27/02/20

Kac-Moody Algebras I

Motivation, root data, definitions; root systems & Weyl grp.

§0 Intro

- Kac-Moody algebras are a generators-&-relations presentation of finite type, affine type, & indefinite type Lie algebras.
- We care mostly about finite & affine type, all of which are symmetrisable & so are particularly nice. For instance, they have associated quatum groups. We will always assume symmetrisability.
- Another big bonus is treating their representation theory in a unified way, & is a nice stepping stone into categorical representation theory, or quantum group representations

Today we will motivate the Kac-Moody construction (& orient ourselves) by reviewing the classification & Serre's presentation of the split semisimple Lie algebras. We will then generalise Cartan matrices & define Kac-Moody algebras, which is almost Serre's construction, but with a modified Cartan. We will briefly review some examples & root systems, & the Weyl group of this generalised root system.

The references for this talk can be found on the student algebra seminar web page, but in brief are:

- Infinite-dimensional Lie algebras by Victor Kac.
- Kac-Moody groups, their flag varieties & Representation theory by Shrawan Kumar.
- An introduction to Kac-Moody groups over fields, by Timothée Marquis.

The second lecture in this series will take place at a later day, some weeks away, & will focus more in-depth at different constructions of the affine algebras & how they relate to each other.

§1 Classification (finite case), Serre's presentation

Recall that the complex semisimple finite dimensional Lie algebras are classified by root systems (in the sense of Bourbaki), & these in turn are classified by Cartan matrices.

$$\underset{\substack{\text{Split semisimple}\\\text{Lie alg}}}{(\underline{g}, \underline{h})} \rightsquigarrow \underset{\text{Root System}}{\Delta \subseteq \underline{h}^*} \rightsquigarrow \underset{\text{Simple system}}{\Pi \subseteq \Delta} \rightsquigarrow \underset{\text{Cartan matrix.}}{a_{ij} = \langle \alpha_j, \alpha_i^\vee \rangle}$$

Serre

Serre's presentation gives a way to recover an algebra isomorphic to $\underline{g}$ from the Cartan matrix A. We will quickly review these steps, since it will provide good motivation for later.

① Since $\underline{h} \subseteq \underline{g}$ is a Cartan & $\mathbb{C}$ is alg closed, $\underline{h} \overset{ad}{\curvearrowright} \underline{g}$ diagonalises.

② Set $\underline{g}_\lambda = \{ x \in \underline{g} \mid [h, x] = \langle \lambda, h \rangle x$ for all $h \in \underline{h} \}$, $\lambda \in \underline{h}^*$.
Simultaneous eigenspace ; $\lambda(h)$.

③ $\underline{g} = (\underline{g}_0 = \underline{h}) \oplus \bigoplus_{\alpha \in \Delta} \underline{g}_\alpha$ ← each has dimension 1.
← roots = nonzero weights of adjoint.

④ For $\alpha \in \Delta$ form the $\underline{sl}_2$-triple $(x_\alpha, x_{-\alpha}, \alpha^\vee)$ where
- $\alpha^\vee \in [\underline{g}_\alpha, \underline{g}_{-\alpha}]$ s.t $\langle \alpha, \alpha^\vee \rangle = 2$. ($[\underline{g}_\alpha, \underline{g}_{-\alpha}]$ 1-dim, so unique).
- $x_\alpha \in \underline{g}_\alpha$ and $x_{-\alpha} \in \underline{g}_{-\alpha}$ s.t $[x_\alpha, x_{-\alpha}] = \alpha^\vee$.

⑤ Set $\underline{s}_\alpha = \text{span}(x_\alpha, x_{-\alpha}, \alpha^\vee)$, a subalg iso to $\underline{sl}_2$.

⑦ Define $r_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$, $r_\alpha \in Aut(\underline{h}^*)$.

Now it turns out that $(\Delta \subseteq \underline{h}^*, \Delta^\vee \subseteq \underline{h})$ form a root system.
~~We can show $\langle \alpha, \beta^\vee \rangle \in \mathbb{Z}$ by considering $\underline{s}_\alpha \curvearrowright \underline{g}$ an $\underline{sl}_2$ module, where $\beta^\vee \in \underline{g}$ has weight $\langle \alpha, \beta^\vee \rangle$~~

- $\langle \alpha, \beta^\vee \rangle \in \mathbb{Z}$ by considering $\underline{g}$ as an $\underline{s}_\beta$-module. As it is finite-dim, it is semisimple, and x_α has weight $\langle \alpha, \beta^\vee \rangle$ which must be an integer.
- A similar argument involving $\underline{sl}_2$-strings shows $r_\beta(\Delta) = \Delta$.

⑧ Choose simple roots $\Pi = \{ \alpha_i \mid i \in I \} \subseteq \Delta$ for some indexing set I.

⑨ Let $A \in Mat_I(\mathbb{Z})$, $a_{ij} = \langle \alpha_j, \alpha_i^\vee \rangle$ the Cartan

Now, it turns out that $\mathfrak{g}$ is generated as a Lie alg by the $\{\mathfrak{s}_\alpha \mid \alpha \in \Pi\}$. So we might try to artificially construct $\mathfrak{g}$ from these $\mathfrak{sl}_2$-triples.

Defn (Serre) Given $A \in Mat_I(\mathbb{Z})$, let $\mathfrak{g}_A$ be the Lie alg gen by $\{e_i, f_i, h_i \mid i \in I\}$, with rels

$\mathfrak{h}$ was commutative $\rightsquigarrow$ $[h_i, h_j] = 0$

$[x_\alpha, x_{-\alpha}] = \alpha^\vee$ $\rightsquigarrow$ $[e_i, f_i] = h_i$

$[\alpha^\vee, x_\beta] = \langle \beta, \alpha^\vee \rangle x_\beta$ $\rightsquigarrow$ $[h_i, e_j] = a_{ij} e_j$, $[h_i, f_j] = -a_{ij} f_j$

$\alpha_i - \alpha_j \notin \Delta \cup 0$ $\rightsquigarrow$ $[e_i, f_j] = 0$ for $i \neq j$

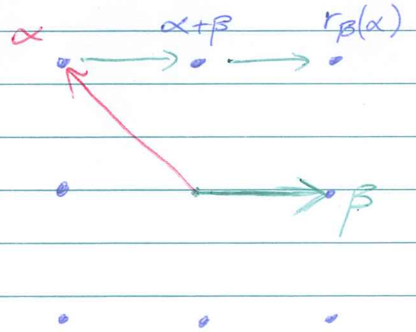
Serre Relations: $(\operatorname{ad} e_i)^{1+|a_{ij}|} e_j = 0$ for $i \neq j$

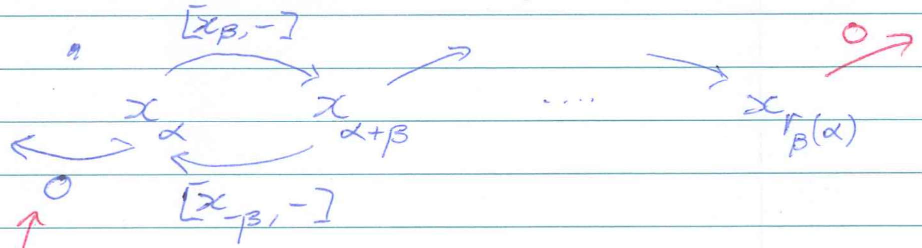
$(\operatorname{ad} f_i)^{1+|a_{ij}|} f_j = 0$ for $i \neq j$.

Thm (Serre) If $\mathfrak{g}$ is classified by A, then $\mathfrak{g} \cong \mathfrak{g}_A$.

On the Serre relations: What are they & where do they come from? Consider a B_2 example with simple roots α, β.

Now consider $\mathfrak{s}_\beta \curvearrowright \mathfrak{g}$, an $\mathfrak{sl}_2$-rep. Take the subrep generated by x_α.





$[x_{-\beta}, x_\alpha] \in \mathfrak{g}_{\alpha-\beta} = 0$
because α, β simple roots
so $\alpha - \beta$ not a weight.

In fact, $x_\alpha, x_{\alpha+\beta}, \ldots, x_{r_\beta(\alpha)}$ is a full $\mathfrak{sl}_2$-string. Hence

$$(\operatorname{ad} x_\beta)^{|a_{\beta\alpha}|+1} x_\alpha = 0$$

Number of arrows in string must be $|\langle \alpha, \beta^\vee \rangle| = |a_{\beta\alpha}|$.

§2 GCMs & Root Data

The Kac-Moody presentation is essentially Serre's, but run with a slightly modified Cartan to remove some degeneracies arising when A is not full rank.

From this point on, we are always over $\mathbb{C}$, I is always a finite set, and $A \in \mathrm{Mat}_I(\mathbb{Z})$, and $[a_{ij}]_{i,j \in I}$ entries of A.

<u>Defn</u> (GCM) A <u>generalised Cartan matrix</u> is $A \in \mathrm{Mat}_I(\mathbb{Z})$ satisfying
① $a_{ii} = 2$, ② $a_{ij} \leq 0$ for $i \neq j$, ③ $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$.

We say A is <u>decomposable</u> if a permutation σ makes $\sigma A \sigma^{-1}$ a block matrix with two or more blocks.

<u>Defn</u> A <u>realisation</u> of the GCM A is $(\underline{h}, \Pi, \Pi^\vee)$ where
- $\underline{h}$ is a $\mathbb{C}$-vect space,
- $\Pi = \{\alpha_i \mid i \in I\} \subseteq \underline{h}^*$ a LI set,
- $\Pi^\vee = \{\alpha_i^\vee \mid i \in I\} \subseteq \underline{h}$ a LI set,

such that $\langle \alpha_j, \alpha_i^\vee \rangle = a_{ij}$.

<u>Eg</u> $A = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$ is a GCM (B_2). Set $\underline{h} = \mathbb{C}\{\alpha_1^\vee, \alpha_2^\vee\}$, and then the α_1, α_2 are completely determined by the pairing condition:

$$\alpha_1 = \begin{pmatrix} \alpha_1^\vee \longmapsto 2 \\ \alpha_2^\vee \longmapsto -2 \end{pmatrix} \qquad \alpha_2 = \begin{pmatrix} \alpha_1^\vee \longmapsto -1 \\ \alpha_2^\vee \longmapsto 2 \end{pmatrix}$$

Write this on a clean board

<u>Eg</u> $A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$ is a GCM $(A_1^{(1)})$. Doing the same thing as before gets bad results: $-\alpha_0 = \alpha_1$.

$$\underline{h} = \mathbb{C}\{\alpha_0^\vee, \alpha_1^\vee, d\} \qquad \alpha_0 = \begin{pmatrix} \alpha_0^\vee \longmapsto 2 \\ \alpha_1^\vee \longmapsto -2 \\ d \longmapsto 1 \end{pmatrix} \qquad \alpha_1 = \begin{pmatrix} \alpha_0^\vee \longmapsto -2 \\ \alpha_1^\vee \longmapsto 2 \\ d \longmapsto 1 \end{pmatrix}$$

So insert an element d into $\underline{h}$, & choose a root (α_0) to perturb. Now α_0 & α_1 are LI.

<u>Ex</u> If $(\underline{h}, \Pi, \Pi^\vee)$ a realisation of A, then $\dim_{\mathbb{C}} \underline{h} \geq |I| + \mathrm{corank}(A)$.

§3 Kac-Moody Algs

The definition of a Kac-Moody alg is exactly Serre's presentation, with the h_i replaced by a specific realisation in the obvious way.

<u>Defn</u> Let A a GCM, and $(\underline{h}, \Pi, \Pi^\vee)$ a realisation where $\dim_{\mathbb{C}} \underline{h} = |I| + \operatorname{corank}(A)$. Define $\underline{g}(A)$ as the Lie alg gen by $\underline{h}$, $\{e_i, f_i \mid i \in I\}$ subject to:

$$[\underline{h}, \underline{h}] = 0$$
$$[h, e_i] = \langle \alpha_i, h \rangle e_i \quad \forall\, h \in \underline{h}$$
$$[h, f_i] = \cancel{h} - \langle \alpha_i, h \rangle f_i \quad \forall\, h \in \underline{h}$$
$$[e_i, f_j] = \delta_{ij} \alpha_i^\vee$$
(Serre rels).

<u>Comment</u>: Kac' definition excludes the Serre rels, but is instead a quotient by a certain ideal containing the Serre rels. It is conjectured that these two definitions are equivalent, $\exists$ is true whenever A is symmetrisable.

We usually get a handle on $\underline{g}(A)$ by its root space decomposition. Let $Q = \operatorname{Span}_{\mathbb{Z}} \{\alpha_i \mid i \in I\}$, $Q^+ = \operatorname{Span}_{\mathbb{N}} \{\alpha_i \mid i \in I\}$.

<u>Facts</u> ① $\underline{g}(A) = \bigoplus_{\alpha \in Q} \underline{g}_\alpha$, where $\underline{g}_0 = \underline{h}$ and $\dim \underline{g}_\alpha < \infty$.

② $\underline{n}^+(A) = \bigoplus_{\alpha \in Q^+ \setminus 0} \underline{g}_\alpha$ is freely gen by the e_i, subject only to the Serre rel.

③ $\underline{n}^-(A) = \bigoplus_{\alpha \in Q^+ \setminus 0} \underline{g}_{-\alpha}$ —— " —— f_i —— " ——.

④ $\underline{h} \hookrightarrow \underline{g}(A)$ is an inclusion, so we may confuse $\underline{h}$ with a subspace of $\underline{g}(A)$.

We will need a little more terminology before being able to classify roots, but for now lets see an example.

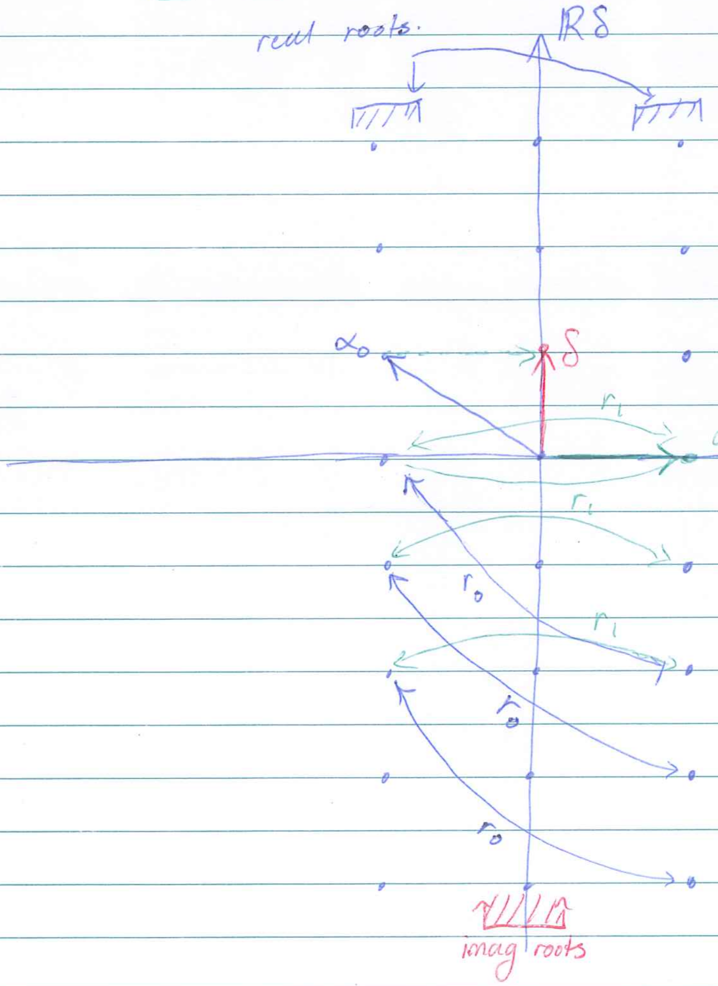
§4 Affine (untwisted) sl_2

Let $A = \begin{array}{c|cc} & \alpha_0 & \alpha_1 \\ \hline \alpha_0^\vee & 2 & -2 \\ \alpha_1^\vee & -2 & 2 \end{array}$ be the GCM for $A_1^{(1)}$. Take the realisation from before, where

$\underline{h} = \mathbb{C}\{\alpha_0^\vee, \alpha_1^\vee, d\}$ $\quad \alpha_0 = \left| \begin{array}{l} \alpha_0^\vee \mapsto 2 \\ \alpha_1^\vee \mapsto -2 \\ d \mapsto 1 \end{array} \right| \quad \alpha_1 = \left| \begin{array}{l} \alpha_0^\vee \mapsto -2 \\ \alpha_1^\vee \mapsto 2 \\ d \mapsto 0 \end{array} \right|.$

Call $\alpha_0 + \alpha_1 = \delta$ the null root, characterised by $\langle \delta, - \rangle \big|_{\underline{h}'} = 0$.

Then (α_1, δ) form a basis of $\mathbb{R}Q$:



$S = \{r_0, r_1\}$.
$W = \langle r_0, r_1 \rangle$
Fact (W, S) a Coxeter System.

$\Delta^{re} := W \cdot \Pi$
$\Delta^{im} := \Delta \setminus \Delta^{re}$.

In $A_1^{(1)}$, we have
$\Delta^{re} = (\alpha_0 + \mathbb{Z}\delta) \cup (\alpha_1 + \mathbb{Z}\delta)$.
$\Delta^{im} = \mathbb{Z}\delta \setminus \{0\}$.

Symmetrisable:
$\Delta^{re} = \{\alpha \in \Delta \mid (\alpha, \alpha) > 0\}$
$\Delta^{im} = \{\alpha \in \Delta \mid (\alpha, \alpha) \leq 0\}$.

By some general theory (which we will cover next time) the roots Δ of $\mathfrak{g}(A)$ look as above. We have

$r_0(\alpha_0) = -\alpha_0 \qquad\qquad = -\alpha_0$	$r_1(\alpha_0) = \alpha_0 + 2\alpha_1$
$r_0(\alpha_1) = \alpha_1 - \langle \alpha_1, \alpha_0^\vee \rangle \alpha_0 = \alpha_1 + 2\alpha_0$	$r_1(\alpha_1) = -\alpha_1$
$\Rightarrow r_0(\delta) = r_0(\alpha_0 + \alpha_1) = \delta$.	$r_1(\delta) = \delta$.

§5 Symmetrisability

When the Cartan matrix is induced from a symmetric bilinear form, $\mathfrak{g}(A)$ is particularly nice, & many theorems are simplified. This is always true in the finite & affine cases.

Defn $\mathfrak{g}(A)$ is called symmetrisable if any of the following equivalent conditions hold:

a) There is an invertible diagonal matrix D & a symmetric matrix B s.t $A = DB$.

b) There exists a nondegenerate bilinear form $(-|-) : \mathfrak{g}(A) \times \mathfrak{g}(A) \to \mathbb{C}$ which is invariant, ie $(x|[y,z]) = ([x,y]|z)$.

c) There exists a nondegenerate symmetric bilinear form $(-,-)$ on $\underline{h}^*$ such that $a_{ij} = \langle \alpha_j, \alpha_i^\vee \rangle = 2\frac{(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}$

I find (c) easiest to remember. Given such a form, we get an iso $\nu : \underline{h} \xrightarrow{\sim} \underline{h}^*$ where we can pull back to get $(-|-)|_{\underline{h}}$, and invariance extends that uniquely to $(-|-) : \mathfrak{g}(A) \times \mathfrak{g}(A) \to \mathbb{C}$.

Ex • All 2×2 GCMs are symmetrisable.
• There is a non-symmetrisable 3×3 GCM.

Prop For an indecomposable A, TFAE:

a) A symmetrisable with $(-,-)$ pos-definite

b) $|W| < \infty$

c) $|\Delta| < \infty$

d) $\mathfrak{g}(A)$ simple finite-dim

e) $\exists$ a highest root, ie $\alpha \in \Delta_+$ s.t $\alpha + \alpha_i \notin \Delta$ $\forall$ $i \in I$.

The real roots of $\mathfrak{g}(A)$ resemble the classic case. If $\alpha \in \Delta^{re}$, then

a) ~~$\dim \mathfrak{g}_\alpha$~~ $\operatorname{mult} \alpha := \dim \mathfrak{g}_\alpha = 1$

b) $k\alpha \in \Delta \Rightarrow k = \pm 1$.

c) Define $\alpha^\vee = w(\alpha_i^\vee)$ when $w(\alpha_i) = \alpha$.

d) $\alpha^\vee = 2\frac{\nu^{-1}(\alpha)}{(\alpha,\alpha)}$

§6 Some data in rank 2.

Listed here are the finite & affine type rank 2 Cartan matrices, together with their Dynkin diagrams, & particular choices of symmetrising matrix B. Remember, there is an arrow towards i if $|a_{ij}| \geq 2$, the number of lines is ~~1/a~~ $\max\{|a_{ij}|, |a_{ji}|\}$, & for all finite & affine diagrams we have $a_{ij}a_{ji} \leq 4$. On the B side, the arrow always points to the shorter root.

Name	Diagram	Cartan	~~B~~ Symmetric Form
A_2		$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$	$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$
B_2		$\begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$	$\begin{pmatrix} 4 & -2 \\ -2 & 2 \end{pmatrix}$
G_2		$\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$	$\begin{pmatrix} 6 & -3 \\ -3 & 2 \end{pmatrix}$
$A_1^{(1)}$		$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$	$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$
$A_1^{(2)}$		$\begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$	$\begin{pmatrix} 2 & -4 \\ -4 & 8 \end{pmatrix}$

Note the symmetric forms are not unique, any scaling by $\mathbb{R}_{>0}$ gives another form that works. However, these forms are special in that the diagonals are all in $\{2, 4, 6, \ldots\}$, & so are appropriate to use when defining a quantum group.