

GETTIN' DIRTY WITH EXT

①

Let A be an abelian category; linear over a ring k .

Suppose A either has enough projectives or enough injectives.

Recall:

A has enough projectives (resp. enough injectives)

• if for every $X \in A$ $\exists P$ projective & a surjection $P \rightarrow X$.

(• if for every $X \in A$ $\exists I$ injective & an inclusion $I \rightarrow X$).

e.g. $A = R\text{-Mod}$, $\text{Coh}(X, k)$, $\text{Loc}(X, k)$, $\text{Sh}(X, k)$
 $\text{Mod}_{k\pi_0(X)}$ $\text{Coh}(k_X)$

Defn.

Let $X, Y \in A$. Then

$$\text{Ext}_A^i(X, Y) = \mathcal{H}^i(\text{RHom}_A(X, Y))$$

i.e. to compute $\text{Ext}^i(X, Y)$ we do one of the following:

(1) Take a projective resolution of X :

$$\cdots \rightarrow P^{-1} \rightarrow P^0 \rightarrow X \rightarrow 0$$

• Apply $\text{Hom}(-, Y)$ to P^\bullet :
(this is contravariant)

$$0 \rightarrow \text{Hom}(P^0, Y) \rightarrow \text{Hom}(P^{-1}, Y) \rightarrow \cdots$$

• Take i -th cohomology.

or (2) Take an injective resolution of Y :

$$0 \rightarrow Y \rightarrow I^0 \rightarrow I^1 \rightarrow \cdots$$

• Apply $\text{Hom}(X, -)$ to I^\bullet :

$$0 \rightarrow \text{Hom}(X, I^0) \rightarrow \text{Hom}(X, I^1) \rightarrow \cdots$$

• Take i -th cohomology.

NB.

(i) it is nontrivial that these two procedures give the same result.

~~(ii) One should~~

(ii) If A is k -linear (i.e. each Hom space is a k -module), then any $\text{Ext}_A^i(X, Y)$ is a k -module

Example 1

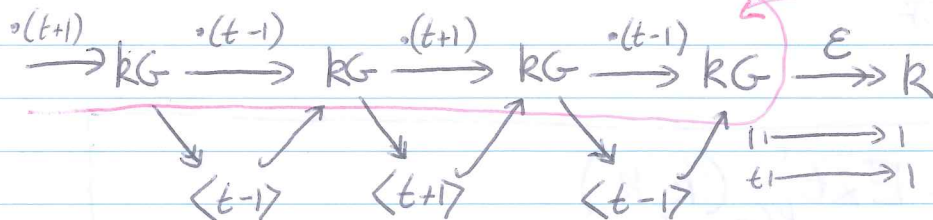
Let $G = \mathbb{Z}/2\mathbb{Z}$.

So $KG = K[t]/(t^2-1)$

Let $A = KG\text{-Mod}$.

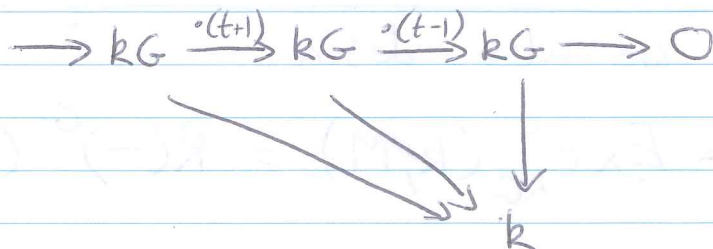
Compute $\text{Ext}_A^i(k, k)$; where k is the trivial module.

(i) Find a projective resolution of k :



projective resolution of k .

(ii) Apply $\text{Hom}_{KG}(-, k)$:

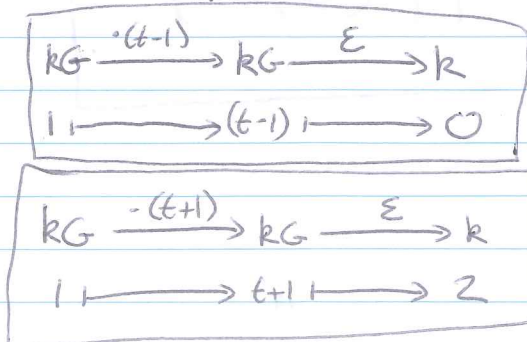


Now, $\text{Hom}_{KG}(KG, k) = k\epsilon$

So,

$$\text{Hom}(P^i, k) = 0 \rightarrow k\epsilon \xrightarrow{0} k\epsilon \xrightarrow{\cdot 2} k\epsilon \xrightarrow{0} k\epsilon \xrightarrow{\cdot 2} \dots$$

Indeed, to compute the differentials:



(iii) Computing Cohomology:

• If $\text{char } k = 2$ then:

$$\text{Ext}_{kG}^i(k, k) = \bigoplus_{i=0}^{\infty} k$$

• If $\text{char } k > 2$ or 0 then

$$\text{Ext}_G^i(k, k) = \begin{cases} k & \text{if } i \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

Group Cohomology

More generally,

$$H^i(G; k) = \text{Ext}_{kG}^i(k, k)$$

is called the Group Cohomology of G (with coefficients in k).

For $kG \curvearrowright M$ we also define:

$$H^i(G; M) = \text{Ext}_{kG}^i(k, M) = R(-)^G(M)$$

Fact:

If X is $K(G, 1)$ then

$$H^i(X; M) = H^i(G; M)$$

Example 2

$$A = \mathbb{Z}\text{-Mod} = \left\{ \begin{array}{l} \text{cat. of} \\ \text{abelian grps} \end{array} \right\}$$

Let $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$

Compute $\text{Ext}^0(\mathbb{Z}_n, \mathbb{Z}_m)$

① What is $\text{Hom}(\mathbb{Z}_n, \mathbb{Z}_m)$?

So For $\varphi: \mathbb{Z}_n \rightarrow \mathbb{Z}_m$ we have

$$\begin{array}{ccc}
 1 & \xrightarrow{\quad} & \frac{m}{d} \\
 \vdots & & \vdots \\
 0 = n & \xrightarrow{\quad} & \left(\frac{n}{d}\right)m = 0
 \end{array}
 \quad \longleftarrow d \text{ a common divisor of } m, n.$$

Fact: For any such d ,

$$\frac{m}{d} = k \left(\frac{m}{\text{gcd}(m,n)} \right) \text{ mod } m$$

So

$$\text{Hom}(\mathbb{Z}_n, \mathbb{Z}_m) = \mathbb{Z}_{\text{gcd}(m,n)}$$

So $\text{Ext}^0(\mathbb{Z}_n, \mathbb{Z}_m) = 0$ if m, n are coprime.

(2) Compute $\text{Ext}^i(\mathbb{Z}_n, \mathbb{Z}_m)$ for n, m not coprime.

\mathbb{Z}_n has projective resolution:

$$\rightarrow 0 \rightarrow \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \rightarrow \mathbb{Z}_n \rightarrow 0$$

\downarrow
 $n\mathbb{Z}$
 \uparrow

Apply $\text{Hom}(-, \mathbb{Z}_m)$:

$$0 \rightarrow \text{Hom}(\mathbb{Z}, \mathbb{Z}_m) \xrightarrow{\cdot n} \text{Hom}(\mathbb{Z}, \mathbb{Z}_m) \rightarrow 0 \rightarrow \dots$$

\downarrow \downarrow
 \mathbb{Z}_m \mathbb{Z}_m
 \longmapsto n

So $\text{Ext}^i(\mathbb{Z}_n, \mathbb{Z}_m) = 0$ if $i \geq 2$.

What is $\text{Ext}^1(\mathbb{Z}_n, \mathbb{Z}_m)$?

$$\text{Ext}^1(\mathbb{Z}_n, \mathbb{Z}_m) = \text{cok}(\mathbb{Z}_m \xrightarrow{\cdot n} \mathbb{Z}_m)$$

Consider the map:

$$\mathbb{Z} \longrightarrow \mathbb{Z}_m \longrightarrow \text{cok}(\mathbb{Z}_m \xrightarrow{\cdot n} \mathbb{Z}_m)$$

This map is surjective w/ kernel

$$m\mathbb{Z} + n\mathbb{Z} = \gcd(m, n)\mathbb{Z}$$

So $\text{Ext}^1(\mathbb{Z}_n, \mathbb{Z}_m) = \mathbb{Z}_{\gcd(m, n)}$

Bonus Question:

What is the group cohomology of \mathbb{Z} ?
(w coefficients in k)

~~Exts~~

Now,

$$R\mathbb{Z} = k[T, T^{-1}]$$

We need

Compute $\text{Ext}_{R[T, T^{-1}]}^i(k, k)$:

k has projective resolution:

$$\begin{array}{ccccccc}
 \rightarrow 0 & \rightarrow & k[T, T^{-1}] & \xrightarrow{\circ(T-1)} & k[T, T^{-1}] & \xrightarrow{\epsilon} & k \rightarrow 0 \\
 & & \searrow & & \swarrow & & \\
 & & \langle T^n - T^m \mid n, m \in \mathbb{Z} \rangle & & T^n & \xrightarrow{\parallel} & 1 \\
 & & \parallel & & & & \\
 & & \langle T-1 \rangle_{k[T, T^{-1}]} & & & &
 \end{array}$$

Apply $\text{Hom}(-, k)$:

$$\begin{array}{ccccccc}
 0 & \rightarrow & \text{Hom}(k[T, T^{-1}], k) & \xrightarrow{0} & \text{Hom}(k[T, T^{-1}], k) & \rightarrow & 0 \\
 & & \parallel & & \parallel & & \\
 & & k & & k & & \\
 & & \parallel & & \parallel & & \\
 & & k\epsilon & & k\epsilon & &
 \end{array}$$

$$\text{So } \text{Ext}_{k[T, T^{-1}]}^i(k, k) = \begin{cases} k & \text{if } i=0, 1 \\ 0 & \text{otherwise} \end{cases}$$

which is the cohomology of S^1 !

Part II. YONEDA EXTS

More generally compute:

$H^i(\mathbb{Z}, M)$ For M a $k[\mathbb{Z}]$ -Module we can compute

$$H^0(\mathbb{Z}, M) = \text{Ext}_{k[\mathbb{Z}]}^0(k, M)$$

↑ group cohomology of \mathbb{Z} with coefficients in M .

Using the projective resolution of k as above we get:

$\text{Ext}^0(k, M)$ is cohomology of:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \text{Hom}(k[\mathbb{Z}], M) & \xrightarrow{\cdot(T-1)} & \text{Hom}(k[\mathbb{Z}], M) & \longrightarrow & 0 \\
 & & \parallel & & \parallel & & \\
 & & M & & M & & \\
 & & & \downarrow \mu(T)-1 & & & \\
 & & & \text{for } \mu: k[\mathbb{Z}] \rightarrow \text{End}(M) & & &
 \end{array}$$

i.e.

i.e.

$$\text{Ext}^i(k, M) = \begin{cases} M^{\mathbb{Z}} & \text{if } i=0 \\ M_{\mu} & \text{if } i=1 \\ 0 & \text{otherwise} \end{cases}$$

where μ is the action of T on $k[\mathbb{Z}]$ on M .

Interpretation:

If \mathcal{L} is a local system on S' ,
 then:
$$H^i(S'; \mathcal{L}) = \begin{cases} V^\mu & i=0 \\ V_\mu & i=1 \\ 0 & \text{otherwise} \end{cases}$$

Where $\mu \subset V$ is the monodromy representation corresponding to \mathcal{L} .

(Recall $\pi_1(S') = \mathbb{Z}$)

Long Exact Sequence in Ext.

GENERAL FACT ABOUT DERIVED FUNCTORS

Thm.

Given a short exact sequence

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

There is a long exact sequence (for any object Y):

$$0 \rightarrow \text{Ext}^0$$

$$0 \rightarrow \text{Hom}(C, Y) \rightarrow \text{Hom}(B, Y) \rightarrow \text{Hom}(A, Y) \rightarrow$$

$$\rightarrow \text{Ext}^1(C, Y) \rightarrow \text{Ext}^1(B, Y) \rightarrow \text{Ext}^1(A, Y) \rightarrow$$

$$\rightarrow \text{Ext}^2(C, Y) \rightarrow \dots$$

Ext¹ as a Cokernel.

In particular, for any $X \in A$, we can form a projective presentation:

$$0 \rightarrow R \xleftarrow{\text{projective}} P \rightarrow X \rightarrow 0$$

Then we have the sequence:

$$0 \rightarrow \text{Hom}(X, Y) \rightarrow \text{Hom}(P, Y) \rightarrow \text{Hom}(R, Y) \rightarrow \text{Ext}^1(X, Y) \rightarrow 0$$

(for any Y)

Hence, we can think of $\text{Ext}^1(X, Y)$ as the cokernel

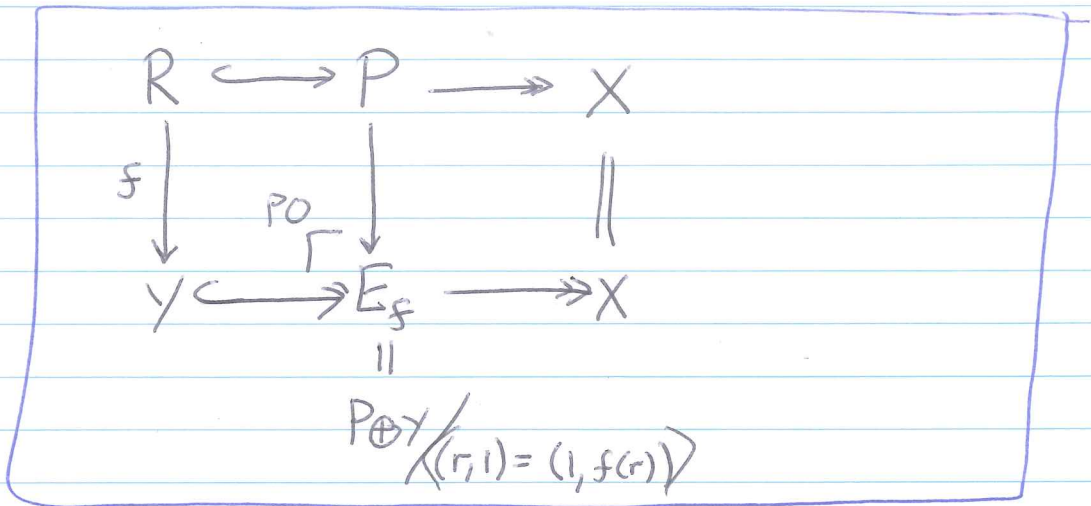
of

$$\text{Hom}(P, Y) \rightarrow \text{Hom}(R, Y)$$

Ext'(X, Y) as extensions.

Given $f \in \text{Hom}(R, Y)$ we can build an extension

$$Y \longrightarrow E_f \longrightarrow X \quad \text{as follows:}$$



This process defines a bijection

$$\text{Ext}'(X, Y) \xrightarrow{\sim} \left\{ \begin{array}{l} \text{set of extensions } Y \longrightarrow E \longrightarrow X \\ \text{modulo the relation} \\ (Y \longrightarrow E \longrightarrow X) = (Y \longrightarrow F \longrightarrow X) \\ \text{if } \exists \varphi: E \xrightarrow{\sim} F \text{ s.t.} \\ \begin{array}{ccccc} Y & \longrightarrow & E & \longrightarrow & X \\ \parallel & \wr & \downarrow \varphi & \wr & \parallel \\ Y & \longrightarrow & F & \longrightarrow & X \end{array} \end{array} \right.$$

think of as equivalence classes of maps $R \rightarrow Y$.

Using the linear structure on $\text{Ext}'(X, Y)$ we can define a linear structure on extensions.

e.g. $0 \longmapsto (Y \longrightarrow X \oplus Y \longrightarrow X) \leftarrow \text{"zero extension"}$

But this is a topic for another talk...