1. Complex Reflection groups

Let $V = \mathbb{C}^n$.

Def: A (complex) reflection $r \in GL(V)$ has finite order and fixes a hyperplane of V.

A finite complex reflection group W is finite group generated by complex reflections.

Examples: - Any finite real real reflection group complexified.

$V = \mathbb{R}^n$, $V_{\mathbb{C}} = \mathbb{R}^n \otimes \mathbb{C}$

- $G = \langle e^{2\pi i/n} \rangle$ on $\mathbb{C}$

Classification of Irreducibles:

- 3 infinite families
- 34 exceptional cases.

Theorem: (Shephard-Todd-Chevalley)

Fix a group W.

W is finite complex reflection group $\Leftrightarrow$

The ring $\mathbb{C}[V]^W$ of W-invariant polynomials is a polynomial algebra, (generated by a collection of alg. ind. homogeneous polynomials $f_1, \dots, f_n$).

Let $d_1, \dots, d_n$ be the degrees of $f_1, \dots, f_n$

Properties:

- $\prod d_i = |W|$
- $\sum (d_i - 1) = \#$ reflections in W.

Example: A_2, $V = \mathbb{C}^3 / x_1 + x_2 + x_3 = 0$

$$f_1(x_1, x_2, x_3) = x_1 x_2 + x_1 x_3 + x_2 x_3$$

$$f_2(x_1, x_2, x_3) = x_1 x_2 x_3 .$$

$d_1 = 2$, $d_2 = 3$. $\quad 2 \times 3 = 6$

$\#$ reflections of A_2 = $\#$ of positive roots
$= 3$

2. Lehrer Springer Theory

Let $g \in W$ and ν be a primitive dth root of unity.

$$V(g,\nu) = \ker(g - \nu\, \mathrm{Id}_V)$$

Theorem: If $V(g,\nu)$ is maximal dimension amongst all ν-eigenspaces of elements of W, then:

- $\dim V(g,\nu) = \left| \{ d_i \mid \nu^{d_i} = 1 \} \right|$
- $W(g,\nu) := N_G(V(g,\nu)) / C_W(V(g,\nu))$ is a finite complex reflection group on $V(g,\nu)$.

Properties of $W(g,\nu)$

- Degrees of $W(g,\nu)$ are $\{ d_i \mid \nu^{d_i} = 1 \}$
- If W irred on V, then $W(g,\nu)$ irred on $V(g,\nu)$.
- Hyperplanes of $W(g,\nu)$ are $V(g,\nu) \cap H$ s.t. $V(g,\nu) \not\subseteq H$, where H is a hyperplane of V.

Examples: - E_6, $r = -1$, g s.t. $V(g,-1)$ is maximal dimension.

E_6 : $2, 5, 6, 8, 9, 12$

$W(g,-1)$: $2, 6, 8, 12$

$W(g,-1) \cong W(F_4)$.

- A_2, $r = e^{2\pi i/3}$, $g = s_1 s_2$

$W(g,r) = \langle e^{2\pi i/3} \rangle$

Def: - A vector $v \in V$ is regular if it lies on no reflecting hyperplanes of W.

- An element $g \in W$ r-regular if $V(g,r)$ contains a regular vector.

Example: Coxeter elements are always regular.

Theorem (Springer)

If g is r-regular then $V(g,r)$ is maximal dimension and

$$W(g,r) = C_W(g).$$

3. Sylow Φ_d-tori of G^F

G is ss algebraic group over $\overline{\mathbb{F}_p}$.

F is a Steinberg endomorphism w.r.t some $\mathbb{F}_q$-structure of G. Assume non-twisted.

$$|G^F| = q^{|\Phi^+|} \prod_{i=1}^{rk(G)} (q^{d_i} - 1)$$

where d_i are the degrees of $W = N_G(T)/C_G(T)$.

$$|G^F| = q^{|\Phi^+|} \prod \Phi_d(q)^{a(d)}$$

Reminder: - $\Phi_d(q) = \prod_{\substack{1 \leq k \leq d \\ \gcd(k,d)=1}} \left(q - e^{2\pi i k/d}\right)$

- $x^n - 1 = \prod_{d|n} \Phi_d(x)$

Def: Let S be a torus of G. Call S a Sylow d-torus of G, F if it is F-stable and $|S^F| = \Phi_d(q)^{a(d)}$

Theorem: $N_G(S)/C_G(S) \cong W(g, \nu)$

where S is Sylow d-torus, ν is primitive dth root of unity and $V(g,\nu)$ maximal dimension.

Applications: - ℓ-Sylow subgroup of G^F.

- Rep theory of G^F.