Algebraic Invariants of Links

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Preface

This book is intended as an introduction to links and a reference for the invariants of abelian coverings of link exteriors, and to outline more recent work, particularly that related to free coverings, nilpotent quotients and concordance. Knot theory has been well served with a variety of texts at various levels, but essential features of the multicomponent case such as link homotopy, I-equivalence, the fact that not all links are boundary links, longitudes, the role of the lower central series as a source of invariants and the homological complexity of the many-variable Laurent polynomial rings are all generally overlooked. Moreover, it has become apparent that for the study of concordance and link homotopy it is more convenient to work with disc links; the distinction is imperceptible in the knot theoretic case.

Invariants of these types play an essential role in the study of such difficult and important problems as the concordance classification of classical knots and the questions of link concordance arising from the Casson-Freedman analysis of topological surgery problems, and particularly in the applications of knot theory to other areas of topology. For instance, the extension of the Disc Embedding Lemma to groups of subexponential growth by Freedman and Teichner derived from computations using link homotopy and the lower central series. Milnor's interpretation of the multivariable Alexander polynomial as a Reidemeister-Franz torsion was refined by Turaev, to give "sign-determined" torsions and Alexander polynomials. These were used by Lescop to extend the Casson invariant to all closed orientable 3-manifolds, and by Meng and Taubes to identify the Seiberg-Witten invariant for 3-manifolds. The multivariable Alexander polynomial also arises in McMullen's lower bound for the Thurston norm of a torally bounded 3-manifold.

Links and the main equivalence relations relating them are defined in Chapter 1. (In particular, we include a proof of Giffen's theorem relating F-isotopy and I-equivalence via shift-spinning.) In Chapter 2 we review homology and cohomology with local coefficients, and Poincaré duality for covering spaces. The most useful manifestations of duality are the Blanchfield pairings for abelian coverings (considered in this chapter) and for free coverings of homology boundary links (considered in Chapter 9). Most of Chapter 3 is on the determinantal invariants of modules over a commutative noetherian ring (including the Reidemeister-Franz torsion for chain complexes), but it also considers some special features of low-dimensional rings and Witt groups of hermitean pairings on torsion modules. These results are applied to the homology of abelian covers of link exteriors in the following five chapters. Chapter 4 is on the maximal abelian cover. Some results well-known for knots are extended to the many component case, and the connections between various properties of boundary links are examined. Relations with the invariants of sublinks, the total linking number cover, fibred links and finite abelian branched covers are considered in Chapter 5.

In the middle of the book (Chapters 6-8) the above ideas are applied in some special cases. Chapters 6 and 7 consider in more detail invariants of knots and of 2-component links, respectively. Here there are some simplifications, both in the algebra and the topology. In particular, surgery is used to describe the Blanchfield pairing of a classical knot (in Chapter 6) and to give Bailey's theorem on presentation matrices of the modules of 2-component links (in Chapter 7). Symmetries of links and link types, as reflected in the Alexander invariants, are studied in Chapter 8.

The later chapters (9-12) describe some invariants of nonabelian coverings and their application to questions of concordance and link homotopy. The links of greatest interest here are those concordant to sublinks of homology boundary links ($cSHB$ links). The exteriors of homology boundary links have covers with nontrivial free covering group. As free groups have cohomological dimension 1, the ideas used in studying knot modules extend readily to the homology modules and duality pairings of such covers. This is done in Chapter 9, which may be considered as an introduction to the work of Sato, Du Val and Farber on high dimensional boundary links. We also give a new proof of Gutiérrez' unlinking theorem for n-links, which holds for all $n \geq 3$ and extends, modulo s-cobordism, to the case $n = 2$.

Although $cSHB$ links do not always have such free covers, their groups have nilpotent quotients isomorphic to those of a free group. More generally, the quotients of a link group by the terms of its lower central series are concordance invariants of the link. (The only other such invariants known are the Witt classes of duality pairings on covering spaces.) Chapter 10 considers the connections between the nilpotent quotients, Lie algebra, cohomology algebra and minimal model of a group and more particularly the relations between Massey products and Milnor invariants for a link group. Although we establish the basic properties of the Milnor invariants here, we refer to Cochran's book for further details on geometric interpretations, computation and construction of examples.

The final two chapters are intended as an introduction to the work of Levine (on algebraic closure and completions), Le Dimet (on high dimensional disc links) and Habegger and Lin (on string links). As this work is still evolving, and the directions of further development may depend on the outcome of unproven conjectures, some arguments in these chapters are only sketched, if given at all. One of the difficulties in constructing invariants for links from the duality pairings of covering spaces is that, in contrast to the knot theoretic case, link groups do not in general share a common quotient with reasonable homological properties. The groups of all μ-component 1-links with all Milnor invariants 0 and the groups of all μ-component n-links for any $n \geq 2$ share the same tower of nilpotent quotients. The projective limit of this tower is the nilpotent completion of the free group on μ generators, and is uncountable. This is related to other notions of completion in Chapter 11. Another problem is that the set of concordance classes of links does not have a natural group structure. However "stacking" with respect to the last coordinate endows the set of concordance classes of n-disc links with such a structure. Chapter 12 considers disc links and their relation to spherical links.

The emphasis is on establishing algebraic invariants and their properties, and constructions for realizing such invariants have been omitted, for the most part. The reader is assumed to know some algebraic and geometric topology, and some commutative algebra (to the level of a first graduate course in each). We occasionally use spectral sequence arguments. Commutative and homological algebra are used systematically, and we avoid as far as possible accidental features, such as the existence of Wirtinger presentations. While the primary focus is on links in S^3, links in other homology spheres and higher dimensions and disc links in discs are also considered.

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Added in February 2012. The topic considered in this book that has expanded most rapidly in the past decade is that of *Twisted Polynomial Invariants.* The final section of the former Chapter 5 has become a new Chapter 6, on this topic. In addition, Chapter 2 has been rewritten, and there is a new Chapter 10, on *Singularities of Plane Curves.* Material has been added to the chapters on *Knot Modules* and on *Nilpotent Quotients.* The errors noticed to date have been corrected, and equations have been displayed more often.

I would like to thank C.Livingston and S.Naik, and D.Silver and S.Williams for our recent collaborations, and J.C.Cha, T.D.Cochran and S.A.Melikhov for their observations.

Part 1

Abelian Covers

CHAPTER 1

Links

In this chapter we shall define knots and links and the standard equivalence relations used in classifying them. We shall also outline the most important geometric aspects. The later chapters shall concentrate largely on the algebraic invariants of covering spaces.

1.1. Basic notions

The standard orientation of $\mathbb{R}^n$ induces an orientation on the unit n-disc $D^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \Sigma x_i^2 \leq 1\}$ and hence on its boundary $S^{n-1} = \partial D^n$, by the convention "outward normal first". We shall assume that standard discs and spheres have such orientations. Qualifications shall usually be omitted when there is no risk of ambiguity. In particular, we shall often abbreviate $X(K)$, $M(K)$ and πK (defined below) as X, M and π, respectively. If μ is a positive integer and Y is a topological space $\mu Y = Y \times \{1, \ldots, \mu\}$, the disjoint union of μ copies of Y.

All manifolds and maps between them shall be assumed PL unless otherwise stated. The main exceptions arise when considering 4-dimensional issues.

A *μ-component n-link* is an embedding $L : \mu S^n \to S^{n+2}$ which extends to an embedding j of $\mu S^n \times D^2$ onto a closed neighbourhood N of L, such that $j(\mu S^n \times \{0\}) = L$ and ∂N is bicollared in S^{n+2}. (We may also use the terms *classical link* when $n = 1$, *higher dimensional link* when $n \geq 2$ and *high dimensional link* when $n \geq 3$.) With this definition and the above conventions on orientations, each link is oriented. It is determined up to (ambient) isotopy by its image $L(\mu S^n)$, considered as an oriented codimension 2 submanifold of S^{n+2}, and so we may let L also denote this submanifold. The i^{th}

component of L is the *n-knot* (1-component n-link) $L_i = L|_{S^n \times \{i\}}$. Most of our arguments extend to links in homology spheres.

Links are locally flat by definition. (However, PL embeddings of higher dimensional manifolds in codimension 2 need not be locally flat. The typical singularity is the cone over an $(n-1)$-knot; there are no nontrivial 0-knots.) We may assume that the embedding j of the product neighbourhood is orientation preserving, and it is then unique up to isotopy *rel* $\mu S^n \times \{0\}$. The *exterior* of L is the compact $(n+2)$-manifold $X(L) = S^{n+2} \setminus int\, N$ with boundary $\partial X(L) \cong \mu S^n \times S^1$, and is well defined up to homeomorphism. It inherits an orientation from S^{n+2}. Let $M(L) = X(L) \cup \mu D^{n+1} \times S^1$ be the closed manifold obtained by surgery on L in S^{n+2}, with framing 0 on each component if $n = 1$. (Since $\pi_n(\mathrm{O}(2)) = 0$ if $n > 1$, the framing is then essentially unique.)

The *link group* is $\pi L = \pi_1(X(L))$. A *meridianal curve* for the i^{th} component of L is an oriented curve in $\partial X(L_i) \subseteq \partial X(L)$ which bounds a 2-disc in $S^{n+2} \setminus X(L_i)$ having algebraic intersection $+1$ with L_i. The image of such a curve in πL is well defined up to conjugation, and any element of πL in this conjugacy class is called an i^{th} *meridian*. A *basing* for a link L is a homomorphism $f : F(\mu) \to \pi L$ determined by a choice of one meridian for each component of L. The homology classes of the meridians form a basis for $H_1(X(L);\mathbb{Z}) \cong \mathbb{Z}^\mu$, while $H_{n+1}(X(L);\mathbb{Z}) \cong \mathbb{Z}^{\mu-1}$ and $H_q(X(L);\mathbb{Z}) = 0$ for $1 < q < n+1$, by Alexander duality.

A *Seifert hypersurface* for L is a locally flat, oriented codimension 1 submanifold V of S^{n+2} with (oriented) boundary L. By a standard argument these always exist. (Using obstruction theory it may be shown that the projection of $\partial X \cong \mu S^n \times S^1$ onto S^1 extends to a map $q : X \to S^1$ [**Ke65**]. By transversality (TOP if $n = 2$!) we may assume that $q^{-1}(1)$ is a bicollared, proper codimension 1 submanifold of X. The union $q^{-1}(1) \cup j(S^n \times [0,1])$ is then a Seifert hypersurface for L.) In general there is no canonical choice of Seifert surface. However, there is one important special case. A link L is *fibred* if there is such a map $q : X \to S^1$ which is the projection of a fibre bundle. The exterior is then the mapping torus of a self homeomorphism θ of the fibre F of q. The isotopy class of θ is called the geometric monodromy of the bundle. Such a map q extends to a fibre bundle projection $\hat{q} : M(L) \to S^1$, with fibre $\widehat{F} = F \cup \mu D^{n+1}$, called the *closed fibre* of L. Higher dimensional links with more than one component are never fibred. (See Theorem 5.12.)

An n-link L is *trivial* if it bounds a collection of μ disjoint locally flat 2-discs in S^n. It is *split* if it is isotopic to one which is the union of nonempty sublinks L_1 and L_2 whose images lie in disjoint discs in S^{n+2}, in which case we write $L = L_1 \amalg L_2$, and it is a *boundary* link if it bounds a collection of μ disjoint orientable hypersurfaces in S^{n+2}. Clearly a trivial link is split, and a split link is a boundary link; neither implication can be reversed if $\mu > 1$. Knots are boundary links, and many arguments about knots that depend on Seifert hypersurfaces extend readily to boundary links.

1.2. The link group

If m_i is a meridian for L_i, represented by a simple closed curve on ∂X then $X \cup_{\{m_i\}} \bigcup D^2$ is a deformation retract of $S^{n+2} \setminus \mu\{*\}$ and so is 1-connected. (This is the only point at which we need the ambient homology sphere to be 1-connected.) Hence $\pi = \pi L$ is the normal closure of a set of meridians. (If S is a subset of a group G the *normal closure* $\langle\langle S\rangle\rangle_G$, or just $\langle\langle S\rangle\rangle$, is the smallest normal subgroup of G containing S, and G has *weight* m if $G = \langle\langle S\rangle\rangle_G$ for some subset S with m elements.) By Hopf's theorem, $H_2(\pi;\mathbb{Z})$ is the cokernel of the Hurewicz homomorphism from $\pi_2(X)$ to $H_2(X;\mathbb{Z})$.

If π is the group of a μ-component n-link L in S^{n+2} then

(1) π *finitely presentable;*
(2) π *is of weight* μ*;*
(3) $H_1(\pi;\mathbb{Z}) = \pi/\pi' \cong \mathbb{Z}^\mu$*; and*
(4) *(if* $n > 1$*)* $H_2(\pi;\mathbb{Z}) = 0$.

Conversely, any group satisfying these conditions is the group of an n-link, for every $n \geq 3$. If (4) is replaced by the stronger condition that π has deficiency μ then π is the group of a 2-link, but this stronger condition is not necessary [**Ke65'**]. If subcomplexes of aspherical 2-complexes are aspherical then a higher-dimensional link group π has geometric dimension at most 2 if and only $\mathrm{def}(\pi) = \mu$ (in which case it is a 2-link group).

The group of a classical link has geometric dimension at most 2. Moreover it has a Wirtinger presentation of deficiency 1 and satisfies (1-3), but satisfies (4) if and only if the link splits completely as a union of knots in disjoint balls. This is related to the presence of longitudes, nontrivial elements which commute with meridians. By the Loop Theorem, every 1-link L has a connected Seifert surface whose fundamental group injects into πL. The image is a non-abelian free subgroup of πL unless the Seifert surface is a disc or an annulus. In fact the unknot and the *Hopf link* Ho (2_1^2 in the tables of [**Rol**]) are the only 1-links with solvable link group.

Let L be a μ-component 1-link. An i^{th} *longitudinal curve* for L is a closed curve in $\partial X(L_i)$ which intersects an i^{th} meridianal curve transversely in one point and which is null homologous in $X(L_i)$. The i^{th} meridian and i^{th} longitude of L are the images of such curves in πL, and are well defined up to simultaneous conjugation. If $*$ is a basepoint for $X(L)$ then representatives for the conjugacy classes of the meridians and longitudes may be determined on choosing paths joining each component of $\partial X(L)$ to the basepoint. The *linking number* $\ell_{ij} = \mathrm{lk}(L_i, L_j)$ is the image of the i^{th} longitude in $H_1(X(L_j);\mathbb{Z}) \cong \mathbb{Z}$; in particular, $\ell_{ii} = 0$. It is not hard to show that $\ell_{ij} = \ell_{ji}$.

When chosen as above, the i^{th} longitude and i^{th} meridian commute, since they both come from $\pi_1(\partial X(L_i)) \cong \mathbb{Z}^2$. In classical knot theory ($\mu = 1$) the longitudes play no role in connection with abelian invariants, as they always lie in the second commutator subgroup $(\pi K)''$. In higher dimensions there is no analogue of the longitude in the link group; there are longitudinal n-spheres, but these represent classes in $\pi_n(X(L))$ and so are generally inaccessible to computation. Let $F(r)$ denote the free group on r letters.

THEOREM 1.1. *A* 1*-link L is trivial if and only if πL is free.*

PROOF. The condition is clearly necessary. If πL is free then the i^{th} longitude and i^{th} meridian must lie in a common cyclic group, for each $1 \leq i \leq \mu$, since a free group has no non-cyclic abelian subgroups. On considering the images in $H_1(X(L_i);\mathbb{Z}) \cong \mathbb{Z}$ we conclude that the i^{th} longitude must be null homotopic in $X(L)$. Hence using the Loop Theorem inductively we see that the longitudes bound disjoint discs in S^3. $\square$

In Chapter 11 we shall show that if $n \geq 3$ an n-link L is trivial if and only if πL is freely generated by meridians and the homotopy groups $\pi_j(X(L))$ are all 0, for $2 \leq j \leq [\frac{n+1}{2}]$. These conditions are also necessary when $n = 2$, and if moreover $\mu = 1$ then L is topologically unknotted, by TOP surgery, since $F(1) = \mathbb{Z}$ is "good" [**FQ**]. However, it is not yet known whether such a knot is (PL) trivial, nor whether these conditions characterize triviality of 2-links with $\mu > 1$. (We show instead that such a 2-link is s-concordant to a trivial link. See §5 below re s-concordance.) The condition on meridians cannot be dropped if $n > 1$ and $\mu > 1$ ([**Po71**] – see §7 of Chapter 8 below).

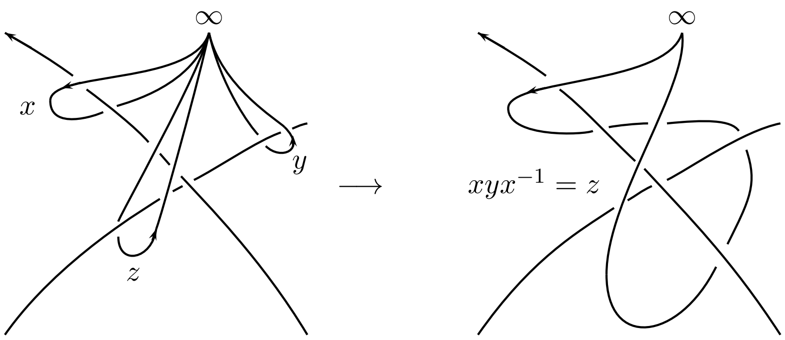

Figure 1.

Any 1-link is ambient isotopic to a link L with image lying strictly above the hyperplane $\mathbb{R}^2 \times \{0\}$ in $\mathbb{R}^3 = S^3 \setminus \{\infty\}$ and for which composition with the projection to $\mathbb{R}^2$ is a local embedding with finitely many double points. Given such a link, the *Wirtinger presentation* is obtained as follows. For each component of the link minus the lower member of each double point pair assign a generator. (This corresponds to a loop coming down on a vertical line from ∞, going once around this component, and returning to ∞.) For the double point corresponding to the arc x crossing over the point separating arcs y and z, there is a relation $xyx^{-1} = z$, where the arcs are oriented as in Figure 1. Thus πL has a presentation

$$\langle x_{i,j} \mid u_{i,j} x_{i,j} u_{i,j}^{-1} = x_{i,j+1}, 1 \leq j \leq j(i), 1 \leq i \leq \mu\rangle,$$

where $u_{i,j} = x_{p,q}^{\pm 1}$ for some p, q and $x_{i,j(i)+1} = x_{i,1}$. It is not hard to see that one of these relations is redundant, and so πL has a presentation of deficiency 1. For an unsplittable link this is best possible.

THEOREM 1.2. *Let L be a 1-link. Then the following are equivalent:*

(1) *L is splittable;*
(2) *πL is a nontrivial free product;*
(3) $\mathrm{def}(\pi L) > 1$.

PROOF. Clearly (1) implies (2) and (2) implies (3). If C is the finite 2-complex determined by a presentation of deficiency ≥ 2 for πL then $\beta_2(\pi L) \leq \beta_2(C) \leq \mu - 2 < \beta_2(X(L)) = \mu - 1$. Hence $\pi_2(X(L)) \neq 0$ and so there is an essential embedded S^2 in $X(L)$, which must split L, by the Sphere Theorem. $\square$

There is not yet a good splitting criterion in higher dimensions.

The centre of a 1-link group is infinite cyclic or trivial, except for the Hopf link, which has group $\mathbb{Z}^2$ [**Mu65**]. The argument of [**HK78**] extends to show that any finitely generated abelian group can be the centre of the group of a boundary 3-link. However, the group of a 2-link with more than one component has no abelian normal subgroup of rank > 0. (See page 42 of [**Hil**]. In all known examples the centre is trivial.)

If G is a group and $x, y \in G$ let $[x, y] = xyx^{-1}y^{-1}$ be the commutator, and let $G' = [G, G]$ be the commutator subgroup. Define the lower central series $\{G_q\}_{q\geq 1}$ for G inductively by $G_1 = G$, $G_2 = G' = [G, G_1]$ and $G_{q+1} = [G, G_q]$. Let $G_\omega = \cap_{q\geq 1} G_q$. A group homomorphism $f : G \to H$ induces homomorphisms $f_q : G/G_q \to H/H_q$, for all $1 \leq q \leq \omega$. It is *homologically 2-connected* if $H_1(f; \mathbb{Z})$ is an isomorphism and $H_2(f;\mathbb{Z})$ is an epimorphism. These notions are related in the following result of Stallings.

THEOREM 1.3. [**St65**] *Let $f : G \to H$ be a homologically 2-connected group homomorphism. Then $f_q : G/G_q \to H/H_q$ is an isomorphism, for all $q \geq 1$. If f is an epimorphism then $f_\omega : G/G_\omega \to H/H_\omega$ is also an isomorphism.*

PROOF. The LHS spectral sequence for G as an extension of G/G_q by G_q gives an exact sequence

$$H_2(G;\mathbb{Z}) \to H_2(G/G_q;\mathbb{Z}) \to G_q/G_{q+1} \to 0,$$

for all $q \geq 1$. Since G/G_{q+1} is a central extension of G/G_q by G_q/G_{q+1} the result follows by the Five-Lemma and induction. □

The group of a link L may be given a *pre-abelian* presentation

$$\langle x_i, y_{ij} \mid [v_{ij}, x_i]y_{ij}, [w_i, x_i], 2 \leq j \leq j(i), 1 \leq i \leq \mu\rangle,$$

where the words x_i and w_i represent i^{th} meridians and longitudes. The images of the x_is generate the nilpotent quotients; for links there is a more precise result due to Milnor.

THEOREM 1.4. [**Mi57**] *Let π be the group of a μ-component 1-link. Then π/π_q has a presentation*

$$\langle x_i, 1 \leq i \leq \mu \mid [w_{i,q}, x_i] = 1,\ 1 \leq i \leq \mu,\ F(\mu)_q\rangle,$$

where x_i and $w_{i,q}$ represent the images in π/π_q of the i^{th} meridian and longitude, respectively. There are words $y_i \in F(\mu)$ such that $\Pi y_i[w_{i,q}, x_i]y_i^{-1} \in F(\mu)_q$.

PROOF. (From [**Tu76**].) Fix a basepoint $* \in X$ and choose arcs α_i from $*$ to $\partial X(L_i)$ which meet only at $*$. Let N be a closed regular neighbourhood of $\cup\alpha_i$ in X and let $D_i = N \cap \partial X(L_i)$. Then $N \cong D^3$, $D_i \cong D^2$ and $\overline{\partial X(L_i) \setminus D_i}$ is a punctured torus. Let $W = \overline{X \setminus N}$ and $G = \pi_1(W, *)$. Since $H_1(W;\mathbb{Z}) \cong \mathbb{Z}^\mu$ and $H_2(W;\mathbb{Z}) = 0$ the inclusion of meridians induces isomorphisms from $F(\mu)/F(\mu)_q$ to G/G_q for all $q \geq 1$, by Theorem 1.3. Since $X = W \cup (\cup_{i=1}^{i=\mu} D_i) \cup N$ we see that $\pi \cong G/\langle\langle \partial D_i \mid 1 \leq i \leq \mu\rangle\rangle$. Clearly ∂D_i represents the commutator of curves in W whose images in π are an i^{th} meridian-longitude pair.

The final assertion follows from the fact that $W \cap N = \overline{\partial N \setminus \partial X}$ is a punctured sphere with boundary $\cup_{i=1}^{i=\mu} \partial D_i$. $\square$

If L is the closure of a pure braid we may take $w_{i,q} = w_i$ for all q, for πL then has a presentation $\langle x_i, 1 \leq i \leq \mu \mid [w_i, x_i], 1 \leq i \leq \mu \rangle$. (See Theorem 2.2 of [**Bir**].)

1.3. Homology boundary links

Classical boundary links were characterized by Smythe, and his result was extended to higher dimensions by Gutiérrez.

Theorem 1.5. [**Sm66, Gu72**] *A μ-component link L is a boundary link if and only if there is an epimorphism $f : \pi L \to F(\mu)$ which carries a set of meridians to a free basis.*

Proof. Suppose that L has a set of disjoint Seifert hypersurfaces U_j, with disjoint product neighbourhoods $N_j \cong U_j \times (-1, 1)$ in X. Let $p : X \to \vee^\mu S^1$ be the map which sends $X \setminus \cup N_j$ to the basepoint and which sends $(n, t) \in N_j$ to $e^{\pi i(t+1)}$ in the j^{th} copy of S^1, for $1 \leq j \leq \mu$. Then $f = \pi_1(p)$ sends a set of meridians for L to the standard basis of $\pi_1(\vee^\mu S^1) \cong F(\mu)$.

Conversely, such a homomorphism $f : \pi L \to F(\mu)$ may be realized by a map $F : X \to \vee^\mu S^1$, since $\vee^\mu S^1$ is aspherical. We may also assume that $F|_{\partial X}$ is standard, since f sends meridians to generators, and that F is transverse to $\mu e^{-\pi i}$, the set of midpoints of the circles. The inverse image $F^{-1}(\mu e^{-\pi i})$ is then a family of disjoint hypersurfaces spanning L. $\square$

An equivalent characterization that is particularly useful in questions of concordance and surgery is that a μ-component n-link L is a boundary link if and only if there is a degree 1 map of pairs from $(X(L), \partial X(L))$ to the exterior of the trivial link which restricts to a homeomorphism on the boundary [**CS80**]. A boundary n-link L is *simple* if there is such a degree 1 map which is $[\frac{n+1}{2}]$-connected. (Thus L is simple if πL is freely generated by meridians and $\pi_j(X(L)) = 0$ for $1 < j < [\frac{n+1}{2}]$, and so every such degree 1 map is $[\frac{n+1}{2}]$-connected.)

If the condition on meridians is dropped L is said to be a *homology boundary link*. Smythe showed also that a classical link L is a homology boundary link if and only if there are μ disjoint oriented codimension 1 submanifolds $U_i \subset X(L)$ with $\partial U_i \subset \partial X(L)$ and such that the image of ∂U_i in $H_n(\partial X(L); \mathbb{Z})$ is homologous to the image of the i^{th} longitudinal n-sphere. This characterization extends to all higher dimensions. In Chapter 2 such *singular Seifert hypersurfaces* are used to construct covering spaces of $X(L)$.

If L is a homology boundary link the epimorphism from $\pi = \pi L$ to $F(\mu)$ satisfies the hypotheses of Stallings' Theorem, and so $\pi/\pi_q \cong F(\mu)/F(\mu)_q$ for all $q \geq 1$. Moreover $\pi/\pi_\omega \cong F(\mu)$, since free groups are residually nilpotent.

If L is a higher dimensional link $H_2(\pi L; \mathbb{Z}) = H_2(F(\mu); \mathbb{Z}) = 0$ and hence a basing f induces isomorphisms on all the nilpotent quotients $F(\mu)/F(\mu)_q \cong \pi L/(\pi L)_q$, and a monomorphism $F(\mu) \to \pi L/(\pi L)_\omega$, by Stallings' Theorem, since in any case $H_1(f; \mathbb{Z})$ is an isomorphism. (In particular, if $\mu \geq 2$ then πL contains a non-abelian free subgroup.) The latter map is an isomorphism if and only if L is a homology boundary link.

An *SHB link* is a sublink of a homology boundary link. Although sublinks of boundary links are clearly boundary links, SHB links need not be homology boundary links. (See Chapter 8 below.)

1.4. $Z/2Z$-boundary links

A μ-component n-link is a *$Z/2Z$-boundary link* if there is an embedding $P : U = \amalg_{i=1}^{i=\mu} U_i \to S^{n+2}$ of μ disjoint $(n+1)$-manifolds U_i such that $L = P|_{\partial U}$. (We do not require that the hypersurfaces are orientable.) The simplest nontrivial example is spanned by two simply linked Möbius bands. (See the link 9^2_{61} of the tables of [**Rol**].)

THEOREM 1.6. *A link L is a $Z/2Z$-boundary link if and only if there is an epimorphism from πL to $*^\mu(Z/2Z)$ which carries some i^{th} meridian to the generator of the i^{th} factor, for all $1 \leq i \leq \mu$.*

PROOF. Let L be a $Z/2Z$-boundary n-link with spanning surfaces U_i, and let ν_i be the normal bundle of U_i in X. Crushing the complement of a disjoint family of open regular neighbourhoods of the U_i to a point collapses X onto the wedge of Thom spaces $\vee T(\nu_i)$. The bundles ν_i are induced from the canonical line bundle η_N over $\mathbb{RP}^N$ (for N large) by classifying maps $n_i : U_i \to \mathbb{RP}^N$, and these maps induce a map $T(n) : \vee T(\nu_i) \to \vee^\mu T(\eta_N)$. Now $T(\eta_N)$ is homeomorphic to $\mathbb{RP}^{N+1}$ by a homeomorphism carrying the zero section to the hyperplane at infinity. Hence we obtain a map from X to $\vee^\mu \mathbb{RP}^\infty = K(*^\mu(Z/2Z), 1)$, which determines a homomorphism $f : \pi L \to *^\mu(Z/2Z)$. The map from X to $\vee^\mu \mathbb{RP}^{N+1}$ carries a loop which meets U_i transversely in one point and is disjoint from U_j for $j \neq i$ to the Thom space of the restriction of η_N over a point, in other words to a curve which meets $\mathbb{RP}^N$ in one point. Thus this curve is essential in $\mathbb{RP}^{N+1}$, and so in $\mathbb{RP}^\infty$. Hence the image of the corresponding meridian generates the i^{th} factor of $*^\mu(Z/2Z)$.

Conversely, such a homomorphism $f : \pi L \to *^\mu(Z/2Z)$ may be realized by a map $F : X \to \vee^\mu \mathbb{RP}^\infty$. Since $X(L)$ has the homotopy type of an $(n+1)$-dimensional complex, we may assume that F maps X to $\vee^\mu \mathbb{RP}^{n+1}$. We may also assume that $F|_{\partial X}$ is standard, since f sends meridians to generators, and that F is transverse to $\amalg^\mu \mathbb{RP}^n$, the disjoint union of the hyperplanes at infinity. Then $F^{-1}(\amalg^\mu \mathbb{RP}^n)$ is a family of disjoint hypersurfaces spanning L. $\square$

The normal bundles for orientable hypersurfaces are trivial, and the universal trivial line bundle $\mathbb{R}$ (with base space a point) has Thom space $T(\mathbb{R}) = S^1 = K(\mathbb{Z}, 1)$. In the characterization of boundary links this plays the part which $T(\eta) = \mathbb{RP}^\infty$ plays here. Finite dimensional approximations $\mathbb{RP}^N$ have been used to emphasize the distinction between the base space ($\mathbb{RP}^N$) and the Thom space ($\mathbb{RP}^{N+1}$) of the universal line bundle.

A similar application of transversality to high dimensional lens spaces shows that L has μ disjoint spanning complexes, the i^{th} being a Z/p_iZ-manifold with no singularities on the boundary, if and only if there is an epimorphism from π to $*_{i=1}^{i=\mu}(Z/p_iZ)$ which carries meridians to generators of the factors.

Smythe's characterization of homology boundary links suggests several possible definitions for the unoriented analogue. The most useful seems to be as follows. A link L is a *$Z/2Z$-homology boundary link* if and only if there are μ disjoint codimension 1 submanifolds $U_i \subset X(L)$ with $\partial U_i \subset \partial X(L)$ and such that the images of ∂U_i and the i^{th} longitudinal n-sphere are homologous in $H_n(\partial X(L); \mathbb{F}_2)$. There is an analogous characterization, which we shall not prove.

THEOREM 1.7. *A link L is a $Z/2Z$-homology boundary link if and only if there is an epimorphism from πL to $*^\mu(Z/2Z)$ such that composition with abelianization carries some i^{th} meridian to the generator of the i^{th} summand of $(Z/2Z)^\mu$, for all $1 \leq i \leq \mu$.* $\square$

1.5. Isotopy, concordance and I-equivalence

A *link type* is an ambient isotopy class of links. A locally flat isotopy is an ambient isotopy, but even an isotopy of 1-links need not be locally flat. For instance, any knot is isotopic to the unknot, but no such isotopy of a nontrivial knot can be ambient. However, a theorem of Rolfsen [**Ro72**] shows that the situation for links is no more complicated.

Two μ-component n-links L and L' are *locally isotopic* if there is an embedding $j : D^{n+2} \to S^{n+2}$ such that $D = L^{-1}(j(D^{n+2}))$ is an n-disc in one component of μS^n and $L|_{(\mu S^n)\setminus D} = L'|_{(\mu S^n)\setminus D}$.

THEOREM. [**Ro72**] *Two n-links L and L' are isotopic if and only if L' may be obtained from L by a finite sequence of local isotopies and an ambient isotopy.* $\square$

In other words, L and L' are isotopic if and only if L' may be obtained from L by successively suppressing or inserting small knots in one component at a time.

An *I-equivalence* between two embeddings $f, g : A \to B$ is an embedding $F : A \times [0,1] \to B \times [0,1]$ such that $F|_{A\times\{0\}} = f$, $F|_{A\times\{1\}} = g$ and $F^{-1}(B \times \{0,1\}) = A \times \{0,1\}$. Here we do *not* assume the embeddings are PL. Clearly isotopy implies I-equivalence. The next result is clear.

THEOREM 1.8. *Let $\mathcal{L}$ be an I-equivalence between μ-component n-links L and L'. Then the inclusions of $X(L)$ and $X(L')$ into $X(\mathcal{L})$ induce isomorphisms on homology.* $\square$

A *concordance* between two μ-component n-links L and L' is a locally flat PL I-equivalence $\mathcal{L}$ between L and L'. Let $C_n(\mu)$ denote the set of concordance classes of such links, and let $C_n = C_n(1)$. The concordance is an *s-concordance* if its exterior is an s-cobordism (*rel* ∂) from $X(L)$ to $X(L')$. In high dimensions this is equivalent to ambient isotopy, by the s-cobordism theorem, but this is not known when $n = 2$. (s-Concordant 1-links are isotopic, by standard 3-manifold topology.) A link L is *null concordant* (or *slice*) if it is concordant to a trivial link. Thus L is a slice link if and only if it extends to a locally flat embedding $C : \mu D^{n+1} \to D^{n+3}$ such that $C^{-1}(S^{n+2}) = \mu S^n$. It is an attractive conjecture that every even-dimensional link is a slice link. This has been verified under additional hypotheses on the link group. In particular, even-dimensional SHB links are slice links [**Co84, De81**].

A μ-component n-link L is *doubly null concordant* or *doubly slice* if there is a trivial μ-component $(n+1)$-link U which is transverse to the equatorial $S^{n+2} \subset S^{n+3}$ and such that U_i meets S^{n+2} in L_i, for $1 \leq i \leq \mu$. Doubly slice links are clearly boundary links, as they are spanned by the intersections of S^{n+2} with μ disjoint $(n+2)$-discs spanning U.

THEOREM. [**Ro85**] *Two n-links L and L' are PL I-equivalent if and only if L' may be obtained from L by a finite sequence of local isotopies and a concordance.* $\square$

A concordance between boundary links L and L' is a *boundary concordance* if it extends to an embedding of disjoint orientable $(n+2)$-manifolds which meet $S^{n+2} \times \{0\}$ and $S^3 \times \{1\}$ transversely in systems of disjoint spanning surfaces for L and L', respectively. There is a parallel notion of $Z/2Z$-boundary concordance.

The process of replacing L_i by a knot K contained in a regular neighbourhood N of L_i (disjoint from the other components) such that K is homologous to L_i in N is called an *elementary F-isotopy* on the i^{th} component of L. (The elementary F-isotopy is *strict* if the maximal abelian covering space of $N \setminus K$ is acyclic.) Two μ-component n-links L and L' are *(strictly) F-isotopic* if they may be related by a sequence of (strict) elementary F-isotopies.

Giffen found a beautiful elementary construction which related F-isotopy and I-equivalence. As his "shift-spinning" construction has never been published, we present it here.

THEOREM 1.9. [**Gi76**] *F-isotopic 1-links are I-equivalent.*

PROOF. Let K be a knot in the interior of $S^1 \times D^2$ which is homologous to the core $S^1 \times \{0\}$. Let Δ be a 2-disc properly embedded in $S^1 \times D^2$, with $\partial\Delta$ essential in $S^1 \times S^1$, and which is transverse to K. Assume that the number $w = |K \cap \Delta|$ is minimal. (This is the geometric winding number of K in $S^1 \times D^2$.) Suppose that $K \subset S^1 \times \rho D^2$, where $0 < \rho < 1$, and split $S^1 \times D^2$ along Δ to obtain a copy of $D^2 \times [0,1]$, with a 1-submanifold L.

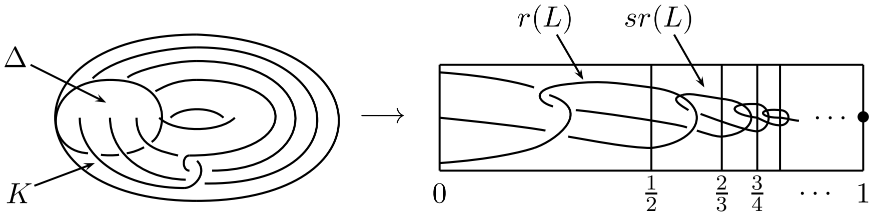

Figure 2.

Let $f : [0,1]^2 \to [\frac{1}{2}, 1]$ be a continuous function such that $f(x,t) = \frac{1}{2}$ if $0 \le x \le (1-t)\rho$ and $f(x,t) = 1$ if $(1 + (1-t)\rho)/2 \le x \le 1$. Let r and s be the self maps of $D^2 \times [0,1]$ given by $r(z,t) = (2^{-t}z, \frac{t}{2})$ and $s(z,t) = (f(|z|,t)z, 1/(2-t))$, for all $(z,t) \in D^2 \times [0,1]$, and let $\kappa = (\cup_{n \ge 0} s^n r(L)) \cup \{(0,1)\}$. Then κ is the union of finitely many arcs in $D^2 \times [0, \frac{1}{2})$ with a "periodic" Fox-Artin arc which tapers towards the core as the interval coordinate increases and converges to $(0,1)$, which is the one wild point, and $s(\kappa) \subset \kappa$. (See Figure 2.)

Now form the mapping torus of the pair $(s, s|_\kappa)$. The result is a wild annulus in $M(s) \cong S^1 \times D^2 \times [0,1]$ with boundary $K \amalg C$, where $C = S^1 \times \{(0,1)\}$ is the core of $S^1 \times D^2 \times \{1\}$. On embedding this solid torus appropriately in $S^3 \times [0,1]$, we obtain an I-equivalence from K to C. (See Figure 3.) This clearly implies the theorem. $\square$

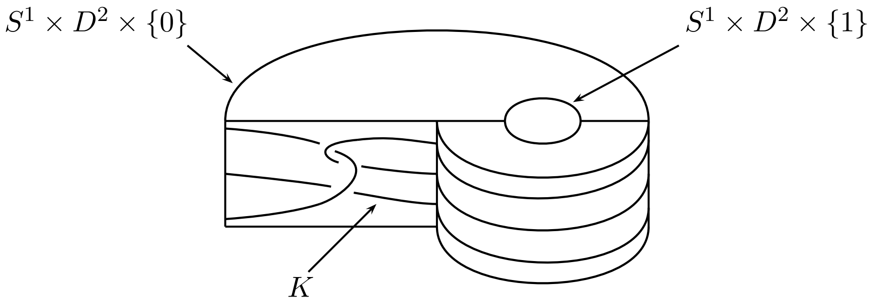

Figure 3.

If K is as depicted in Figure 2 then $K \amalg \partial\Delta$ is I-equivalent to the Hopf link $C \amalg \partial\Delta$, but there is no PL I-equivalence - see §3 of Chapter 8.

1.6. Link homotopy and surgery

Two μ-component n-links L and L' are *link homotopic* if they are connected by a map $H : \mu S^n \times [0,1] \to S^{n+2}$ such that $H|_{\mu S^n \times \{0\}} = L$, $H|_{\mu S^n \times \{1\}} = L'$ and $H(S^n \times \{(i,t)\}) \cap H(S^n \times \{(j,t)\}) = \emptyset$ for all $t \in [0,1]$ and all $1 \leq i < j \leq \mu$. Thus a link homotopy is a homotopy of the maps L and L' such that at no time do the images of distinct spheres intersect (although self intersections are allowed). This is interesting only in the classical case as every higher dimensional link is link homotopic to a trivial link [**BT99, Ba01**]. (However, the link homotopy classification of higher-dimensional "link maps" is nontrivial. See also [**Kai**].)

Theorem 1.10. [**Gi79, Go79**] *Concordant* 1*-links are link homotopic.*

Proof. Let $\mathcal{L}$ be a concordance from L to L'. After an isotopy if necessary, we may assume that $\mathcal{L}$ has an embedded handle decomposition in which the levels at which the handles are added increase with the degree. Since the domain of $\mathcal{L}$ is a product, we may assume that the 0-handles cancel with the 1-handles added below level $\frac{1}{2}$ (and that none are added at this level). It can then be shown that the link at this level is link homotopic to L. Viewed from the other end, the duals of the remaining 1-handles cancel the duals of the 2-handles, and so this link is also link homotopic to L'. See [Ha92] for details. $\square$

It is well known that a 1-knot K may be unknotted by "replacing certain of the undercrossings by overcrossings"; this idea is made precise and extended to links in the following lemma.

Let L be a 1-link and $D \subset S^3$ an oriented 2-disc which meets one component L_i transversely in two points, with opposite orientations, and is otherwise disjoint from L. Let N be an open regular neighbourhood of ∂D in $X(L)$, Then there is an orientation preserving homeomorphism $D^2 \times S^1 \cong S^3 \setminus N$. Fix such a homeomorphism f and define a self homeomorphism h of $S^3 \setminus N$ by $hf(z,s) = f(sz,s)$, for all $s \in S^1$ and $z \in D^2$. The links L and $h(L)$ are then said to be obtained from each other by an *elementary surgery.*

Lemma 1.11. *Let L and L' be 1-links. Then the following are equivalent:*

(1) *L and L' are link homotopic;*
(2) *there is a sequence $L(0) = L, \ldots, L(n) = L'$ of links such that $L(i)$ is obtained from $L(i-1)$ by an elementary surgery, for all $1 \leq i \leq n$.*

Proof. Up to isotopy, any link homotopy may be achieved by a sequence of elementary homotopies, involving the crossing of two arcs in a small ball B. Clearly such an elementary homotopy is equivalent to an elementary surgery. $\square$

The correct choices of "twisting" homeomorphisms h are important here.

The disc D used in such an elementary surgery may be isotoped to avoid a finite set of disjoint discs, and so the surgeries of (2) can be performed simultaneously. Thus the conditions of the lemma imply

Addendum. *If L and L' are link homotopic then there is an embedding T of $mS^1 \times D^2$ in $X(L)$ with core $T|_{mS^1 \times \{0\}}$ a trivial link and a self homeomorphism h of $X(T) = S^3 \setminus int\, T(mS^1 \times D^2)$ such that*

(1) $\mathrm{lk}(T_i, L_j) = 0$, *for each* $1 \leq i \leq m$ *and each component* L_j *of* L*;*
(2) $h(T_i(z, s)) = T_i(s^{e(i)}z, s)$, *for some* $e(i) = \pm 1$ *and for all* $(z, s) \in \partial(S^1 \times D^2)$ *and* $1 \leq i \leq m$*;*
(3) $h \circ L = L'$. $\square$

We shall say that two links L and L' related by such surgeries are *surgery equivalent*. The requirement that the core be trivial ensures that the 3-manifold resulting from the surgeries is again S^3; the linking number condition implies that the surgery tori lift to abelian covers of L. Surgery equivalent links need not be link homotopic, as the cores of the surgery tori may link distinct components of L.

1.7. Ribbon links

A *μ-component n-ribbon* is a map $R : \mu D^{n+1} \to S^{n+2}$ which is locally an embedding and whose only singularities are transverse double points, the double point sets being a disjoint union of discs, and such that $R|_{\mu S^n}$ is an embedding. A μ-component n-link L is a *ribbon* link if there is a ribbon R such that $L = R|_{\mu S^n}$.

If D is a component of the singular set of R then either D is disjoint from $\partial(\mu D^{n+1})$ or $\partial D = D \cap \partial(\mu D^{n+1})$: we call such a component a *slit* or a *throughcut*, respectively. We may assume that each component of the graph with vertices the components of the complement of the throughcuts and edges the throughcuts has at most one vertex of degree > 2, and that the slits are in components corresponding to terminal vertices [**Yj69**]. In our examples below (here and in Chapter 8) each vertex has degree ≤ 2.

An n-link L is a *homotopy ribbon* link if it bounds a properly embedded $(n+1)$-disc in D^{n+3} whose exterior W has a handlebody decomposition consisting of 0-, 1- and 2-handles. The dual decomposition of W relative to $\partial W = M(L)$ has only $(n+1)$- and $(n+2)$-handles, and so the inclusion of M into W is n-connected. (The definition of "homotopically ribbon" for 1-knots given in Problem 4.22 of [**Ki97**] requires only that this latter condition be satisfied.) Every ribbon link is homotopy ribbon and hence slice [**Ht79**]. It is unknown whether every classical slice knot is ribbon, but in higher dimensions there are slice knots which are not even homotopy ribbon.

THEOREM 1.12. *Let L be a μ-component ribbon n-link. Then L is a sublink of a ν-component ribbon n-link $\widehat{L}$ such that $M(\widehat{L}) \cong \sharp^\nu(S^1 \times S^{n+1})$. In particular, $\widehat{L}$ is a homology boundary link and L is an SHB link.*

PROOF. Let R be a ribbon for L, with slits $\{S_i \mid 1 \leq i \leq \sigma\}$. Choose disjoint regular neighbourhoods N_i for each slit in the interior of the corresponding $(n+1)$-disc. Let $\nu = \mu+\sigma$ and let $\widehat{L} = L \cup R|_{\partial N}$, where $N = \cup N_i$. Let $W = D^{n+3} \cup \nu(D^{n+1} \times D^2)$ be the trace of surgery on $\widehat{L}$ (with framing 0 on each component if $n = 1$). Then $M(\widehat{L}) = \partial W$.

Now $\widehat{L}$ may be replaced by a ribbon link with one less singularity, by adding a pushoff of $\widehat{L}|_{\partial N_i}$ to the component of L bounding the $(n+1)$-disc containing N_i. Moreover, if $n = 1$ each component of the new link still has framing 0. Continuing thus, $\widehat{L}$ may be replaced by a ribbon link $\tilde{L}$ for which the only singularities are those corresponding to the components ∂N_i. These may be slipped off the ends of the other components of the new ribbon and so $\tilde{L}$ is trivial. Adding pushoffs of link components to one another corresponds to sliding $(n + 1)$-handles of W across one another, which leaves unchanged the topological type of W. Hence $M(\widehat{L}) \cong M(\tilde{L}) \cong \sharp^\nu(S^1 \times S^{n+1})$, and $\pi\widehat{L}$ maps onto $F(\nu)$. $\square$

If $n = 1$ the homomorphism $\pi\widehat{L} \to F(\nu)$ is an isomorphism if and only if $\widehat{L}$ is trivial, in which case L is also trivial. If $n > 1$ this homomorphism is an isomorphism, but need not carry any set of meridians to a basis. This is so if and only if $\widehat{L}$ is a boundary link. If $n > 2$ it is then trivial and so L is also trivial.

If $M(K) \cong S^1 \times S^2$ then K is the unknot [**Ga87**]. Is there a nontrivial boundary 1-link L such that $M(L) \cong \sharp^\mu(S^1 \times S^2)$?

There is the following partial converse.

THEOREM 1.13. *If L is a ν-component n-link such that $M(L) \cong \sharp^\nu(S^1 \times S^{n+1})$ then L is a homology boundary link and is null concordant. Hence also any sublink of L is nullconcordant.*

Proof. That L is a homology boundary link is clear. Let $U(L)$ be the trace of the surgeries on L, so $\partial U(L) = S^{n+2} \amalg \sharp^{\nu}(S^1 \times S^{n+1})$. The $(n+3)$-manifold $D(L) = U(L) \cup \natural^{\nu}(D^2 \times S^{n+1})$ is contractible and has boundary S^{n+2}, and so is homeomorphic to D^{n+3}. The link L clearly bounds ν disjoint $(n+1)$-discs in $D(L)$. $\square$

This argument rests on the TOP 4-dimensional Poincaré conjecture when $n = 1$. This dependance can be partially sidestepped. A relatively simple argument using the TOP Schoenflies Theorem shows that if the result of 0-framed surgery on the first ρ components of L is $\sharp^{\rho}(S^1 \times S^2)$, for each $\rho \leq \nu$, then L is TOP null concordant [**Ru80**]. Is every slice link an SHB link?

Theorem 1.14. *A finitely presentable group G is the group of a μ-component sublink of a ν-component n-link L with group $\pi L \cong F(\nu)$ (for some ν and any $n \geq 2$) if and only if it has deficiency μ and weight μ.*

Proof. The conditions are clearly necessary. Suppose that G has a presentation $\langle x_i, 1 \leq i \leq \nu \mid r_j, 1 \leq j \leq \nu - \mu \rangle$ and that the images of $s_1, \dots, s_\mu \in F(\nu)$ in G generate G normally. The words r_j and s_k may be represented by disjoint embeddings ρ_j and σ_k of $S^1 \times D^{n+1}$ in $\sharp^{\nu}(S^1 \times S^{n+1})$. If surgery is performed on all the ρ_j and σ_k the resulting manifold is a homotopy $(n+2)$-sphere, and $Y = \sharp^{\nu}(S^1 \times S^{n+2}) \setminus \cup_{j=1}^{j=\nu-\mu} \rho_j(S^1 \times D^{n+1}) \setminus \cup_{k=1}^{k=\mu} \sigma_k(S^1 \times D^{n+1})$ is the complement of a ν-component n-link in this homotopy sphere, with link group $F(\nu)$. Therefore if surgery is performed on the ρ_j only, the space $Y \cup (\nu - \mu)(D^2 \times S^n)$ is the complement of a μ-component sublink with group G. $\square$

When $n = 2$ the resulting link is merely TOP locally flat.

Let $G(i,j)$ be the group with presentation $\langle x, y, z \mid x[z^i, x][z^j, y] \rangle$. Then the generators y and z determine a homomorphism from $F(2)$ to $G(i,j)$ which induces isomorphisms on all nilpotent quotients, and $G(i,j)_\omega = 1$, but $G(i,j)$ is not free unless $ij = 0$ [**Ba69**]. As $G(i,j)$ is the normal closure of the images of y and z, and the presentation $\langle x, y, z \mid x[z^i, x][z^j, y], y, z \rangle$ of the trivial group is AC-equivalent to the empty presentation, this group can be realized by a PL locally flat link in S^4. The higher dimensional links constructed from this presentation as in Theorem 1.14 are sublinks of 3-component homology boundary links but are not homology boundary links. See Chapter 8 for examples of ribbon 1-links which are not homology boundary links (although they are SHB links, by Theorem 1.12).

An immediate consequence of Theorems 1.12 and 1.14 is that if $n > 1$ the group of a μ-component ribbon n-link has a presentation of deficiency μ. Thus the 2-twist spin of the trefoil knot is slice [**Ke65**], but not ribbon [**Yj64**]. (See also Theorems 4.3 and 4.6 below.)

We may use a ribbon map R extending a 1-link L to construct a concordance $\mathcal{C}$ from L to a trivial link U, such that the only singularities of the composite $f = pr_2 \circ \mathcal{C} : \mu D^2 \to S^3 \times [0,1] \to [0,1]$ are saddle points corresponding to the throughcuts. Capping off the components of U in D^4 and doubling gives a μ-component 2-link DR. The *ribbon group* of R is $H(R) = \pi DR$. Each throughcut T determines a conjugacy class $g(T) \subset \pi L$ represented by the oriented boundary of a small disc neighbourhood in R of the corresponding slit. (The standard orientation on D^2 induces an orientation on this neighbourhood via the local homeomorphism R.) Let TC be the normal subgroup determined by the throughcuts of R.

THEOREM 1.15. *Let L be a ribbon 1-link with group $\pi = \pi L$ and R a ribbon map extending L. Then $H(R) = \pi L/TC$ and has a Wirtinger presentation of deficiency μ. The longitudes of L are in the normal subgroup TC, which is contained in π_ω. Hence the projection of π onto π/π_ω factors through $H(R)$.*

PROOF. It is clear from the description of the construction in the above paragraph that the inclusion of $X(L)$ into $X(\mathcal{C})$ induces an isomorphism from $\pi L/TC$ to $\pi_1(X(\mathcal{C}))$. Hence $\pi DR \cong \pi L/TC$, by Van Kampen's Theorem.

Each longitude is represented up to conjugacy by a curve on and near the boundary of the corresponding disc, which is clearly homotopic to a product of conjugates of loops about the slits on the disc. Thus the longitudes of L are in TC.

We may show by induction on q that $TC \leq \pi_q$ for all $q \geq 1$. This is clear for $q = 1$. If $TC \leq \pi_n$ the image of each conjugacy class $g(T)$

in π/π_{n+1} is a central element $g_{n+1}(T)$. If T and T' are adjacent representatives of $g(T)$ and $g(T')$ differ only by commutators involving loops around slits in the segment of the ribbon between T and T', and so $g_{n+1}(T) = g_{n+1}(T')$. Moving along the ribbon, we find that $g_{n+1}(T) = 1$, and so $g(T) \subset \pi_{n+1}$, for all T. Thus $TC \leq \pi_\omega$.

We may choose a generic projection of the ribbon with no triple points. The Wirtinger generators of the link group corresponding to the subarcs of the link which "lie under" a segment of the ribbon may be deleted, and the two associated relations replaced by one stating that either adjacent generator is conjugate to the other by a loop around the overlying segment.

Any loop about a segment of the ribbon dies in $H(R)$, for the only obstructions to deforming it onto a loop around the throughcut at an end of the segment are elements in the conjugacy classes of the throughcuts between the loop and that end. Hence the remaining generators corresponding to subarcs of the boundary of a given component of the complement of the throughcuts coalesce in $H(R)$. Conversely the presentation obtained from the Wirtinger presentation by making such deletions and identifications is as claimed, and presents a group in which the image of each $g(T)$ is trivial, for the image of $g(T)$ is trivial if and only if the pair of generators corresponding to arcs meeting the projection of T are identified. Thus the group is exactly $H(R)$. $\square$

Conversely, any such presentation can be realized by some ribbon map $R : \mu D^2 \to S^3$. A similar argument shows that a group G is the group of a μ-component ribbon n-link for any $n \geq 2$ if and only if G has a Wirtinger presentation of deficiency μ and $G/G' \cong Z^\mu$. The generators correspond to meridianal loops transverse to the components of the complements of the throughcuts, and there is one relation for each throughcut. Thus although the group of an unsplittable 1-link has no presentation of deficiency > 1, the groups of ribbon links have quotients with deficiency μ. (See [**Si80**] for some connections between Wirtinger presentations and homology.)

Much of this theorem can be deduced from Theorem 1.12, by arguing as in Theorem 1.14 to adjoin $\nu - \mu$ relations to $F(\nu)$. In general, π/π_ω, $H(R)$ and $\pi/\langle\langle longitudes\rangle\rangle$ are distinct groups, even when $\mu = 1$. (Consider the square knot $3_1\sharp - 3_1$.) If one ribbon R_1 is obtained from another R_2 by knotting the ribbon or inserting full twists then $H(R_1) = H(R_2)$, as such operations do not change the pattern of the singularities.

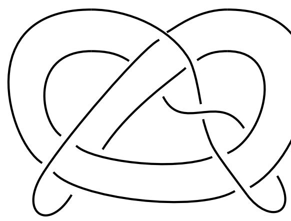

Figure 4.

Let R be the ribbon disc of Figure 4 and let $K = \partial R$. Then $H(R)$ has the presentation $\langle a, b, c, d \mid aca^{-1} = d, dad^{-1} = b, dcd^{-1} = b\rangle$, and so $H(R) \cong Z$. It can be shown that there is a homomorphism from πK to $\mathrm{SL}(2, \mathbb{F}_7)$ with non-abelian image. The corresponding ribbon 2-knot is trivial, and so K is a nontrivial slice of a trivial 2-knot [**Yn70**]. (It is in fact the Kinoshita-Terasaka 11-crossing knot with Alexander polynomial 1 [**KT57**]. See Figure 3(a) of [**Wa94**].)

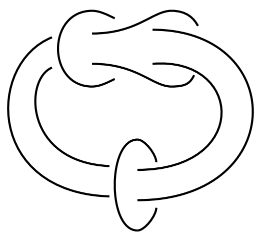

Figure 5.

Similarly, $\langle a, w, x, y, z \mid axa^{-1} = y, wyw^{-1} = z, zwz^{-1} = x\rangle$ leads to a 2-component homology boundary link which is a slice of a 2-link with group $F(2)$. (See Figure 5.) This 1-link is not a boundary link ([**Cr71**] - see also §7 of Chapter 8 below). Hence the 2-link with group $F(2)$ of which it is a slice is not one either, illustrating the result of Poenaru [**Po71**].

The above results may be usefully extended by the notion of fusion. A fusion band for an n-link L is a pair $\beta = (b, u)$, where $b : [0,1] \to S^{n+2}$ is an embedded arc with endpoints on L and u is a unit normal vector field along b such that $u|_{\{0,1\}}$ is normal to L, and such that the orientations are compatible. These data determine a band $B : [0,1] \times D^n \to S^{n+2}$ which may be used to form the connected sum of two of the components of L. The resulting $(\mu - 1)$-component link is called the *fusion* of L (along β). The *strong fusion* is the μ-component link obtained by adjoining to the fusion the boundary of an $(n+1)$-disc transverse to b.

When $n > 1$ the normal vector field u is unique up to isotopy, but in the classical case any two choices differ by an element of $\pi_1(SO(2)) \cong Z$, and so it determines the twisting of the band B.

Ribbon links are fusions of trivial links. The argument of Theorem 1.12 can be extended to show that a fusion of a boundary link is an SHB link [**Co87**]. Moreover any SHB link is concordant to a fused boundary link [**CL91**]. If a strong fusion of a link is an homology boundary link then so was the original link [**Ka93**].

Concordance of 1-links is generated as an equivalence relation by fusions $L \to L +_\beta \partial R$, where $R : D^2 \to X(L)$ is a ribbon map with image disjoint from L and where $+_\beta$ denotes fusion along a band β from some component of L to ∂R [**Tr69**].

1.8. Link-symmetric groups

Let $r_n : S^n \to S^n$ be the map which changes the sign of the last coordinate. Then every (PL) homeomorphism of S^n is isotopic to id_{S^n} or r_n, depending on whether it preserves or reverses the orientation. An n-knot K is *invertible, +amphicheiral* or *−amphicheiral* if it is ambient isotopic to $K\rho = K \circ r_n$, $rK = r_{n+2} \circ K$ or $-K = rK\rho$, respectively. If a knot has two of these properties then it has all three. Conway has suggested the alternative terminology *reversible, obversible, inversible*, as $-K$ represents the inverse of the class of K in the knot concordance group [**Co70**].

These notions have been extended to links as follows. The extended symmetric group on μ symbols is the semidirect product

$(Z/2Z)^\mu \rtimes S_\mu$, where S_μ acts on the normal subgroup $(Z/2Z)^\mu$ by permutation of the symbols. Then the *link-symmetric group of degree* μ is $LS(\mu) = (Z/2Z) \times ((Z/2Z)^\mu \rtimes S_\mu)$. A μ-component n-link L *admits* $\gamma = (\epsilon_0, \ldots, \epsilon_\mu, \sigma) \in LS(\mu)$ if L is ambient isotopic to $\gamma L = r_{n+2}^{\epsilon_0} \circ L \circ (\amalg r_n^{\epsilon_i}) \circ \tilde{\sigma}$, (where $\tilde{\sigma}$ permutes the components, and where $Z/2Z$ is identified with $\{\pm 1\}$). A link L is *invertible* if it admits $(1, -1, \ldots, -1, id)$, ϵ-*amphicheiral* if it admits $(-1, \epsilon, \ldots, \epsilon, id)$, and *interchangeable* if it admits γ with image $\sigma \in S_\mu$ not the identity permutation.

The group of symmetries of a link L is the subgroup $\Sigma(L) \leq LS(\mu)$ consisting of the elements admitted by L. This group depends only on the ambient isotopy type of L. Changing the orientation of one component or the order of the components replaces $\Sigma(L)$ by a conjugate subgroup. (See [**Wh69**].)

1.9. Link composition

Let L be a μ-component n-link, and choose homeomorphisms ϕ_i from $S^n \times D^2$ onto disjoint regular neighbourhoods of the components L_i, for $1 \leq i \leq \mu$. If $n = 1$ assume that the circles $\phi_i(S^1 \times \{d\})$ corresponding to different values of $d \in D^2$ have mutual linking number 0. Let $K(i)$ be a ν_i-component link in $S^n \times D^2$ and let $K(i)^+$ be the $(\nu_i + 1)$-component n-link in S^{n+2} obtained by adjoining $S^n \times \{1\} \subset \partial(S^n \times D^2)$. Then the *composite* of L with $\mathcal{K} = \{K(i)\}_{1 \leq i \leq \mu}$ is $L \circ \mathcal{K} = \bigcup_{1 \leq i \leq \mu} \phi_i \circ K(i)$. (This link has $\nu = \Sigma \nu_i$ components.) As $X(L \circ \mathcal{K}) \cong X(L) \cup \bigcup_{1 \leq i \leq \mu} X(K(i)^+)$, this construction is well adapted to applications of the Van Kampen and Mayer-Vietoris Theorems. If $K(i) = S^n \times \{0\}$ for all i then $L \circ \mathcal{K} = L$. We shall assume henceforth that $K(i) = S^n \times \{0\}$ for $i \neq j$.

If $\mu = 1 = \nu$ then $L \circ \mathcal{K}$ is a *satellite* of L; in particular, if $K = K(1)$ has geometric winding number 1 in $S^n \times D^2$ (i.e., intersects some disc $\{s\} \times D^2$ transversely in one point) this gives the *sum* $K \sharp L$ of the knots K and L.

If $\nu_j = 1$ and $K(j)$ is homologous to $S^1 \times \{0\}$ in $S^1 \times D^2$ then $L \circ \mathcal{K}$ is obtained from L by an elementary F-isotopy on the j^{th}

component. If L' is obtained from L by an elementary F-isotopy then $X(L)$ is a retract of $X(L')$, since $\partial X(h)$ is a retract of $X(h)$ for any 2-component link h with linking number 1.

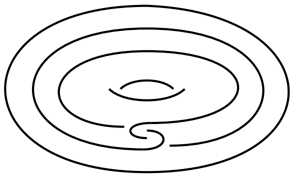

Figure 6. $\theta \circ Wh_2$

Let $Wh : 2S^1 \to S^3$ be the *Whitehead link* (5_1^2 in the tables of [**Rol**]), and let $\theta : X(Wh_1) \to S^1 \times D^2$ be a homeomorphism such that $\theta(\phi_1(u, v)) = (v, u)$, for all $u, v \in S^1$. If $K(j) = \theta \circ Wh_2$ then $L \circ \mathcal{K}$ is obtained from L by *Whitehead doubling* the j^{th} component. (See Figure 6.) When $\mu = 1$ this is an untwisted double of the knot L. Since each component of the Whitehead link bounds a punctured torus in the complement of the other component, Whitehead doubling every component of a link gives a boundary link.

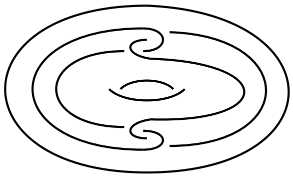

Figure 7. $\theta \circ Bo_{2,3}$

Similarly, if $Bo : 3S^1 \to S^3$ is the *Borromean ring* link (6_2^3 in the tables of [**Rol**]) let $\theta : X(Bo_1) \to S^1 \times D^2$ be a homeomorphism such that $\theta(\phi_1(u, v)) = (v, u)$, for all $u, v \in S^1$. If $Bo_{2,3}$ is the union of the second and third components of Bo and $K(j) = \theta \circ Bo_{2,3}$ then $L \circ \mathcal{K}$ is obtained from L by *Bing doubling* the j^{th} component. (See Figure 7.) In the latter two cases there are further mild ambiguities, related to the definition of the Whitehead link, etc.