Cuspons and peakons vis-a-vis regular solitons and collapse in a three-wave system

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ABSTRACT

We introduce a general model of a one-dimensional three-component wave system with cubic nonlinearity. Linear couplings between the components prevent intersections between the corresponding dispersion curves, which opens two gaps in the system's linear spectrum. Detailed analysis is performed for zero-velocity solitons, in the reference frame in which the group velocity of one wave is zero. Disregarding the self-phase-modulation (SPM) term in the equation for that wave, we find an analytical solution which shows that there simultaneously exist two different families of generic solitons: regular ones, which may be regarded as a smooth deformation of the usual *gap solitons* in the two-wave system, and *cuspons* with a singularity in the first derivative at the center, while their energy is finite. Even in the limit when the linear coupling of the zero-group-velocity wave to the other two components is vanishing, the soliton family remains drastically different from that in the linearly uncoupled system: in this limit, regular solitons whose amplitude exceeds a certain critical value are replaced by *peakons*. While the regular solitons, cuspons, and peakons are found in an exact analytical form, their stability is tested numerically, showing that they all may be stable. In the case when the cuspons are unstable, the instability may trigger onset of *spatio-temporal collapse* in the system. If the SPM terms are retained, we find that there again simultaneously exist two different families of generic stable soliton solutions, which are regular ones and peakons. The existence of the peakons depends, in this case, on the sign of certain parameters of the system. Direct simulations show that both types of the solitons may be stable in this most general case too.

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I. INTRODUCTION

A. Gap-soliton models

Gap solitons (GS) is a common name for solitary waves in nonlinear systems which feature one or more gaps in their linear spectrum [1]. A soliton may exist if its frequency belongs to the gap, as then it does not decay into linear waves.

Gaps in the linear spectrum are a generic phenomenon in two- or multicomponent systems, as intersection of dispersion curves belonging to different components is, generically, prevented by a linear coupling between the components. Excluding cases when the zero solution in the system is unstable [2], the intersection avoidance alters the spectrum so that a gap opens in place of the intersection. Approximating the two dispersion curves, that would intersect in the absence of the linear coupling, by straight lines, and assuming a generic cubic nonlinearity, one arrives at a *generalized massive Thirring model* (GMTM) for two wave fields $u_{1,2}(x,t)$:

$$i(\frac{\partial u_1}{\partial t} - \frac{\partial u_1}{\partial x}) + u_2 + \left(\sigma|u_1|^2 + |u_2|^2\right) u_1 = 0\,, \tag{1}$$

$$i(\frac{\partial u_2}{\partial t} + \frac{\partial u_2}{\partial x}) + u_1 + \left(\sigma|u_2|^2 + |u_1|^2\right) u_2 = 0\,, \tag{2}$$

where the group velocities of the two waves are normalized to be ± 1, the linear-coupling constant and the coefficient of the nonlinear *cross-phase-modulation* (XPM) coupling may also be normalized to be 1, and $\sigma \geq 0$ is the *self-phase-modulation* (SPM) coefficient.

The model based on Eqs. (1) and (2) with $\sigma = 1/2$ has a direct, and very important, application to nonlinear optics, describing co-propagation of left- and right-traveling electromagnetic waves in a fiber with a resonant Bragg grating (BG) written on it [3,4,1]. The version of the model corresponding to $\sigma \to \infty$, i.e., with the SPM nonlinearity only,

$$i(\frac{\partial u_1}{\partial z} - \frac{\partial u_1}{\partial \tau}) + u_2 + |u_1|^2 u_1 = 0\,, \tag{3}$$

$$i(\frac{\partial u_2}{\partial z} + \frac{\partial u_2}{\partial \tau}) + u_1 + |u_2|^2 u_2 = 0\,, \tag{4}$$

may also be realized in terms of nonlinear fiber optics, describing co-propagation of light in a dual-core fiber with a group-velocity mismatch between the cores (which is normalized to be 1), while the intrinsic dispersion of the cores is neglected [5]. In Eqs. (3) and (4), the evolutional variable is not time, but rather the propagation distance z, while the role of x is played by the so-called reduced time, $\tau \equiv t - z/V_0$, where V_0 is the mean group velocity of the carrier wave.

It had been demonstrated more than twenty years ago that the massive Thirring model proper, which corresponds to Eqs. (1) and (2) with $\sigma = 0$, is *exactly integrable* by means of the inverse scattering transform, and, moreover, it can be explicitly transformed into the sine-Gordon equation [6]. On the other hand, it was also demonstrated that GMTM with any $\sigma \neq 0$ is *not* integrable (this conclusion follows, for instance, from an early observation that collisions between solitons are inelastic if $\sigma \neq 0$ [4]. Nevertheless, the general model (1), (2) with an arbitrary value of σ has a family of exact GS solutions that completely fill the gap in its spectrum. Gap solitons, first predicted theoretically [3,4], were observed in experiments with light pulses launched into a short piece of the BG-equipped fiber [7] (in fact, optical solitons that were first observed in the BG fiber [8] were, strictly speaking, not of the GS type, but more general ones, whose central frequency did not belong to the fiber's bandgap).

Models giving rise to GSs are known not only in optics but also in other areas, for instance, in hydrodynamics of density-stratified fluids, where dispersion curves pertaining to different internal-wave modes can readily intersect. Taking into regard the nonlinearity, one can easily predict the occurrence of GS in density-stratified fluids [10].

B. Introducing a three-wave model

In this work, we aim to study GSs in a system of *three* coupled waves, assuming that the corresponding three dispersion curves are close to intersection at a single point, unless linear couplings are taken into regard. Of course, the situation with three curves passing through a single point is degenerate. Our objective is to investigate GS not for this special case, but in its vicinity in the parameter space. We will demonstrate that families of GS solutions in the three-wave systems is drastically different from that in the two-wave GMTM. In particular, generic solutions will include not only regular solitons, similar to those known in GMTM, but also *cuspons* and *peakons*, i.e., solitons with a divergence or jump of the first derivative, but, nevertheless, with finite amplitude and energy. Moreover, we will demonstrate that a part of the cuspon and peakon solutions are completely stable ones. Another principal difference of the three-wave system from its two-wave counterpart is that the former one may give rise to *spatio-temporal collapse*, i.e., formation of a singularity of the wave fields in finite time. We will demonstrate that, in the cases when cuspons or peakons are unstable, their instability may easily provoke the onset of the collapse [9].

Three-wave systems of this type can readily occur in the above-mentioned density-stratified flows [11], and are also possible in optics. For instance, this case takes place in a *resonantly absorbing* BG, which are arranged as a system of thin ($\sim$ 100 nm) parallel layers of two-level atoms, with the spacing between them equal to half the wavelength of light. This system combines the resonant Bragg reflection and self-induced transparency (SIT), see Ref. [12] and references therein. A model describing the BG-SIT system includes equations for three essential fields, viz., local amplitudes of right- and left-traveling electromagnetic waves, and the inversion rate of the two-level atoms (which, obviously, has zero group velocity in the laboratory reference frame). This model indeed produces a linear spectrum with three dispersion curves close to intersecting at one point, so that two gaps open in the system's spectrum.

Another realization of gaps between three dispersion curves is possible in terms of stationary optical fields in a planar nonlinear waveguide equipped with BG in the form of parallel scores [13]. In this case, the resonant Bragg reflection linearly couples waves propagating in two different directions. To induce linear couplings between all the three waves in the system, it is necessary to have a planar waveguide with *two* different BG systems of parallel scores, oriented in different directions. Postponing a consideration of this rather complicated model to another work, we here give a simple example for a case when the single BG is aligned along the axis x, perpendicular to the propagation direction z. Two waves $u_{1,2}$ have opposite incidence angles with respect to the BG, while the third wave u_3 has its wave vector parallel to x, see Fig. 1 in Ref. [13]. Then, assuming that the size of the sample is much smaller than the diffraction length of a broad spatial beam, but is larger than a characteristic length induced by strong artificial diffraction induced by BG, normalized equations governing the spatial evolution of the fields in the planar waveguide with the usual Kerr nonlinearity are

$$i\left(\frac{\partial u_1}{\partial z} - \frac{\partial u_1}{\partial x}\right) + u_2 + \left(\frac{1}{2}|u_1|^2 + |u_2|^2 + |u_3|^2\right) u_1 = 0\,, \tag{5}$$

$$i\left(\frac{\partial u_2}{\partial z} + \frac{\partial u_2}{\partial x}\right) + u_1 + \left(\frac{1}{2}|u_2|^2 + |u_1|^2 + |u_3|^2\right) u_2 = 0\,, \tag{6}$$

$$i\frac{\partial u_3}{\partial z} + \left(\frac{1}{2}|u_3|^2 + |u_1|^2 + |u_2|^2\right) u_3 = k_0 u_3\,, \tag{7}$$

where k_0 is a wavenumber mismatch between the third and first two waves.

The model based on Eqs. (5) - (7) represents a particular case only, as it does not include linear couplings between the waves $u_{1,2}$ and u_3. We aim to introduce a generic model describing a nonlinear system of three waves with linear couplings between all of them. We assume that the system can be derived from a Hamiltonian, and confine attention to the case of cubic nonlinearities. Taking into regard these restrictions, and making use of scaling invariances to diminish the number of free parameters, we arrive at a system

$$i(\frac{\partial u_1}{\partial t} - \frac{\partial u_1}{\partial x}) + u_2 + \kappa u_3 + \alpha\left(\alpha\sigma_1|u_1|^2 + \alpha|u_2|^2 + |u_3|^2\right)u_1 = 0\,, \tag{8}$$

$$i(\frac{\partial u_2}{\partial t} + \frac{\partial u_2}{\partial x}) + u_1 + \kappa u_3 + \alpha\left(\alpha\sigma_1|u_2|^2 + \alpha|u_1|^2 + |u_3|^2\right)u_2 = 0\,, \tag{9}$$

$$i\frac{\partial u_3}{\partial t} + \kappa\,(u_1 + u_2) + \left(\sigma_3|u_3|^2 + \alpha|u_1|^2 + \alpha|u_2|^2\right)u_3 = \omega_0 u_3\,. \tag{10}$$

Here, we consider the evolution in the temporal domain, unlike the spatial-domain evolution in Eqs. (5) - (7), and without loss of generality, we use a reference frame in which the third wave u_3 has zero group velocity. Note that the coefficient of the linear coupling between the first two waves is normalized to be 1, while κ accounts for their linear coupling to the third wave, and it may always be defined to be positive.

We assume full symmetry between the two waves $u_{1,2}$, following the pattern of the GMT model; in particular, the group velocities of these waves are normalized to be ∓ 1. However, we note that this assumption is not essential, and we shall comment later on the case when the group-velocity terms in Eqs. (8) and (9) are generalized as follows:

$$-\frac{\partial u_1}{\partial x} \to -c_1\frac{\partial u_1}{\partial x}\,,\;+\frac{\partial u_2}{\partial x} \to +c_2\frac{\partial u_1}{\partial x}\,, \tag{11}$$

where c_1 and c_2 are different, but have the same sign. Note that the symmetry of the system's dispersion law $\omega = \omega(k)$ is assumed with respect to the sign of k, but not of ω. To this end, the parameter ω_0 was added to Eq. (3). This parameter breaks the "ω-symmetry", that, unlike the "k-symmetry", does not have any natural cause to exist.

The coefficients $\sigma_{1,3}$ and α in Eqs. (8) - (10) account for the nonlinear SPM and XPM nonlinearities, respectively. In particular, α is defined as a relative XPM coefficient between the first two and the third waves, hence it is an irreduceable parameter. As for the SPM coefficients, both σ_1 and σ_3 may be normalized to be ± 1, unless they are equal to zero; however, it will be convenient to keep them as free parameters, see below (note that the SPM coefficients are always positive in the optical models, but in those describing stratified fluids they may have either sign).

Equations (8) - (10) conserve the norm, which is frequently called energy in optics,

$$N \equiv \sum_{n=1,2,3} \int_{-\infty}^{+\infty} |u_n(x)|^2 \, dx, \tag{12}$$

the Hamiltonian,

$$H \equiv H_{\mathrm{grad}} + H_{\mathrm{coupl}} + H_{\mathrm{focus}}, \tag{13}$$

$$H_{\mathrm{grad}} \equiv \frac{i}{2} \int_{-\infty}^{+\infty} \left(u_1^* \frac{\partial u_1}{\partial x} - u_2^* \frac{\partial u_1}{\partial x} \right) dx + \mathrm{c.c.}, \tag{14}$$

$$H_{\mathrm{coupl}} \equiv - \int_{-\infty}^{+\infty} \left[u_1^* u_2 + \kappa u_3^* \left(u_1 + u_2 \right) \right] dx + \mathrm{c.c.}, \tag{15}$$

$$H_{\mathrm{focus}} \equiv - \int_{-\infty}^{+\infty} \left[\frac{1}{2} \alpha^2 \sigma_1 \left(|u_1|^4 + |u_2|^4 \right) + \frac{1}{2} \sigma_3 |u_3|^4 + \alpha^2 |u_1|^2 |u_2|^2 + \alpha |u_3|^2 \left(|u_1|^2 + |u_2|^2 \right) \right] dx \,, \tag{16}$$

and the momentum, which will not be used here. In these expressions, the asterisk and c.c. both stand for complex conjugation, H_{grad}, H_{coupl} and H_{focus} being the gradient, linear-coupling, and self-focusing parts of the Hamiltonian.

C. Solitons in the three-wave models

Our objective is to find various types of solitons existing in the generic three-wave system (8) - (10) and investigate their stability. The existence of various types of the solitons is considered in section 3. Focusing first on the case (suggested by the analogy with GMTM) when the SPM term in Eq. (10) may be neglected (i.e., $\sigma_3 = 0$), we will find a general family of zero-velocity solitons in an exact analytical form. We will demonstrate that the family contains solutions of two drastically different types: regular GSs, and *cuspons*, i.e., solitons with a cusp singularity at the center, while their energy is finite (this singularity assumes that the function remains finite at the cusp point, while its first derivative diverges). Cuspons are known to exist in degenerate models without linear terms (except for the evolution term such as $\partial u/\partial t$), a well-known example being the Camassa-Holm (CH) equation [17,18]. As well as the massive Thirring model (1), (2) with $\sigma = 0$, the CH equation is *exactly integrable* by means of the inverse scattering transform. Our nonintegrable model, as well as the CH one, gives rise to coexisting solutions in the form of regular solitons and cuspons. However, the cause for the existence of cuspons in our model is very different: looking for a zero-velocity soliton solution, one may eliminate the field u_3 by means of an algebraic relation following, in this case, from Eq. (10). The subsequent substitution of this result into the first two equations (8) and (9) produces a *non-polynomial* (in fact, rational) nonlinearity in them. The corresponding rational functions feature a singularity at some (critical) value of the soliton's amplitude. If the amplitude of a formal regular-soliton solution exceeds the critical value, it actually cannot exist, and, in the case when $\sigma_3 = 0$, it is replaced by a cuspon, whose amplitude is exactly equal to the critical value.

In the limit $\kappa \to 0$, which corresponds to the vanishing linear coupling between the first two and third waves, the cuspon resembles a *peakon*, which is a finite-amplitude solitary wave with a jump of its first derivative at the center. Note that peakon solutions, coexisting with regular solitons (they also coexist our model), are known in a slightly different version of the CH equation (which is also integrable by means of the inverse scattering transform), see, e.g., Ref. [17,19,20].

Then, we show that, when the SPM term in Eq. (10) is restored in Eq. (10) (i.e., $\sigma_3 \neq 0$; the presence or absence of the SPM term $\propto \sigma_1$ in Eqs. (8) and (9) is not crucially important), the system supports a different set of soliton families. These are regular GSs and, depending on the sign of certain parameters, a family of peakons, which, this time, appear as generic solutions, unlike the case $\sigma_3 = 0$, when they only exist as a limiting form of the solutions corresponding to $\kappa \to 0$. As far as we know, the model formulated in the present work is the first non-degenerate one (i.e., a model with a nonvanishing linear part) which yields both cuspons and peakons.

D. Stability of the solitons and spatiotemporal collapse

As concerns the dynamical stability of the various solitons in the model (8) -(10), in this work we limit ourselves to direct simulations, as a more rigorous approach, based on numerical analysis of the corresponding linear stability-eigenvalue problem [21], is technically difficult in the case of cuspons and peakons. In fact, direct simulations of perturbed cuspons and peakons is a hard problem too, but we have checked that identical results concerning the stability are produced (see section 3 below) by high-accuracy finite-difference and pseudo-spectral methods (each being implemented in more than one particular form), which lends the results credibility. A general conclusion is that the regular solitons are always stable. As for the cuspons and peakons, they may be either stable or unstable. If the cusp is strong enough, instability of the cuspon initiates formation of a genuine singularity, i.e., onset of a *spatiotemporal collapse* [9] in the present one-dimensional model.

Note that a simple virial-type estimate for the possibility of collapse can be made, assuming that the field focuses itself in a narrow spot with a size $L(t)$, amplitude $\aleph(t)$, and a characteristic value $K(t)$ of the field's wavenumber [9]. The conservation of the norm (12) imposes a restriction $\aleph^2 L \sim N$, i.e., $L \sim N/\aleph^2$. Next, the self-focusing part (13) of the Hamiltonian (13), which drives the collapse, can be estimated as

$$H_{\mathrm{focus}} \sim -\aleph^4 L \sim -N\aleph^2. \tag{17}$$

On the other hand, the collapse can be checked by the gradient term (14) in the full Hamiltonian, that, in the same approximation, can be estimated as $H_{\mathrm{grad}} \sim \aleph^2 KL \sim NK$. Further, Eqs. (8) - (10) suggest an estimate $K \sim \aleph^2$ for a characteristic wavenumber of the wave field (the same estimate for K follows from an expression (24) for the exact stationary-soliton solution given below), thus we have $H_{\mathrm{grad}} \sim N\aleph^2$. Comparing this with the expression (17), one concludes that the parts of the Hamiltonian promoting and inhibiting the collapse scale the same way as $\aleph \to \infty$ (or $L \to 0$), hence a *weak collapse* [9] may be possible (but not necessarily) in systems of the present type. In the models of GSs studied thus far and based on GMTM, collapse has never been reported. The *real existence* of the collapse in the present one-dimensional three-wave GS model is therefore a novel dynamical feature, and it seems quite natural that cuspons and peakons, in the case when they are unstable, play the role of catalysts stimulating the onset of the collapse.

II. ANALYTICAL SOLUTIONS FOR SOLITONS

A. The dispersion relation

The first step in the investigation of the system is to understand its linear spectrum. Substituting $u_{1,2,3} \sim \exp(ikx - i\omega t)$ into Eqs. (8 -10), and omitting nonlinear terms, we arrive at a dispersion equation,

$$(\omega^2 - k^2 - 1)(\omega - \omega_0) = 2\kappa^2(\omega - 1). \tag{18}$$

If $\kappa = 0$, the third wave decouples, and the coupling between the first two waves produces a commonly known gap, so that the solutions to Eq. (18) are $\omega_{1,2} = \pm\sqrt{1+k^2}$ and $\omega_3 = \omega_0$. If $\kappa \neq 0$, the spectrum can be easily understood by treating κ as a small parameter. However, the following analysis is valid for all values of κ in the range $0 < \kappa^2 < 1$.

First, consider the situation when $k = 0$. Then, three solutions of Eq. (18) are

$$\omega = 1, \; \omega = \omega_\pm \equiv (\omega_0 - 1)/2 \pm \sqrt{(\omega_0 + 1)^2/4 + 2\kappa^2}. \tag{19}$$

It can be easily shown that $\omega_- < \min\{\omega_0, -1\} \leq \max\{\omega_0, -1\} < \omega_+$, so that one always has $\omega_- < -1$, while $\omega_+ > 1$ if $1 - \omega_0 > \kappa^2$, and vice versa. Next, it is readily seen that, as $k^2 \to \infty$, either $\omega^2 \approx k^2$, or $\omega \approx \omega_0$. It can also be shown that each branch of the dispersion relation generated by Eq. (18) is a monotonic function of k^2. Generic examples of the spectrum are shown in Fig. 1, where the panels (a) and (b) pertain, respectively, to the cases $\omega_0 < 1 - \kappa^2$ with $\omega_+ < 1$, and $\omega_0 > 1$ with $\omega_+ > 1$. The intermediate case, $1 - \kappa^2 < \omega_0 < 1$, is similar to that shown in panel (a), but with the points ω_+ and 1 at $k = 0$ interchanged. When $\omega_0 < 1$, the upper gap in the spectrum is $\min\{\omega_+, 1\} < \omega < \max\{\omega_+, 1\}$, while the lower gap is $\omega_- < \omega < \omega_0$. When $\omega_0 > 1$, the upper gap is $\omega_0 < \omega < \omega_+$, and the lower one is $\omega_- < \omega < 1$.

B. A generic family of gap solitons

The next step is to search for GS solutions to the full nonlinear system. In this work, we confine ourselves to the case of zero-velocity GS, substituting into Eqs. (8) - (10)

$$u_n(x,t) = U_n(x)\exp(-i\omega t)\,,\; n = 1,2,3, \tag{20}$$

where it is assumed that the soliton's frequency ω belongs to one of the gaps. In fact, even the description of zero-velocity solitons is quite complicated. Note, however, that if one sets $\kappa = 0$ in Eqs. (8) - (10), keeping nonlinear XPM couplings between the first two and third waves, the gap which exists in the two-wave GMT model remains unchanged, and the corresponding family of GS solutions does not essentially alter, in accord with the principle that nonlinear couplings cannot alter gaps or open a new one if the linear coupling is absent [14]; nevertheless, the situation is essentially different if κ is vanishingly small, but not exactly equal to zero, see below.

First, the substitution of (20) into Eqs. (8) and (9) leads to a system of two ordinary differential equations for $U_1(x)$ and $U_2(x)$,

$$iU_1' = \omega U_1 + U_2 + \kappa U_3 + \alpha\left(\alpha\sigma_1|U_1|^2 + \alpha|U_2|^2 + |U_3|^2\right)U_1, \tag{21}$$

$$-iU_2' = \omega U_2 + U_1 + \kappa U_3 + \alpha\left(\alpha\sigma_1|U_2|^2 + \alpha|U_1|^2 + |U_3|^2\right)U_2, \tag{22}$$

where the prime represents d/dx. To solve these equations, we substitute $U_{1,2} = A_{1,2}(x)\exp(\imath\phi_{1,2}(x))$ with real A_n and ϕ_n. After simple manipulations, it can be found that $\left(A_1^2 - A_2^2\right)' = 0$ and $(\phi_1 + \phi_2)' = 0$. With regard to the condition that the soliton fields vanish at infinity, we immediately conclude that

$$A_1^2(x) = A_2^2(x) \equiv S(x); \tag{23}$$

as for the constant value of $\phi_1 + \phi_2$, it may be set equal to zero without restriction of the generality, so that $\phi_1(x) = -\phi_2(x) \equiv \phi(x)/2$, where $\phi(x)$ is the relative phase of the two fields. After this, we obtain two equations for $S(x)$ and $\phi(x)$ from Eqs. (21) and (22),

$$\phi' = -2\omega - 2\cos\phi - 2\alpha^2\left(1+\sigma_1\right)S - S^{-1}U_3^2\left(\omega_0 - \omega - \sigma_3 U_3^2\right), \tag{24}$$

$$S' = -2S\sin\phi - 2\kappa\sqrt{S}U_3\sin\left(\phi/2\right), \tag{25}$$

and Eq. (10) for the third wave U_3 takes the form of a cubic algebraic equation

$$U_3\left(\omega_0 - \omega - 2\alpha S - \sigma_3|U_3|^2\right) = 2\kappa\sqrt{S}\cos\left(\phi/2\right), \tag{26}$$

from which it follows that U_3 is a real-valued function.

This analytical consideration can be readily extended for more general equations (8) and (9) that do not assume the symmetry between the waves u_1 and u_2, i.e., with the group-velocity terms in the equations altered as per Eq. (11). In particular, the relation (23) is then replaced by $c_1 A_1^2(x) = c_2 A_2^2(x) \equiv S(x)$. It can be checked that results for the asymmetric model are not qualitatively different from those presented below for the symmetric one.

Equations (24) and (25) have a Hamiltonian structure, as they can be represented in the form

$$\frac{dS}{dx} = \frac{\partial H}{\partial\phi}, \quad \frac{d\phi}{dx} = -\frac{\partial H}{\partial S}, \tag{27}$$

with the Hamiltonian

$$H = 2S\cos\phi + \alpha^2\left(1+\sigma_1\right)S^2 + 2\omega S + U_3^2\left(\omega_0 - \omega - 2\alpha S\right) - \frac{3}{2}\sigma_3 U_3^4, \tag{28}$$

which is precisely a reduction of the Hamiltonian (13) of the original system (8) - (10) for the solutions of the present type. Note that H is here regarded as a function of S and ϕ, and the relation (26) is regarded as determining U_3 in terms of S and ϕ. For soliton solutions, the boundary conditions at $x = \pm\infty$ yield $H = 0$, so that the solutions can be obtained in an implicit form,

$$2S\cos\phi + \alpha^2 (1+\sigma_1) S^2 + 2\omega S + U_3^2 (\omega_0 - \omega - 2\alpha S) - (3/2)\,\sigma_3 U_3^4 \; = \; 0. \tag{29}$$

In principle, one can use the relations (26) and (29) to eliminate U_3 and ϕ and so obtain a single equation for S. However, this is not easily done unless $\sigma_3 = 0$ [no SPM term in Eq. (10)], therefore we proceed to examine this special, but important, case first. Recall that the zero-SPM case also plays an important role in the case of the two-wave GMTM based on Eqs. (1) and (2), as precisely in this case (which corresponds to the massive Thirring model proper) the equations are exactly integrable by means of the inverse scattering transform [6].

C. Cuspons in the zero-self-phase-modulation case ($\sigma_3 = 0$)

Setting $\sigma_3 = 0$ makes it possible to solve Eq. (26) for U_3 explicitly in terms of S and ϕ,

$$U_3 \; = \; \frac{2\kappa\sqrt{S}\cos(\phi/2)}{\omega_0 - \omega - 2\alpha S} \, . \tag{30}$$

For simplicity, we also set $\sigma_1 = 0$ in Eqs. (8) and (9) and subsequent equations, although the latter assumption is not crucially important for the analysis developed below. If $\sigma_1 \neq 0$ is restored, the conclusions of this subsection will not be substantially altered.

As the next step, one can also eliminate ϕ, using Eqs. (29) and (30), to derive a single equation for S,

$$(dS/dx)^2 \; = \; 4S^2 F(S), \tag{31}$$

$$F(S) \equiv (1 - \omega - \frac{1}{2}\alpha^2 S) \left[2\left(1 + \frac{\kappa^2}{\omega_0 - \omega - 2\alpha S} \right) - (1 - \omega - \frac{1}{2}\alpha^2 S) \right]. \tag{32}$$

The function $F(S)$ has either one or three real zeros S_0. One is

$$S_{01} \; = \; 2\,(1-\omega)\,/\alpha^2, \tag{33}$$

and the remaining two, if they exist, are real roots of the quadratic equation,

$$(2 + 2\omega + \alpha^2 S_0)(\omega_0 - \omega - 2\alpha S_0) + 4\kappa^2 \; = 0. \tag{34}$$

Only the smallest positive real root of Eq. (34), to be denoted S_{02} (if such exists), will be relevant below. Note, incidentally, that $F(S)$ cannot have double roots. A consequence of this fact is that Eq. (31) cannot generate kink solutions, which have different limits as $x \to \pm\infty$. Indeed, if $S(x) \to \mathrm{const} \equiv \overline{S}$ as $x \to \pm\infty$, then one needs to have $dS/dx \sim \left(S - \overline{S}\right)$ in the same limit, which implies that the function $F(s)$ in Eq. (31) must have a double zero at $S = \overline{S}$.

For a soliton solution of (31), we need first that $F(0) > 0$, which can be shown to be exactly equivalent to requiring that ω belongs to either the upper or the lower gap of the linear spectrum. We note that the coupling to the third wave gives rise to the rational nonlinearity in the expression (32), despite the fact that the underlying system (8) - (10) contains only linear and cubic terms. Even if the coupling constant κ is small, it is clear that the rational nonlinearity may produce a strong effect in a vicinity of a *critical value* of the squared amplitude at which the denominator in the expression (32) vanishes,

$$S_{\mathrm{cr}} = (\omega_0 - \omega)/2\alpha. \tag{35}$$

As it follows from this expression, one must have $\alpha(\omega_0 - \omega) > 0$ for the existence of the critical value.

If S_{cr} exists, the structure of the soliton crucially depends on whether, with an increase of S, the function $F(S)$ defined by Eq. (32) first reaches zero at $S = S_0$, or, instead, it first reaches the singularity at $S = S_{\mathrm{cr}}$, i.e., whether $0 < S_0 < S_{\mathrm{cr}}$, or $0 < S_{\mathrm{cr}} < S_0$. In the former case, the existence of S_{cr} plays no role, and the soliton is a regular one, having the amplitude $\sqrt{S_0}$. This *regular soliton* may be regarded as obtained by a smooth deformation from the usual GS known in GMTM at $\kappa = 0$.

As the soliton cannot have an amplitude larger than $\sqrt{S_{\mathrm{cr}}}$, in the case $0 < S_{\mathrm{cr}} < S_0$ the squared amplitude takes the value S_{cr}, rather than S_0. The soliton is singular in this case, being a *cuspon* [see Eqs. (41) and (42) below], but, nevertheless, it is an absolutely relevant solution. If $S_{\mathrm{cr}} < 0$ and $S_0 > 0$ or vice versa, then the soliton may only be, respectively, regular or singular, and no soliton exists if both S_0 and S_{cr} are negative. Further, it is readily shown that for all these soliton solutions, $S(x)$ is symmetric about its center, which may be set at $x = 0$, that is, $S(x)$ is an even function of x. For the cuspon solutions, and for those regular solutions whose squared amplitude is S_{01}, it can also be shown that the phase variable $\psi(x) = \phi(x) - \pi$ and $U_3(x)$ are odd functions of x, while for those regular solutions whose squared amplitude is S_{02} the phase variable $\phi(x)$ and $U_3(x)$ are, respectively, odd and even functions of x.

It is now necessary to determine which parameter combinations in the set $(\omega, \omega_0, \alpha)$ permit the options described above. The most interesting case occurs when $\omega_0 > \omega$ (so that ω belongs to the lower gap, see Fig. 1) and $\alpha > 0$ (the latter condition always holds in the applications to nonlinear optics). In this case, it can be shown that the root S_{02} of Eq. (34) is not relevant, and the options are determined by the competition between S_{01} and S_{cr}. The soliton is a cuspon ($0 < S_{\mathrm{cr}} < S_{01}$) if

$$\alpha(\omega_0 - \omega) < 4(1 - \omega). \tag{36}$$

In effect, the condition (36) sets an upper bound on α for given ω_0 and ω. In particular, the condition is always satisfied if $0 < \alpha < 4$.

If, on the other hand, the condition (36) is not satisfied (i.e., $0 < S_{01} < S_{\rm cr}$), we obtain a regular soliton. In a less physically relevant case, when again $\omega_0 > \omega$ but $\alpha < 0$, cuspons do not occur [as this time $S_{\rm cr} < 0$, see Eq. (35)], and only regular solitons may exist.

Next we proceed to the case $\omega_0 < \omega$, so that ω is located in the upper gap of the linear spectrum. For $\alpha > 0$, we have $S_{\rm cr} < 0$, hence only regular solitons may occur, and indeed it can be shown that there is always at least one positive root S_0, so a regular soliton exists indeed. If $\alpha < 0$, then we have $S_{\rm cr} > 0$, but it can be shown that, if $\omega_0 < 1 - \kappa^2$ (when also $\omega < 1$), there is at least one positive root $S_0 < S_{\rm cr}$; thus, only a regular soliton can exist in this case too. On the other hand, if $\alpha < 0$ and $\omega_0 > 1 - \kappa^2$ (and then $\omega > 1$), there are no positive roots S_0, and so only cuspons occur.

Let us now turn to a detailed description of the cuspon's local structure near its center, when S is close to $S_{\rm cr}$. From the above analysis, one sees that cuspons occur whenever ω lies in the lower gap, with $\omega_0 > \omega$ and $\alpha > 0$, so that the criterion (36) is satisfied, or when ω lies in the upper gap with $1 - \kappa^2 < \omega_0 < \omega$ and $\alpha < 0$. To analyze the structure of the cuspon, we first note that, as it follows from Eq. (29), one has $\cos\phi = -1$ (i.e., $\phi = \pi$) when $S = S_{\rm cr}$, which suggest to set

$$S_{\rm cr} - S \equiv \delta \cdot \kappa^2 R, \qquad 1 + \cos\phi \equiv \delta \cdot \rho, \tag{37}$$

where δ is a small positive parameter, and the stretched variables R and ρ are positive. At the leading order in δ, it then follows from Eq. (29) that $\rho = \rho_0 R$, where

$$\rho_0 \equiv \alpha^3 (S_{01} - S_{\rm cr}). \tag{38}$$

As it follows from the above analysis, ρ_0 is always positive for a cuspon. We also stretch the spatial coordinate, defining $x \equiv \delta^{3/2}\kappa^2 y$, the soliton center being at $x = 0$. Since $S(x)$ is an even function of x, it is sufficient to set $x > 0$ in this analysis. Then, on substituting the first relation from Eq. (37) into Eq. (31), we get, to the leading order in δ, an equation

$$R\,(dR/dy)^2 = \rho_0 S_{\rm cr}^2/\alpha^2 \equiv K^2, \tag{39}$$

so that

$$R = (3Ky/2)^{2/3}. \tag{40}$$

In the original unstretched variables, the relation (40) shows that, near the cusp,

$$S_{\rm cr} - S(x) \approx (3K\kappa x/2)^{2/3}, \tag{41}$$

$$dS/dx \approx (2/3)^{1/3} (K\kappa)^{2/3} \cdot x^{-1/3}, \tag{42}$$

and it follows from Eq. (30) that U_3 is unbounded near the cusp,

$$U_3 \approx (S_{\rm cr}/\alpha)(2\alpha\rho_0 K^2/3\kappa x)^{1/3}. \tag{43}$$

In particular, Eq. (42) implies that, as $K\kappa$ decreases, the cusp gets localized in a narrow region where $|x| \lesssim K^2\kappa^2$ (outside this region, $|dS/dx|$ is bounded and shows no cusp). Note that this limit can be obtained either as $\kappa^2 \to 0$, or as $\rho_0 \to 0$ [recall ρ_0 is defined in Eq. (38)].

It is relevant to mention that, very close to the cusp, the underlying physical model, which is based on the paraxial approximation in the application to optical systems, or on a long-wave expansion in the case of internal waves in stratified fluids, may become irrelevant. However, this circumstance will lead to a modification of the structure of the physical fields inside the cuspon only in a very small vicinity of the singular point (for instance, on a scale of the order of the light wavelength in optical systems, or the layer's depth in the fluids). Thus, the cuspon solutions are quite relevant to applications, provided that they are stable.

An example of the cuspon is shown in Fig. 2. Although the first derivative in the cuspon is singular at its center, as follows from Eq. (42) [see also Fig. 2(a)], it is easily verified that the Hamiltonian (13) (and, obviously, the norm (12) too) are finite for the cuspon solution. These solitons are similar to cuspons found as exact solutions to the Camassa-Holm (CH) equation [17,18], which have a singularity of the type $|x|^{1/3}$ or $|x|^{2/3}$ as $|x| \to 0$, cf. Eqs. (41) and (42). The CH equation is integrable, and it is degenerate in the sense that it has no linear terms except for $\partial u/\partial t$ (which makes the existence of the solution with a cusp singularity possible). Our three-wave system (8) - (10) is not degenerate in that sense; nevertheless, the cuspon solitons are possible in it because of the model's multicomponent structure: the elimination of the third component generates the non-polynomial nonlinearity in Eqs. (21), (22), and, finally, in Eqs. (25) and (31), which gives rise to the cusp. It is noteworthy that, as well as the CH model, ours gives rise to two different *coexisting* families of solitons, viz., regular ones and cuspons. It will be shown below that the solitons of both types may be stable.

In the special case $\kappa \ll 1$, when the third component is weakly coupled to the first two ones in the linear approximation, a straightforward perturbation analysis shows that the cuspons look like *peakons*; that is, except for the above-mentioned narrow region of the width $|x| \sim \kappa^2$, where the cusp is located, they have the shape of a soliton with a discontinuity in the first derivative of $S(x)$ and a jump in the phase $\phi(x)$, which are the defining features of peakons ([17,19]). An important result of our analysis is that the family of solitons obtained in the limit $\kappa \to 0$ is drastically different from that in the model where one sets $\kappa = 0$ from the very beginning. In particular, in the most relevant case, with $\omega_0 > \omega$ and $\alpha > 0$, the family corresponding to $\kappa \to 0$ contains regular solitons whose amplitude is smaller than $\sqrt{S_{\rm cr}}$; however, the solitons whose amplitude at $\kappa = 0$ is larger than $\sqrt{S_{\rm cr}}$, i.e., the ones whose frequencies belong to the region (36) [note that the definition of $S_{\rm cr}$ does not depend on κ at all, see Eq. (35)], are replaced by the peakons which are constructed in a very simple way: drop the part of the usual soliton above the critical level $S = S_{\rm cr}$, and bring together the two symmetric parts which remain below the critical level, see Fig. 2(b). It is interesting that peakons are known as exact solutions to a version of the integrable CH equation slightly different from that which gives rise to the cuspons. As well as in the present system, in that equation the peakons coexist with regular solitons [19]. In the next subsection, we demonstrate that the peakons, which are found only as limit-form solutions in the zero-SPM case $\sigma_3 = 0$, become generic solutions in the case $\sigma_3 \neq 0$.

D. Peakons, the case $\sigma_3 \neq 0$

Before proceeding to the consideration of dynamical stability of various soliton solutions found above, it is relevant to address another issue, viz., *structural stability* of the cuspon solutions. To this end, we restore the SPM term in Eq. (10), that is, we now set $\sigma_3 \neq 0$, but assume that it is a small parameter. Note that, in the application to nonlinear optics, one should expect that $\sigma_3 > 0$, but there is no such a restriction on the sign of σ_3 in the application to the flow of a density-stratified fluid. We still keep $\sigma_1 = 0$, as the inclusion of the corresponding SPM terms in Eqs. (8) and (9) amounts to straightforward changes in details of both the above analysis, and that presented below. On the other hand, we show below that the inclusion of the SPM term in Eq. (10) is a structural perturbation which drastically changes the character of the soliton solutions.

In view of the above results concerning the cuspons, we restrict our discussion here to the most interesting case when $S(x)$ is an even function of x, while $\psi(x) = \phi(x) - \pi$ and $U_3(x)$ are odd functions. In principle, one can use the relations (26) and (29) to eliminate ϕ and U_3 and so obtain a single equation for S (a counterpart to Eq. (31)), as it was done above when $\sigma_3 = 0$. However, when $\sigma_3 \neq 0$, this cannot be done explicitly. Instead, we shall develop an asymptotic analysis valid for $x \to 0$, which will be combined with results obtained by direct numerical integration of Eqs. (24) and (25), subject, of course, to the constraints (26) and (29). Since singularities only arise at the center of the soliton (i.e., at $x = 0$) when $\sigma_3 = 0$, it is clear that the introduction of a small $\sigma_3 \neq 0$ will produce only a small deformation of the soliton solution in the region where x is bounded away from zero.

First, we consider regular solitons. Because the left-hand side of Eq. (26) is not singular at any x, including the point $x = 0$ when $\sigma_3 = 0$, we expect that regular solitons survive a perturbation induced by $\sigma_3 \neq 0$. Indeed, if there exists a regular soliton, with $S_0 \equiv S(x = 0)$, and $\phi(x = 0) = \pi$ and $U_3(x = 0) = 0$, it follows from Eq. (29) that the soliton's amplitude remains exactly the same as it was for $\sigma_3 = 0$, due to the fact that the regular soliton has $U_3(x = 0) = 0$.

Next, we turn to the possibility of singular solutions, that is, cuspons or peakons. Since we are assuming that $S_0 = S(x = 0)$ is finite, and that $\phi(x = 0) = \pi$, it immediately follows from Eq. (26) that when $\sigma_3 \neq 0$, U_3 must remain finite for all x, taking some value $U_0 \neq 0$, say, as $x \to +0$. As it has been established above that U_3 is an odd function of x, and $U_3(x = +0) \equiv U_0 \neq 0$, there must be a discontinuity in U_3 at $x = 0$, i.e., a jump from U_0 to $-U_0$. This feature is in marked contrast to the cuspons for which U_3 is infinite at the center, see Eq. (43). Further, it then follows from Eq. (25) that, as $x \to 0$, there is also a discontinuity in dS/dx, with a jump from $2\kappa U_0\sqrt{S_0}$ to $-2\kappa U_0\sqrt{S_0}$. Hence, if we can find soliton solutions of this type, with $U_0 \neq 0$, they are necessarily *peakons*, and we infer that cuspons do *not* survive the structural perturbation induced by $\sigma_3 \neq 0$.

Further, if we assume that $U_0 \neq 0$, then Eq. (26), taken in the limit $x \to 0$, immediately shows that

$$2\alpha(S_{\mathrm{cr}} - S_0) = \sigma_3 U_0^2 \tag{44}$$

(recall that S_{cr} is defined by Eq. (35)). Next, the Hamiltonian relation (29), also taken in the limit $x \to 0$, shows that

$$-\frac{\rho_0}{\alpha} S_0 - \alpha^2 S_0 (S_{\mathrm{cr}} - S_0) = \frac{1}{2}\sigma_3 U_0^4, \tag{45}$$

where we have used Eq. (44) (recall that ρ_0 is defined by Eq. (38)). Elimination of U_0 from (44,45) yields a quadratic equation for S_0, whose positive roots represent the possible values of the peakon's amplitude.

We recall that for a cuspon which exists at $\sigma_3 = 0$ one has $\rho_0 > 0$, i.e., the amplitude of the corresponding formal regular soliton exceeds the critical value of the amplitude, see Eq. (38). Then, if we retain the condition $\rho_0 > 0$, it immediately follows from Eqs. (44) and (45) that no peakons may exist if the SPM coefficient in Eq. (10) is positive, $\sigma_3 > 0$. Indeed, Eq. (44) shows that $S_{\rm cr} - S_0 > 0$ if $\sigma_3 > 0$, which, along with $\rho_0 > 0$, leads to a contradiction in the relation (45).

Further, it is easy to see that a general condition for the existence of peakons following from Eqs. (44) and (45) is

$$\sigma_3 \rho_0 < 0 \,, \tag{46}$$

hence peakons are possible if $\sigma_3 < 0$, or if we keep $\sigma_3 > 0$ but allow $\rho_0 < 0$. In the remainder of this subsection, we will show that peakons may exist only if $\rho_0 > 0$. Hence, it follows from the necessary condition (46) that peakons may indeed be possible solely in the case $\sigma_3 < 0$. On the other hand, regular solitons do exist in the case $\sigma_3 > 0$ (i.e., in particular, in nonlinear-optics models), as they have $U_0 = 0$, hence neither Eq. (44) nor its consequence in the form of the inequality (46) apply to regular solitons. The existence of (stable) peakons for $\sigma_3 < 0$, and of (also stable) regular solitons for $\sigma_3 > 0$ will be confirmed by direct numerical results presented in the next section.

To obtain a necessary condition (which will take the form of $\rho_0 > 0$) for the existence of the peakons, we notice that existence of any solitary wave implies the presence of closed dynamical trajectories in the phase plane of the corresponding dynamical system, which is here based on the ordinary differential equations (24) and (25), supplemented by the constraint (26). Further, at least one stable fixed point (FP) must exist inside such closed trajectories, therefore the existence of such a stable FP is, finally, a necessary condition for the existence of any solitary wave.

The FP are found by equating to zero the right-hand sides of Eq. (24) and (25), which together with Eq. (26) give three equations for the three coordinates ϕ, S and U_3 of the FP. First of all, one can find a trivial unstable FP of the dynamical system,

$$\cos\phi = -\frac{\omega + \kappa^2/(\omega_0 - \omega)}{1 + \kappa^2/(\omega_0 - \omega)} \,, \quad S \, = \, 0 \,,$$

which does not depend on σ_3. Then, three nontrivial FPs can be found, with their coordinates ϕ_*, S_* and U_{3*} given by the following expressions:

$$\phi_*^{(1)} \, = \pi, \quad S_*^{(1)} \, = \frac{1-\omega}{\alpha^2} \, = \frac{1}{2} S_{01}, \quad U_{3*}^{(1)} \, = 0, \tag{47}$$

$$\phi_*^{(2)} = \pi, \quad (2-\sigma_3)S_*^{(2)} = 2S_{\rm cr} - \frac{\sigma_3}{2}S_{01}, \quad (2-\sigma_3)\left[\alpha U_{3*}^{(2)}\right]^2 = \rho_0 - \alpha^3 S_{\rm cr}\,, \tag{48}$$

$$(2-\sigma_3)S_*^{(3)} = 2S_{\rm cr} - \frac{1}{2}\sigma_3 S_{01} + \frac{\kappa^2}{\alpha}, \quad (2-\sigma_3)\left[\alpha U_{3*}^{(3)}\right]^2 = \rho_0 - \alpha^3 S_{\rm cr} - \alpha^2\kappa^2\,,$$

$$\cos\left(\phi_*^{(3)}/2\right) = -\frac{1}{2}\kappa U_{3*}^{(3)}\,/\sqrt{S_*^{(3)}}\,. \tag{49}$$

To be specific, we now consider the case of most interest, when both $S_{01} > 0$ and $S_{\rm cr} > 0$. In this case, the FP given by Eqs. (47) exists for all σ_3 and all ρ_0. However, for small σ_3 (in fact $\sigma_3 < 2$ is enough) and small κ, the FPs given by Eqs. (48) and (49) exist only when $\rho_0 > 0$. Indeed, they exist only for $\rho_0 > \alpha^3 S_{01}$ and $\rho_0 > \alpha^3 S_{01} + \kappa^2$, respectively, or, on using the definition (38) of ρ_0, when $S_{01} > 2S_{\rm cr}$ and $S_{01} > 2S_{\rm cr} + \kappa^2/\alpha$, respectively.

Let us first suppose that $\rho_0 < 0$. Then there is only the single non-trivial FP, namely the one given by Eqs. (47). This FP is clearly associated with the regular solitons, whose squared amplitude is S_{01}. Hence, we infer that for $\rho_0 < 0$ there are no other solitary-wave solutions, and in particular, no peakons (and no cuspons either when $\sigma_3 = 0$, in accordance with what we have already found in subsection 2.3 above). Combining this with the necessary condition (46) for the existence of peakons, we infer that there are no peakons when $\sigma_3 > 0$, thus excluding peakons from applications to the nonlinear-optics models, where this SPM coefficient is positive. However, peakons may occur in density-stratified fluid flows, where there is no inherent restriction on the sign of σ_3. This case is considered below, but first we note that in the case $\rho_0 < 0$ and $\sigma_3 > 0$ (which includes the applications to nonlinear optics), the same arguments suggest that there may be *periodic* solutions with peakon-type discontinuities; indeed, our numerical solutions of the system (24,25) (not displayed in this paper) show that this is the case.

Next, we suppose that $\rho_0 > 0$. First, if $S_{01} < 2S_{\rm cr}$, then there again exists the single non-trivial FP given by (47). But now, by analogy with the existence of cuspons when $\rho_0 > 0$ and $\sigma_3 = 0$, we infer that the solitary-wave solution which is associated with this fixed point is a *peakon*, whose squared amplitude S_0 for small σ_3 is close to $S_{\rm cr}$, while the FP has $S_*^{(1)} = S_{01}/2 < S_{\rm cr}$.

If, on the other hand, $S_{01} > 2S_{\rm cr}$, the FPs given by Eqs.(48) and (49) become available as well. We now infer that the peakon solitary-wave solution continues to exist, and for sufficiently small σ_3 and κ it is associated with the FP given by Eq. (48). Although Eq. (48) implies that $S_*^{(2)} \approx S_{\rm cr}$, and the peakon's squared amplitude S_0, determined by Eqs. (44) and (45), is also approximately equal to $S_{\rm cr}$, we nevertheless have $S_0 > S_*^{(2)}$ as required. Note that, in the present case, the FPs given by Eqs. (47) and

(49) lie outside the peakon's homoclinic orbit. In Fig. 3, we show a plot of a typical peakon obtained, in this case, by numerical solution of Eqs. (24) and (25).

III. NUMERICAL RESULTS

A. Simulation techniques

The objectives of direct numerical simulations of the underlying equations (8) - (10) were to check the dynamical stability of regular solitons, cuspons, and peakons in the case $\sigma_3 = 0$, and the existence and stability of peakons in the more general case, $\sigma_3 \neq 0$. Both finite-difference and pseudo-spectral numerical methods have been used, in order to check that the same results are obtained by methods of both types. We used semi-implicit Crank-Nicholson schemes, in which the nonlinear terms were treated by means of the Adams-Bashforth algorithm.

The presence of singularities required a careful treatment of cuspon and peakon solutions. To avoid numerical instabilities due to discontinuities, we sometimes introduced a weak artificial high-wavenumber viscosity into the pseudospectral code. We have found that viscosities $\sim 10^{-5}$ were sufficient to avoid the Gibbs' phenomenon in long-time simulations. When instabilities occur at a singular point (cusp or peak), it is hard to determine whether the instability is a real one or a numerical artifact. Therefore, we checked the results by means of a finite-difference code which used an adaptive staggered grid; motivated by the analysis of the vicinity of the point $x = 0$ reported above, we introduced the variable $\xi \equiv x^{2/3}$ to define an adaptive grid, and also redefined $U_3 \equiv \sqrt{\xi}\widetilde{U}_3$. In these variables, the cusp seems like a regular point. We stress that this approach was solely used to check the possible occurrence of numerical instabilities.

In the following subsections we present typical examples of the numerical results for both cases considered above, viz., $\sigma_3 = 0$ and $\sigma_3 < 0$, when, respectively, the cuspons and peakons are expected.

B. The case $\sigma_3 = 0$

First, we report results obtained for the stability of regular solitary waves in the case $\sigma_3 = 0$. As initial configurations, we used the corresponding stationary solutions to Eqs. (24) and (25). To test the stability of the regular solitary waves, we added small perturbations to them. As could be anticipated, the regular solitary wave sheds off a small-amplitude dispersive wave (radiation) and relaxes to a stationary soliton, see Fig. 4. If, however, regular solitons are taken at parameter values close to the border of the cuspon region, an initial perturbation does not make the soliton unstable, but it excites persistent internal vibrations in the soliton, see an example in Fig. 5. These and many other simulations clearly show that the regular soliton is *always stable*, and, close to the parameter border with cuspons, it has a persistent *internal mode*.

It was shown analytically above that Eqs. (21) and (22) (with $\sigma_3 = 0$) support peakons when $\rho_0 > 0$ and $\rho_0\kappa^2$ is very small. Direct simulations show that the peakons do exist in this case, and are *stable*. In Fig. 6, we display the time evolution of a typical stable peakon.

An essential result revealed by the simulations is that cuspons may also be *stable*, a typical example being displayed in Fig. 7. A moving weak singularity seen in this figure is, actually, a small shock wave which is initially generated at the cuspon's crest. It seems plausible that this shock wave is generated by some initial perturbation which could be a result of the finite mesh size in the finite-difference numerical scheme employed for the simulations. We have observed that the emission of a small-amplitude shock wave is quite a typical way of the relaxation of both cuspons and peakons to a final stable state.

However, unlike the regular solitons, which were found to be stable, the cuspons are sometimes unstable. Typically, their instability triggers onset of the spatiotemporal collapse, i.e., formation of a singularity in a finite time (see a discussion of the feasible collapse in systems of the present type, given in the Introduction). Simulations of the collapse were possible with the use of an adaptive grid. A typical example of the collapse is shown in Fig. 8, the inset illustrating the fact that the amplitude of the solution indeed diverges in a finite time. In some other cases, which are not displayed here, the instability of peakons could be quite weak, giving rise to their rearrangement into regular solitons by shedding small amounts of radiation.

C. The case $\sigma_3 \neq 0$

The predictions of the analysis developed above for the most general case, when the SPM terms are present in the model ($\sigma_3 \neq 0$), were also checked against direct simulations. As a result, we have found, in accord with the predictions, that only regular solitons exist in the case $\sigma_3 > 0$, while in the case $\sigma_3 < 0$ both regular solitons and peakons have been found as generic solutions. Further simulations, details of which are not shown here, demonstrate that both regular solitons *and* peakons are stable in this case.

IV. CONCLUSION

In this work, we have introduced a generic model of three waves coupled by linear and nonlinear terms, which describes a situation when three dispersion curves are close to intersection at one point. The model was cast into the form of a system of two waves with opposite group velocities that, by itself, gives rise to the usual gap solitons, which is further coupled to a third wave with the zero group velocity (in the laboratory reference frame). Situations of this type are possible in various models of nonlinear optics and density-stratified fluid flows. The consideration was focussed on zero-velocity solitons. In a special case when the self-phase modulation (SPM) is absent in the equation for the third wave, soliton solutions were found in an exact form. It was shown that there are two coexisting generic families of solitons: regular solitons and cuspons. In the special case when the coefficient of the linear coupling between the first two waves and the third one vanishes, cuspons are replaced by peakons. Direct simulations have demonstrated that the regular solitons are stable (in the case when the regular soliton is close to the border of the cuspon region, it has a persistent internal mode). The cuspons and peakons may be both stable and unstable. If they are unstable, they either shed off some radiation and rearrange themselves into regular solitons, or, in most typical cases, the development of the cuspon's instability initiates onset of spatiotemporal collapse. To the best of our knowledge, the present system gives the first explicit example of the collapse in one-dimensional gap-soliton models.

The most general version of the model, which includes the self-phase modulation term in the equation for the third wave, has also been considered. Analysis shows that cuspons cannot exist in this case, i.e., cuspons, although being dynamically stable, are structurally unstable. However, depending on the signs of the SPM coefficient and some combination of the system's parameters, it was shown that a generic family of peakon solutions may exist instead. In accord with this prediction, the peakons have been found in direct simulations. The peakons, as well as the regular solitons, are stable in the system including the SPM term.

The next step in the study of this system should be consideration of moving solitons, which is suggested by the well-known fact that the usual two-wave model gives rise to moving gap solitons too [1]. However, in contrast to the two-wave system, one may expect a drastic difference between the zero-velocity and moving solitons in the present three-wave model. This is due to the reappearance of a derivative term in Eq. (10), when it is written for a moving soliton, hence solitons which assume a singularity or jump in the U_3 component, i.e., both cuspons and peakons, cannot exist if the velocity is different from zero. Nevertheless, one may expect that slowly moving solitons will have approximately the same form as the cuspons and peakons, with the singularity at the central point replaced by a narrow transient layer with a large gradient of the U_3 field. However, detailed analysis of the moving solitons is beyond the scope of this work.

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Fig. 1. Dispersion curves produced by Eq. (18) in the case $\kappa = 0.5$: (a) $\omega_0 < 1 - \kappa^2$; (b) $\omega_0 > 1$. The dashed line in each panel is $\omega = \omega_0$. The case with $1 - \kappa^2 < \omega_0 < 1$ is similar to the case (a) but with the points ω_+ and 1 at $k = 0$ interchanged.

Fig. 2. The shape of the cuspon for $\alpha = 2.0$, $\omega_0 = 0.1$, $\omega = -0.5$, and (a) $\kappa = 0.5$, i.e., in the general case, and (b) $\kappa = 0.1$, i.e., for small κ. In the case (b) we also show the usual gap soliton (by the dashed line), the part of which above the critical value $S = S_{\mathrm{cr}}$ (shown by the dotted line) should be removed and the remaining parts brought together to form the peakon corresponding to $\rho_0\kappa^2 \to 0$.

Fig. 3. The shape of the peakon in for the case when $\sigma_3 < 0$. The parameters are $\sigma_3 = -0.01$, $\kappa = 0.1$, $\alpha = 2.0$, $\omega_0 = 0.1$, and $\omega = -0.5$. In this case, $\rho_0 = 4.8$.

Fig. 4. The shape of an initially perturbed regular soliton in the case $\sigma_3 = 0$ at $t = 5$, which illustrates the stabilization of the soliton via the shedding of small-amplitude radiation waves. The plot displays the field $\mathrm{Re}\, U_1(x)$. The parameters are $\kappa = 0.01, \alpha = 1.0, \omega_0 = 0.2$, and $\omega = 0.9$.

Fig. 5. Internal vibrations of an initially-perturbed regular soliton, which was taken close to the border of the cuspon region. The plot shown is the squared amplitude $a \equiv |U_1(x = 0)|^2$ of the $U_1(x)$ field vs. time. The parameters are $\kappa = 0.01, \alpha = 1.9, \omega_0 = 1.5$, and $\omega = 0.5$, with $\rho_0 = 0.095$ [see Eq. (38].

Fig. 6. An example of a stable peakon. The plot shown is the field $\mathrm{Im}\, U_1$ vs. x and t. The parameters are $\kappa = 1.0, \alpha = 1.95, \omega_0 = 1.5$, and $\omega = 0.5$, with $\rho_0 = 0.04875$.

Fig. 7. An example of a stable cuspon. The plot shown is the field $\mathrm{Im}\, U_1$ vs. x and t. The parameters are $\kappa = 1.0, \alpha = 1.0, \omega_0 = 1.5$, and $\omega = 0.5$, with $\rho_0 = 0.5$.

Fig. 8. The spatial profile is shown for an unstable cuspon in terms of $\mathrm{Im}\, U_1$ at $t = 10^{-3}$. The transition to collapse is additionally illustrated by the inset which shows the growth of the amplitude of the field $|U_1|^2$ with time. The parameters are $\kappa = 0.01, \alpha = 1.1, \omega_0 = 0.1$, and $\omega = -0.3$, with $\rho_0 = 2.618$.