What is the universal enveloping algebra?

§1. Motivation. (Do not worry if this section makes no sense to you)

For a semi-simple simply connected Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ we will construct a unital associative algebra $U(\mathfrak{g})$. This algebra will be an $\mathbb{N}$-filtered algebra and:

$$\mathfrak{g}\text{-mod} = \left\{ \begin{array}{l} \text{homomorphisms} \\ \rho : \mathfrak{g} \longrightarrow \mathrm{End}(V) \\ \text{of Lie algebras} \\ \text{for a } \mathbb{C}\text{-vector space } V \end{array} \right\} \overset{\sim}{\longleftrightarrow} \left\{ \begin{array}{l} \text{unital } k\text{-algebra} \\ \text{homomorphisms} \\ \rho : U(\mathfrak{g}) \longrightarrow \mathrm{End}(V) \\ \text{for a } \mathbb{C}\text{-vector space } V \end{array} \right\} = U(\mathfrak{g})\text{-mod}.$$

There is a third equivalence by using the sheaf of differential operators on the flag variety $\mathcal{Fl}(\mathfrak{g})$ of $\mathfrak{g}$.

Let A be a $\mathbb{C}$-algebra, $\mathrm{End}_{\mathbb{C}}(A)$ is a Lie algebra.

$$\underbrace{\mathrm{End}_{\mathbb{C}}(A)}_{\text{a huge algebra}} \supset \underbrace{\mathrm{Diff}_A}_{\text{differential operators}} \supset \underbrace{\mathrm{Der}_{\mathbb{C}}(A)}_{\mathbb{C}\text{-derivations}} \supset \underbrace{\mathrm{End}_A(A)}_{\text{small algebra}} \simeq A$$

The Beilinson–Bernstein localisation theorem gives the equivalence

$$U(\mathfrak{g})/_{\mathcal{Z}_+}\text{—mod} = \mathrm{Diff}(\mathcal{O}_{\mathcal{Fl}(\mathfrak{g})})\text{—mod} \qquad \mathcal{Z}_+ = \mathrm{Ann}_{U(\mathfrak{g})}(\mathbb{C})$$

We can obtain irreducible $\mathfrak{g}$-mod by considering Verma modules V_λ for a weight λ.

<u>Philosophy</u>: Understand f.d. irreducible $\mathfrak{g}$-mod by using ∞-dimensional modules (the V_λ's) which are quotients of $U(\mathfrak{g})$.

§2. Definition and first properties of $U(\mathfrak{g})$.

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$. The tensor algebra $T(\mathfrak{g})$ is the free algebra

$$T(\mathfrak{g}) = \bigoplus_{i=0}^{\infty} \mathfrak{g}^{\otimes i}$$

$\otimes := \otimes_{\mathbb{C}}$ $\mathfrak{g}^{\otimes 1} \simeq \mathfrak{g}$

$\mathfrak{g}^0 := \mathbb{C}$

i.e. elements are $v_1 \otimes \cdots \otimes v_n$ for some $v_i \in \mathfrak{g}$ and some $n \in \mathbb{N}$ or in $\mathbb{C}$, and multiplication

$$(v_1 \otimes \cdots \otimes v_n) \cdot (w_1 \otimes \cdots \otimes w_m) := v_1 \otimes \cdots \otimes v_n \otimes w_1 \otimes \cdots \otimes w_m$$

It is a unital associative algebra over $\mathbb{C}$.

The universal enveloping algebra $U(\mathfrak{g})$ is the quotient of vector spaces

$$T(\mathfrak{g}) \Big/ \underbrace{\langle a \otimes b - b \otimes a - [a,b] \in \mathfrak{g} \oplus (\mathfrak{g} \otimes \mathfrak{g}) \mid a, b \in \mathfrak{g} \rangle}_{} =: I$$

where $\langle X \rangle$ means ideal generated by X. With multiplication induced by the multiplication in $T(\mathfrak{g})$.

It is a unital associative algebra over $\mathbb{C}$.

<u>Theorem</u> (Poincaré - Birkoff - Witt) Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$ with ordered basis $(e_i)_{i \in I}$. An ordered tensor is a simple tensor of the form $e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_p}$ such that $e_{i_1} \leq e_{i_2} \leq \cdots \leq e_{i_p}$.
The collection of all ordered tensors modulo I is a basis of $U(\mathfrak{g})$.

<u>Proof:</u> Very hard!

<u>Corollary</u>: Let $\mathfrak{g}$ be a Lie algebra. The natural map

$$i : \mathfrak{g} \longrightarrow U(\mathfrak{g})$$
$$x \longmapsto x + I \qquad \text{is injective}$$

Theorem: There is an equivalence of $\otimes$-categories

$$\operatorname{Rep} \mathfrak{g} \xrightarrow{\sim} \operatorname{Rep} U(\mathfrak{g}).$$

Before proving this we need a lemma.

Remark: A $\mathbb{C}$-algebra is naturally a Lie algebra by setting

$$[x,y] := xy - yx.$$

A map φ between a Lie algebra and a $\mathbb{C}$-algebra is called a Lie algebra map if

$$\varphi([x,y]) = \varphi(x)\varphi(y) - \varphi(y)\varphi(x).$$

Lemma: The map $i : \mathfrak{g} \longrightarrow U(\mathfrak{g})$ is a Lie algebra map and satisfies the following universal property:

- For any unital associative algebra A and any Lie algebra map $\phi : \mathfrak{g} \longrightarrow A$ there is a map $(*)$

$$U(\mathfrak{g}) \xrightarrow{\bar{\phi}} A$$

of associative k-algebras such that

$$\begin{tikzcd} U(\mathfrak{g}) \arrow[rr, dashed, "\bar{\phi}"] & & A \\ & \mathfrak{g} \arrow[ul, hook, "i"] \arrow[ur, "\phi"'] & \end{tikzcd}$$

commutes.

Proof: ϕ naturally extends to a morphism $\phi' : T(\mathfrak{g}) \longrightarrow A$ of k-algebras by the rule

$$\phi'(a \otimes b \otimes \cdots \otimes c) = \phi(a)\phi(b)\cdots\phi(c)$$

The map $\tilde{i} : \mathfrak{g} \hookrightarrow T(\mathfrak{g})$, $x \longmapsto x + I$ is injective and

$$\begin{tikzcd}
T(\mathfrak{g}) \arrow[rr, "\phi'"] & & A \\
& \mathfrak{g} \arrow[ul, "\tilde{i}"] \arrow[ur, "\phi"'] &
\end{tikzcd}$$

is commutative.

Let us show that $\ker \phi' \supset I$, where I is the defining ideal of $U(\mathfrak{g})$, i.e., $U(\mathfrak{g}) = T(\mathfrak{g})/I$. We have

$$\phi'(a\otimes b - b\otimes a - [a,b]) = \phi(a)\phi(b) - \phi(b)\phi(a) - \phi([a,b])$$
$$= 0 \text{ by } (*).$$

By the fundamental theorem of homomorphisms of algebras there exists $\bar{\phi} : U(\mathfrak{g}) \longrightarrow A$ s.t.

$$\begin{tikzcd}
T(\mathfrak{g}) \arrow[rr, "\phi'"] \arrow[dr] & & A \\
& U(\mathfrak{g}) \arrow[ur, "\bar{\phi}"'] &
\end{tikzcd}$$

commutes.

Therefore

$$\begin{tikzcd}
& U(\mathfrak{g}) \arrow[ddr, "\bar{\phi}"] & \\
& T(\mathfrak{g}) \arrow[u, "\pi"] \arrow[dr, "\phi'"] & \\
\mathfrak{g} \arrow[uur, "i"] \arrow[ur, "\tilde{i}"] \arrow[rr, "\phi"'] & & A
\end{tikzcd}$$

commutes

$\square$

This lemma provides us an arrow

$$\mathrm{Ob}(\mathrm{Rep}\ \mathfrak{g}) \longrightarrow \mathrm{Ob}(\mathrm{Rep}\ U(\mathfrak{g}))$$

$$\rho : \mathfrak{g} \longrightarrow \mathrm{End}(V) \longmapsto \bar{\rho} : U(\mathfrak{g}) \longrightarrow \mathrm{End}(V)$$

For an arrow to the left, note that if $f : U(\mathfrak{g}) \longrightarrow \mathrm{End}(V)$ is a representation of $U(\mathfrak{g})$, we can compose

$$\begin{tikzcd} U(\mathfrak{g}) \arrow[r, "f"] & \mathrm{End}(V) \\ \mathfrak{g} \arrow[u, "i"] \arrow[ur, "f \circ i"'] & \end{tikzcd}$$

The composition is a Lie map.

There are still missing some details I don't think they are particularly hard (please correct me if I am wrong):

- Showing the map between arrows
$$\mathrm{Arr}(\mathrm{Rep}\ \mathfrak{g}) \longrightarrow \mathrm{Arr}(\mathrm{Rep}\ U(\mathfrak{g})).$$
- Showing the induced map $\mathrm{Rep}\ \mathfrak{g} \longrightarrow \mathrm{Rep}\ U(\mathfrak{g})$ is a $\otimes$-functor, which is an equivalence of categories.

Example: $\mathfrak{sl}_2(\mathbb{C}) = \left\{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} : a, b, c \in \mathbb{C} \right\}$

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$ is a Lie algebra with

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

is a 3-dimensional Lie algebra.

$$U(\mathfrak{sl}_2(\mathbb{C})) = T(\mathfrak{sl}_2(\mathbb{C})) \Big/ \langle 2e = h \otimes e - e \otimes h,\ -2f = h \otimes f - f \otimes h,\ h = e \otimes f - f \otimes e \rangle$$

This is an ∞-dim algebra w/ basis $\{ e^{\otimes i} \otimes f^{\otimes j} \otimes h^{\otimes k} \}_{i,j,k \in \mathbb{N}}$.

$\boxed{\triangle\ \underline{\text{warning!}}\quad e \lozenge e = 0, \text{ but } \quad e \otimes e + I \neq I .}$

matrix multiplication

Thanks for listening!