ and $\mathcal{T}''$.