GETTIN' DIRTY WITH EXT

Let $\mathcal{A}$ be an abelian category; linear over a ring k.

Suppose $\mathcal{A}$ either has enough projectives or enough injectives.

Recall:

$\mathcal{A}$ has enough projectives (resp. enough injectives)

- if for every $X \in \mathcal{A}$ $\exists P$ projective & a surjection $P \twoheadrightarrow A$.

(• if for every $X \in \mathcal{A}$ $\exists I$ injective & an inclusion $I \hookrightarrow X$).

e.g. $\mathcal{A} = R\text{-Mod}$, $\mathrm{Coh}(X,k)$, $\underset{k\pi_1(X)\text{-Mod}}{\underset{\shortparallel}{\mathrm{Loc}(X,k)}}$, $\underset{\mathrm{Coh}(\underline{k}_X)}{\underset{\shortparallel}{\mathrm{Sh}(X,k)}}$

Defn.

$$\boxed{\begin{array}{l}\text{Let } X, Y \in \mathcal{A}. \text{ Then} \\ \quad \mathrm{Ext}^i_{\mathcal{A}}(X,Y) = \mathcal{H}^i(R\mathrm{Hom}_{\mathcal{A}}(X,Y))\end{array}}$$

i.e. to compute $\mathrm{Ext}^i(X,Y)$ we do one of the following:

(1) • Take a projective resolution of X:

$$\cdots \to P^{-1} \to P^0 \twoheadrightarrow X \to 0$$

• Apply $\mathrm{Hom}(-,Y)$ to $P^\bullet$:
(this is contravariant)

$$0 \to \mathrm{Hom}(P^0, Y) \to \mathrm{Hom}(P^{-1}, Y) \to \cdots$$

• Take i-th cohomology.

or (2) • Take an injective resolution of Y:

$$0 \to Y \hookrightarrow I^0 \to I^1 \to \cdots$$

• Apply $\mathrm{Hom}(X,-)$ to $I^\bullet$:

$$0 \to \mathrm{Hom}(X, I^0) \to \mathrm{Hom}(X, I^1) \to \cdots$$

• Take i-th cohomology.

NB.
(i) it is nontrivial that these two procedures give the same result.
~~(ii) one should~~

(ii) $\boxed{\text{If } \mathcal{A} \text{ is } k\text{-linear (i.e. each Hom space is a } k\text{-module), then any } \mathrm{Ext}^i_{\mathcal{A}}(X,Y) \text{ is a } k\text{-module}}$

Example 1.

Let $G = \mathbb{Z}/2\mathbb{Z}$.

So $kG = k[t]/(t^2-1)$

Let $\mathcal{A} = kG\text{-Mod}$.

Compute $\operatorname{Ext}^{\bullet}_{\mathcal{A}}(k,k)$, where k is the trivial module.

(i) Find a projective resolution of k:

$$\begin{tikzcd}
{} \arrow[r, "\cdot(t+1)"] & kG \arrow[rr, "\cdot(t-1)"] \arrow[dr] & & kG \arrow[rr, "\cdot(t+1)"] \arrow[dr] & & kG \arrow[rr, "\cdot(t-1)"] \arrow[dr] & & kG \arrow[r, two heads, "\varepsilon"] & k \\
& & \langle t-1\rangle \arrow[ur] & & \langle t+1\rangle \arrow[ur] & & \langle t-1\rangle \arrow[ur] & &
\end{tikzcd}$$

$1 \longmapsto 1$

$t \longmapsto 1$

projective resolution of k.

(ii) Apply $\operatorname{Hom}_{kG}(-,k)$:

$$\begin{tikzcd}
{} \arrow[r] & kG \arrow[r, "\cdot(t+1)"] \arrow[drr] & kG \arrow[r, "\cdot(t-1)"] \arrow[dr] & kG \arrow[r] \arrow[d] & 0 \\
& & & k &
\end{tikzcd}$$

$\leftarrow$ compute these Hom spaces.

Now, $\operatorname{Hom}_{kG}(kG,k) = k\varepsilon$

So,

$$\boxed{\operatorname{Hom}(P^{\bullet},k) = 0 \longrightarrow k\varepsilon \xrightarrow{0} k\varepsilon \xrightarrow{\cdot 2} k\varepsilon \xrightarrow{0} k\varepsilon \xrightarrow{\cdot 2} \cdots}$$

Indeed, to compute the differentials:

$$\boxed{\begin{array}{l} kG \xrightarrow{\cdot(t-1)} kG \xrightarrow{\varepsilon} k \\ 1 \longmapsto (t-1) \longmapsto 0 \end{array}}$$

$$\boxed{\begin{array}{l} kG \xrightarrow{\cdot(t+1)} kG \xrightarrow{\varepsilon} k \\ 1 \longmapsto t+1 \longmapsto 2 \end{array}}$$

(iii) Computing Cohomology:

- If char $k = 2$ then:

$$\mathrm{Ext}^{\bullet}_{kG}(k,k) = \bigoplus_{i=0}^{\infty} k$$

- If char $k > 2$ or 0 then

$$\mathrm{Ext}^{i}_{G}(k,k) = \begin{cases} k & \text{if } i \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

Group Cohomology

More generally,

$$\boxed{H^{i}(G;k) = \mathrm{Ext}^{i}_{kG}(k,k)}$$

is called the Group Cohomology of G (with coefficients in k).

For $kG \circlearrowright M$ we also define:

$$H^{i}(G;M) = \mathrm{Ext}^{i}_{kG}(k,M) = R(-)^{G}(M)$$

Fact:

If X is $K(G,1)$ then

$$H^{i}(X;M) = H^{i}(G;M)$$

Example 2

$A = \mathbb{Z}\text{-Mod} = \left\{ \begin{array}{c}\text{cat. of} \\ \text{abelian grps}\end{array} \right\}$

Let $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$

Compute $Ext^0(\mathbb{Z}_n, \mathbb{Z}_m)$

① What is $Hom(\mathbb{Z}_n, \mathbb{Z}_m)$?

~~So~~ For $\varphi : \mathbb{Z}_n \to \mathbb{Z}_m$ we have

$$\begin{array}{ccc} 1 & \longmapsto & \frac{m}{d} \\ \vdots & & \vdots \\ 0 = n & \longmapsto & \left(\frac{n}{d}\right) m = 0 \end{array}$$

d a common divisor of m, n.

Fact: For any such d,

$$\frac{m}{d} = k\left(\frac{m}{\gcd(m,n)}\right) \bmod m$$

So $$\boxed{Hom(\mathbb{Z}_n, \mathbb{Z}_m) = \mathbb{Z}_{\gcd(m,n)}}$$

So $Ext^0(\mathbb{Z}_n, \mathbb{Z}_m) = 0$ if m, n are coprime.

② Compute $\mathrm{Ext}^\circ(\mathbb{Z}_n, \mathbb{Z}_m)$ for n, m not coprime.

$\mathbb{Z}_n$ has projective resolution:

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[rr, "\cdot n"] \arrow[dr, two heads] & & \mathbb{Z} \arrow[r] & \mathbb{Z}_n \arrow[r] & 0 \\
& & & n\mathbb{Z} \arrow[ur] & & &
\end{tikzcd}$$

Apply $\mathrm{Hom}(-, \mathbb{Z}_m)$:

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}_m) & \xrightarrow{\cdot n} & \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}_m) & \longrightarrow 0 \rightarrow \cdots \\
& \wr\!\wr & & \wr\!\wr & \\
& \mathbb{Z}_m & & \mathbb{Z}_m & \\
& 1 & \longmapsto & n &
\end{array}$$

So $\boxed{\mathrm{Ext}^i(\mathbb{Z}_n, \mathbb{Z}_m) = 0 \text{ if } i \geq 2.}$

What is $\mathrm{Ext}^1(\mathbb{Z}_n, \mathbb{Z}_m)$?

$$\mathrm{Ext}^1(\mathbb{Z}_n, \mathbb{Z}_m) = \mathrm{cok}(\mathbb{Z}_m \xrightarrow{\cdot n} \mathbb{Z}_m)$$

Consider the map:

$$\mathbb{Z} \longrightarrow \mathbb{Z}_m \longrightarrow \mathrm{cok}(\mathbb{Z}_m \xrightarrow{\cdot n} \mathbb{Z}_m)$$

This map is surjective w kernel

$$m\mathbb{Z} + n\mathbb{Z} = \cancel{g\mathbb{Z}}.\, \gcd(m,n)\mathbb{Z}.$$

So $\boxed{\mathrm{Ext}^1(\mathbb{Z}_n, \mathbb{Z}_m) = \mathbb{Z}_{\gcd(m,n)}}$

Bonus Question:

What is the group cohomology of $\mathbb{Z}$?
(w coefficients in k).

~~Ext~~

Now,
$$k\mathbb{Z} = k[T, T^{-1}].$$

~~We need~~
Compute $\mathrm{Ext}^{\bullet}_{k[T,T^{-1}]}(k, k)$:

k has projective resolution:

$$\begin{tikzcd}
\to 0 \arrow[r] & k[T,T^{-1}] \arrow[rr, "\cdot(T-1)"] \arrow[dr] & & k[T,T^{-1}] \arrow[r, "\varepsilon"] & k \arrow[r] & 0 \\
& & \langle T^n - T^m \mid_{n,m\in\mathbb{Z}} \rangle \arrow[ur] & & &
\end{tikzcd}$$

$$1 \longmapsto 1$$
$$T^n \longmapsto 1$$

$$\langle T^n - T^m \mid_{n,m\in\mathbb{Z}} \rangle = \langle T-1 \rangle_{k[T,T^{-1}]}$$

Apply $\mathrm{Hom}(-, k)$:

$$0 \longrightarrow \mathrm{Hom}(k[T,T^{-1}], k) \xrightarrow{\ 0\ } \mathrm{Hom}(k[T,T^{-1}], k) \to 0 \to$$
$$\mathrm{Hom}(k[T,T^{-1}], k) \cong k\varepsilon, \qquad \mathrm{Hom}(k[T,T^{-1}], k) \cong k\varepsilon$$

So
$$\mathrm{Ext}^i_{k[T,T^{-1}]}(k,k) = \begin{cases} k & \text{if } i = 0,1 \\ 0 & \text{otherwise} \end{cases}$$

which is the cohomology of S^1!

~~Part II. YONEDA EXTS~~

More generally compute:

~~$H^i(G,$~~ For M a $k[\mathbb{Z}]$-Module we can compute

$$H^\bullet(\mathbb{Z};M) = \mathrm{Ext}^\bullet_{k[T,T^{-1}]}(k,M)$$

group cohomology of $\mathbb{Z}$ with coefficients in M.

Using the projective resolution of k as above we get:

$\mathrm{Ext}^\bullet(k,M)$ is cohomology of:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Hom}(k[T,T^{-1}],M) & \xrightarrow{\cdot(T-1)} & \mathrm{Hom}(k[T,T^{-1}],M) & \rightarrow & 0 \\ & & \| & & \| & & \\ & & M & \mu(T)-1 & M & & \end{array}$$

for $\mu: k[T,T^{-1}] \rightarrow \mathrm{End}(M)$

i.e.

$$\mathrm{Ext}^i(k,M) = \begin{cases} M^\mu & \text{if } i=0 \\ M_\mu & \text{if } i=1 \\ 0 & \text{otherwise} \end{cases}$$

where μ is the action of T on $k[T,T^{-1}]$ on M.

Interpretation:

If $\mathcal{L}$ is a local system on S^1,

then:
$$H^i(S^1;\mathcal{L}) = \begin{cases} V^{\rho} & i=0 \\ V_{\rho} & i=1 \\ 0 & \text{otherwise} \end{cases}$$

where $\rho \circlearrowright V$ is the monodromy representation corresponding to $\mathcal{L}$.

(Recall $\pi_1(S^1) = \mathbb{Z}$)

Long Exact Sequence in Ext.

GENERAL FACT ABOUT DERIVED FUNCTORS

Thm. Given a short exact sequence

$$0 \to A \to B \to C \to 0$$

There is a long exact sequence (for any object Y):

~~$0 \to Ext$~~

$$0 \to \mathrm{Hom}(C,Y) \to \mathrm{Hom}(B,Y) \to \mathrm{Hom}(A,Y) \to$$
$$\to \mathrm{Ext}^1(C,Y) \to \mathrm{Ext}^1(B,Y) \to \mathrm{Ext}^1(A,Y) \to$$
$$\to \mathrm{Ext}^2(C,Y) \to \cdots$$

Ext' as a Cokernel.

In particular, for any $X \in \mathcal{A}$, we can form a projective presentation:

$$0 \to R \hookrightarrow \underset{\text{projective}}{P} \twoheadrightarrow X \to 0$$

Then we have the sequence:

$$0 \to \mathrm{Hom}(X,Y) \to \mathrm{Hom}(P,Y) \to \mathrm{Hom}(R,Y) \to \mathrm{Ext}^1(X,Y) \to 0$$

(for any Y)

Hence, we can think of $\mathrm{Ext}^1(X,Y)$ as the cokernel of

$$\mathrm{Hom}(P,Y) \to \mathrm{Hom}(R,Y)$$

$\mathrm{Ext}^1(X,Y)$ as extensions.

Given $f \in \mathrm{Hom}(R,Y)$ we can build as extension

$$Y \to E_f \to X \quad \text{as follows:}$$

$$\begin{tikzcd} R \arrow[r, hook] \arrow[d, "f"'] & P \arrow[r, two heads] \arrow[d] \arrow[dl, phantom, "PO"] & X \arrow[d, equal] \\ Y \arrow[r, hook] & E_f \arrow[r, two heads] \arrow[d, equal] & X \\ & P \oplus Y / \langle (r,1) = (1, f(r)) \rangle & \end{tikzcd}$$

This process defines a bijection

$$\mathrm{Ext}^1(X,Y) \xrightarrow{\sim} \left\{ \begin{array}{l} \cancel{\text{Set}}\text{ extensions } Y \to E \to X \\ \text{modulo the relation} \\ (Y \to E \to X) = (Y \to F \to X) \\ \text{if } \exists \varphi : E \xrightarrow{\sim} F \text{ s.t.} \\ \begin{tikzcd} Y \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d, "\varphi"] \arrow[dl, phantom, "\circlearrowright"] & X \arrow[d, equal] \arrow[dl, phantom, "\circlearrowright"] \\ Y \arrow[r] & F \arrow[r] & X \end{tikzcd} \end{array} \right\}$$

think of as equivalence classes of maps $R \to Y$.

Using the linear structure on $\mathrm{Ext}^1(X,Y)$ we can define a linear structure on extensions

e.g. $0 \mapsto (Y \to X \oplus Y \to X)$ ← "zero extension"

But this is a topic for another talk...