

A NEW EVANS FUNCTION FOR QUASI-PERIODIC SOLUTIONS OF THE LINEARISED SINE-GORDON EQUATION

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ABSTRACT

We construct a new Evans function for quasi-periodic solutions to the linearisation of the sine-Gordon equation about a periodic travelling wave. This Evans function is written in terms of fundamental solutions to a Hill's equation. Applying the Evans-Krein function theory of [KM2014] to our Evans function, we provide a new method for computing the Krein signatures of simple characteristic values of the linearised sine-Gordon equation. By varying the Floquet exponent parametrising the quasi-periodic solutions, we compute the linearised spectra of periodic travelling wave solutions of the sine-Gordon equation and track dynamical Hamiltonian-Hopf bifurcations via the Krein signature. Finally, we show that our new Evans function can be readily applied to the general case of the nonlinear Klein-Gordon equation with a non-periodic potential.

1 Introduction

We consider the sine-Gordon equation,

$$u_{tt} - u_{xx} + \sin(u) = 0, \tag{1}$$

where $u(x, t) : \mathbb{R} \times [0, +\infty) \rightarrow \mathbb{R}$. This equation has been used in the modelling of a number of different physical and biological systems. For example, equation (1) models the electrodynamics of a long Josephson junction arising in the theory of superconductors [BP1982, DDKS2012]. Solitary wave solutions to equation (1) have been used to describe the dynamics of DNA as it interacts with RNA-polymerase [DG2011]. More recent research has seen sine-Gordon solitons used as scalar gravitational fields in the theory of general relativity, the solutions of which are solitonic stars and black holes [CFMT2019]. Equation (1) can also be derived from classical mechanics applied to a mechanical transmission line in which pendula are coupled to their nearest neighbours by springs obeying Hooke's law [Kno2000]. See [BEMS1971] for an extensive list of applications of the sine-Gordon equation.

In this paper, we focus on the problem of spectral stability of periodic solutions to the sine-Gordon equation. In [Sco1969], Scott correctly attributed spectral stability and instability to various types of periodic wavetrains, however this was rigorously proved only recently in [JMMP2013] in which the authors related the sine-Gordon equation to a Hill's equation using Floquet theory and known results on Hill's equation [JMMP2013, MW2013]. There are a number of approaches taken in order to determine the stability of travelling waves; in [SS2012], Stanislavova and Stefanov developed a stability index for travelling wave solutions to second order in time PDEs (including the Klein-Gordon-Zakharov system). The Evans function is another tool that has been used to investigate spectral stability of travelling waves. Evans used this eponymous function while studying the equations governing electrical pulses in nerve axons [Eva1972]. Jones was the first to coin the term *Evans function* in [Jones1984], where he used this function to prove the stability of travelling wave solutions to the Fitzhugh-Nagumo equations close to a singular limit of the equations. Alexander, Gardner and Jones in [AGJ1990], and Gardner in [Gar1997], developed much of the theory and approach we use in this paper to construct and apply a periodic Evans function to problems of spectral stability. Evans function theory has been applied to a general nonlinear Klein-Gordon equation [LLM2011], and also more specifically to a perturbed sine-Gordon equation [DDGV2003]. In the previous cases, the Evans functions used in the analyses of nonlinear Klein-Gordon type equations comes from a classical construction of the Evans function based on the exterior product of solutions. In [JMMP2014], the authors instead use Floquet theory to construct a periodic Evans function

for quasi-periodic solutions of the nonlinear Klein-Gordon equation. This paper combines results for Hill's equation in [JMMP2013] and the periodic Evans function from [JMMP2014] in order to construct a new Evans function for quasi-periodic solutions to the linearised sine-Gordon equation. We show how this new Evans function can be used to calculate the Krein signature - a stability index that is then used to detect Hamiltonian-Hopf bifurcations. Our new Evans function leverages the simplicity of Hill's equation, which translates into a more elegant calculation of Krein signatures when compared to previous results [JMMP2014] and allows for tracking of Hamiltonian-Hopf bifurcations in terms of the Floquet exponent. This method distinguishes itself from the stationary methods of [JMMP2013, MM2015] which instead calculate the zeroes of bespoke functions in order to detect the regions of the spectrum that exhibit Hamiltonian-Hopf instabilities.

1.1 Set-up of the eigenvalue problem

In travelling wave coordinates $z = x - ct$, $\tau = t$, a travelling wave solution $\hat{u}(z, \tau)$ to equation (1) satisfies:

$$(c^2 - 1)\hat{u}_{zz} - 2c\hat{u}_{z\tau} + \hat{u}_{\tau\tau} + \sin(\hat{u}) = 0. \quad (2)$$

A standing wave solution $\hat{U}(z)$ in travelling wave coordinates will be independent of τ and hence satisfies:

$$(c^2 - 1)\hat{U}_{zz} + \sin(\hat{U}) = 0, \quad c \neq 1. \quad (3)$$

Integrating equation (3) with respect to z yields:

$$\frac{1}{2}(c^2 - 1)\hat{U}_z^2 + 1 - \cos(\hat{U}) = E,$$

where E is a constant of integration which we interpret as the total energy of the system. We follow the results established in [JMMP2013] and [MM2015], where $\hat{U}(z)$ is assumed to be periodic modulo 2π . We denote the fundamental period of \hat{U} as T , so that $\hat{U}(z + T) = \hat{U}(z) \pmod{2\pi}$. Linearising equation (2) about this periodic standing wave solution, we write $\hat{u} = \hat{U} + \epsilon p(z)e^{\lambda\tau}$ with $\epsilon \ll 1$ and equate $O(\epsilon)$ terms, which yields the spectral problem:

$$(c^2 - 1)p'' - 2c\lambda p' + \left(\lambda^2 + \cos(\hat{U})\right)p = 0. \quad (4)$$

The (Floquet) spectrum σ consists of all λ such that $p: \mathbb{R} \rightarrow \mathbb{C}$ is bounded. In [JMMP2014, Proposition 3.9], Jones et. al. prove that σ has Hamiltonian symmetry $\sigma = \sigma^* = -\sigma = -\sigma^*$. Consequently, any $\lambda \in \sigma$ with non-zero real part implies an unstable eigenvalue. We make the substitution $\lambda = i\zeta$ and henceforth use the phrase *linearised sine-Gordon equation* to mean:

$$p'' - \frac{2ic\zeta}{c^2 - 1}p' + \left(-\frac{\zeta^2}{c^2 - 1} + \frac{\cos(\hat{U})}{c^2 - 1}\right)p = 0. \quad (5)$$

An equivalent condition for $\zeta \in \sigma$ is the existence of a non-trivial solution $p(z)$ which can be written in Bloch form à la [MM2015]:

$$p(z) = e^{-i\theta z/T}P(z),$$

where $\theta \in \mathbb{R}$ is called the *Floquet exponent*, and $P(z) = P(z + T)$. The solutions $p(z)$ are quasi-periodic, since:

$$p(z) = e^{i\theta}p(z + T). \quad (6)$$

We now reframe the spectral problem in equation (5) using the formalism of operator pencils, in particular drawing on the work of Markus [Mar1988] and Kato [Kat1976]. Given linear operators L_0, L_1, \dots, L_n with $L_i: X \rightarrow Y$ for Banach Spaces X and Y , then

$$\mathcal{L}(\lambda) := L_n\lambda^n + L_{n-1}\lambda^{n-1} + \dots + L_0$$

defines a polynomial operator pencil of degree n depending on the complex variable λ in an open set $\lambda \in S \subset \mathbb{C}$ [Mar1988, §12.1]. Equation (5) can be rewritten in the form $\mathcal{L}_{SG}(\zeta)p = 0$ where

$$\mathcal{L}_{SG}(\zeta) := -\frac{\zeta^2}{c^2 - 1}\mathbb{I} - \frac{2ic\zeta}{c^2 - 1}\partial_z + \left(\partial_z^2 + \frac{\cos(\hat{U})}{c^2 - 1}\right) \quad (7)$$

is a quadratic operator pencil. We narrow our focus to operator pencils which are holomorphic families of type (A). Such pencils $\mathcal{L}(\lambda)$ have a domain B that is independent of the spectral variable λ , and for each $u \in B$, $\mathcal{L}(\lambda)u$ is a

holomorphic function of λ . Kollar and Miller note in [KM2014] that \mathcal{L}_{SG} defined in equation (7) is a holomorphic family of type (A) with compact resolvent. Moreover, when $\zeta \in \mathbb{R}$, $\mathcal{L}_{SG}(\zeta)$ is self-adjoint. These qualities are necessary for Krein signatures to be well-defined [KM2014, Theorem 3.3].

As in [KM2014], for $\mathcal{L}(\lambda)$ a holomorphic family of type (A), we say that $\lambda_0 \in \mathbb{C}$ is a *characteristic value* if there exists a *characteristic vector* $u \neq 0$ such that $\mathcal{L}(\lambda_0)u = 0$. The *geometric multiplicity* of λ_0 is $\dim(\ker(\mathcal{L}(\lambda_0)))$. If we also assume that $\mathcal{L}(\lambda)$ is self-adjoint and has compact resolvent, then the eigenvalue problem

$$\mathcal{L}(\lambda)u(\lambda) = \mu(\lambda)u(\lambda)$$

can be solved for analytic functions $u(\lambda), \mu(\lambda)$ at $\lambda = \lambda_0$. In particular, if $\dim(\ker(\mathcal{L}(\lambda_0))) = k$, then there exist exactly k analytic functions $\mu_1(\lambda), \mu_2(\lambda), \dots, \mu_k(\lambda)$ called *eigenvalue branches*, which vanish at $\lambda = \lambda_0$. If we let m_i be the order of vanishing of the eigenvalue branch $\mu_i(\lambda)$ at λ_0 for $1 \leq i \leq k$, that is:

$$\mu_i(\lambda_0) = \mu_i'(\lambda_0) = \dots = \mu_i^{(m_i-1)}(\lambda_0) = 0, \quad \mu_i^{(m_i)}(\lambda_0) \neq 0,$$

then the algebraic multiplicity M of a characteristic value $\lambda_0 \in \mathbb{R}$ is given by:

$$M = \sum_{i=1}^k m_i.$$

Our main objective in this paper is to construct a new Evans function $D(\zeta; \theta)$, $D: \mathbb{C} \times [0, 2\pi) \rightarrow \mathbb{C}$ whose roots coincide exactly with the isolated characteristic values ζ_0 of $\mathcal{L}_{SG}(\zeta)$ for a given θ , with $D(\zeta_0; \theta)$ vanishing to the order of algebraic multiplicity of ζ_0 . Our secondary objective is to use this Evans function to calculate the Krein signatures of simple characteristic values of $\mathcal{L}_{SG}(\zeta)$. A characteristic value is called *simple* when its algebraic and geometric multiplicities are both 1. Kollar and Miller provide a full treatment of Krein signature theory in [KM2014], however the procedure for calculating the Krein signature for simple, isolated characteristic values is straightforward given the pencils in this paper. In particular, for an isolated characteristic value λ_0 and its single eigenvalue branch $\mu(\lambda)$ which vanishes to order 1 at $\lambda = \lambda_0$, the graphical Krein signature can be calculated from [KM2014, Definition 3.5]:

$$\kappa(\lambda_0) = \text{sign} \left(\frac{d}{d\lambda} \mu(\lambda_0) \right). \quad (8)$$

2 Spectrum of the linearised sine-Gordon equation

We begin by surveying the results of [JMMP2013] and [MM2015]. For each $\zeta \in \sigma$, we define the *principal fundamental solution matrix* as:

$$\mathbb{F}(z; \zeta) := \begin{pmatrix} p_1(z; \zeta) & p_2(z; \zeta) \\ p_1'(z; \zeta) & p_2'(z; \zeta) \end{pmatrix}, \quad (9)$$

where $p_1(z; \zeta)$ and $p_2(z; \zeta)$ are the unique solutions of equation (5) satisfying the initial conditions:

$$\mathbb{F}(0; \zeta) = \begin{pmatrix} p_1(0; \zeta) & p_2(0; \zeta) \\ p_1'(0; \zeta) & p_2'(0; \zeta) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (10)$$

We may write any solution $p(z; \zeta)$ as a superposition of the fundamental solutions:

$$\begin{pmatrix} p(z; \zeta) \\ p'(z; \zeta) \end{pmatrix} = \mathbb{F}(z; \zeta) \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}, \quad C_1, C_2 \in \mathbb{C}.$$

Now if $\zeta \in \sigma$, we can use equation (6) with $z = 0$:

$$\mathbb{F}(0; \zeta) \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = e^{i\theta} \mathbb{F}(T; \zeta) \begin{pmatrix} C_1 \\ C_2 \end{pmatrix},$$

from which we have:

$$\mathbb{F}(T; \zeta) \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = e^{-i\theta} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}. \quad (11)$$

The matrix $\mathbb{F}(T; \zeta)$ is called the *monodromy matrix*, and its two eigenvalues ρ_1, ρ_2 are referred to as *Floquet multipliers* [JMMP2014]. From equation (11), we identify $\rho_1 = e^{-i\theta}$, and we apply Abel's identity and the initial conditions in equation (10) to equation (5) to find that:

$$\det(\mathbb{F}(z; \zeta)) = \exp \left(\frac{2ic\zeta}{c^2 - 1} z \right).$$

When $z = T$, we have:

$$\rho_1 \rho_2 = \det(\mathbb{F}(T; \zeta)) = \exp\left(\frac{2icT\zeta}{c^2 - 1}\right),$$

from which we conclude that:

$$\rho_2 = \exp\left(\frac{2icT\zeta}{c^2 - 1} + i\theta\right). \quad (12)$$

We seek to make use of the results in [JMMP2013] which connect the Floquet multipliers ρ_1, ρ_2 of equation (5) to Hill's equation in equation (14). Making the exponential transform [JMMP2013]:

$$q(z) = p(z) \exp\left(\frac{-ic\zeta}{c^2 - 1} z\right), \quad (13)$$

we transform equation (5) into the following form of Hill's equation:

$$q'' + \left(\frac{\zeta^2}{(c^2 - 1)^2} + \frac{\cos(\hat{U})}{c^2 - 1}\right) q = 0. \quad (14)$$

We define the principal fundamental solution matrix for equation (14) as:

$$\mathbb{H}(z; \zeta) := \begin{pmatrix} q_1(z; \zeta) & q_2(z; \zeta) \\ q_1'(z; \zeta) & q_2'(z; \zeta) \end{pmatrix}, \quad \mathbb{H}(0; \zeta) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (15)$$

We write $\mathbb{H}(T; \zeta)$ for the monodromy matrix of equation (14), and its two Floquet multipliers are denoted by η_1, η_2 . In [JMMP2013, Lemma 3.1], Jones et al. prove that ρ_1, ρ_2 are the Floquet multipliers of equation (5) if and only if

$$\eta_1 = \exp\frac{-ic\zeta T}{c^2 - 1} \rho_1, \quad \eta_2 = \exp\frac{-ic\zeta T}{c^2 - 1} \rho_2 \quad (16)$$

are the Floquet multipliers of equation (14). Consequently, the function:

$$D_2(\zeta; \theta) := \text{tr}(\mathbb{H}(T; \zeta)) - (\eta_1 + \eta_2) \quad (17)$$

vanishes, at least to first order, at precisely the values $\zeta = \zeta_0$ which are characteristic values of the linearised sine-Gordon equation for a given Floquet exponent θ . We can calculate η_1 and η_2 directly using equations (12) and (16):

$$\eta_1 = \exp\left(-\frac{ic\zeta T}{c^2 - 1} - i\theta\right), \quad \eta_2 = \exp\left(\frac{ic\zeta T}{c^2 - 1} + i\theta\right),$$

which indeed obeys the condition that $\eta_1 \eta_2 = \det(\mathbb{H}(T; \zeta)) = 1$ by Abel's identity. Equation (17) then simplifies to:

$$D_2(\zeta; \theta) := \text{tr}(\mathbb{H}(T; \zeta)) - 2 \cos\left(\frac{c\zeta T}{c^2 - 1} + \theta\right) \quad (18)$$

We pause here to include a relevant result from [JMMP2014, Definition 3.5]. The authors provide an Evans function for quasi-periodic solutions $p(z) = e^{i\theta} p(z + T)$ to the linearised sine-Gordon equation (5):

$$D_1(\zeta; \theta) := \det(\mathbb{F}(T; \zeta) - e^{-i\theta} \mathbb{I}). \quad (19)$$

Theorem 1. *The function*

$$D_2(\zeta; \theta) := \text{tr}(\mathbb{H}(T; \zeta)) - 2 \cos\left(\frac{c\zeta T}{c^2 - 1} + \theta\right)$$

is an Evans function for characteristic values of $\mathcal{L}_{SG}(\zeta)$ defined in equation (7) parametrised by the Floquet exponent θ .

Proof. Magnus and Winkler proved that $\text{tr}(\mathbb{H}(T; \zeta))$ is an entire function of ζ [MW2013, Theorem 2.2], so $D_2(\zeta; \theta)$ is itself an entire function of ζ . We now prove that $D_1(\zeta; \theta)$ (defined in equation (19)) and $D_2(\zeta; \theta)$ vanish at the same values ζ_0 to precisely the same degree, thus qualifying $D_2(\zeta; \theta)$ as an Evans function. We proceed by rewriting $D_1(\zeta; \theta)$ and $D_2(\zeta; \theta)$ in terms of fundamental solutions in equation (9) and equation (15):

$$\begin{aligned} D_1(\zeta; \theta) &= \det \begin{pmatrix} p_1(T; \zeta) - e^{-i\theta} & p_2(T; \zeta) \\ p_1'(T; \zeta) & p_2'(T; \zeta) - e^{-i\theta} \end{pmatrix}, \\ &= \det(\mathbb{F}(T; \zeta)) - e^{-i\theta} (p_1(T; \zeta) + p_2'(T; \zeta)) + e^{-2i\theta}, \\ &= \exp\left(\frac{2icT\zeta}{c^2 - 1}\right) - e^{-i\theta} (p_1(T; \zeta) + p_2'(T; \zeta)) + e^{-2i\theta}. \end{aligned}$$

Similarly:

$$D_2(\zeta; \theta) = q_1(T; \zeta) + q_2'(T; \zeta) - 2 \cos\left(\frac{c\zeta T}{c^2 - 1} + \theta\right).$$

We define $r_1(z; \zeta)$ and $r_2(z; \zeta)$ by using the exponential transform in equation (13) applied to p_1 and p_2 :

$$\begin{aligned} r_1(z; \zeta) &:= p_1(z; \zeta) \exp\left(\frac{-ic\zeta}{c^2 - 1} z\right), \\ r_2(z; \zeta) &:= p_2(z; \zeta) \exp\left(\frac{-ic\zeta}{c^2 - 1} z\right), \end{aligned}$$

and we note that r_1, r_2 are solutions of equation (14). We use the initial conditions of p_1, p_2 to express r_1, r_2 in the basis of fundamental solutions q_1, q_2 , which yields:

$$\begin{aligned} r_1(z; \zeta) &= q_1(z; \zeta) - \frac{ic\zeta}{c^2 - 1} q_2(z; \zeta) \\ r_2(z; \zeta) &= q_2(z; \zeta). \end{aligned}$$

We can now write p_1, p_2 in terms of q_1, q_2 :

$$\begin{aligned} p_1(z; \zeta) &= \exp\left(\frac{ic\zeta}{c^2 - 1} z\right) \left(q_1(z; \zeta) - \frac{ic\zeta}{c^2 - 1} q_2(z; \zeta)\right) \\ p_2(z; \zeta) &= \exp\left(\frac{ic\zeta}{c^2 - 1} z\right) q_2(z; \zeta). \end{aligned}$$

Finally, we substitute these expressions into equation (20):

$$D_1(\zeta; \theta) = \exp\left(\frac{2icT\zeta}{c^2 - 1}\right) - \exp\left(\frac{ic\zeta T}{c^2 - 1} - i\theta\right) (q_1(T; \zeta) + q_2'(T; \zeta)) + e^{-2i\theta} \quad (20)$$

$$= \exp\left(\frac{ic\zeta T}{c^2 - 1} - i\theta\right) \left(\exp\left(\frac{ic\zeta T}{c^2 - 1} + i\theta\right) - q_1(T; \zeta) - q_2'(T; \zeta) + \exp\left(-\frac{ic\zeta T}{c^2 - 1} - i\theta\right)\right) \quad (21)$$

$$= -\exp\left(\frac{ic\zeta T}{c^2 - 1} - i\theta\right) D_2(\zeta; \theta). \quad (22)$$

It follows that $D_1(\zeta; \theta) = 0 \iff D_2(\zeta; \theta) = 0$. Suppose that ζ_0 is a zero of $D_1(\zeta; \theta)$ with degree of vanishing n :

$$D_1(\zeta_0; \theta) = D_1'(\zeta_0; \theta) = \dots = D_1^{(n-1)}(\zeta_0; \theta) = 0, \quad D_1^{(n)}(\zeta_0; \theta) \neq 0.$$

For some $j \leq n$, with base case $j = 1$, we assume inductively that:

$$D_2(\zeta_0; \theta) = D_2'(\zeta_0; \theta) = \dots = D_2^{(j-1)}(\zeta_0; \theta) = 0. \quad (23)$$

We apply the product rule to equation (22) j times, which yields:

$$D_1^{(j)}(\zeta_0; \theta) = -\exp\left(\frac{icT\zeta_0}{c^2 - 1} - i\theta\right) \sum_{k=0}^j \binom{j}{k} \left(\frac{icT}{c^2 - 1}\right)^{j-k} D_2^{(k)}(\zeta_0; \theta).$$

Using the inductive hypothesis in equation (23), we have:

$$D_1^{(j)}(\zeta_0; \theta) = -\exp\left(\frac{icT\zeta_0}{c^2 - 1} - i\theta\right) D_2^{(j)}(\zeta_0; \theta),$$

and hence $D_1^{(j)}(\zeta_0; \theta) = 0 \implies D_2^{(j)}(\zeta_0; \theta) = 0$. By induction, this is true for $j = 1, \dots, n-1$, and we check that:

$$D_1^{(n)}(\zeta_0; \theta) = -\exp\left(\frac{icT\zeta_0}{c^2 - 1} - i\theta\right) D_2^{(n)}(\zeta_0; \theta),$$

meaning that $D_1^{(n)}(\zeta_0; \theta) \neq 0 \implies D_2^{(n)}(\zeta_0; \theta) \neq 0$. Given ζ_0 a zero of order n for $D_1(\zeta; \theta)$, then ζ_0 is also a zero of order n for $D_2(\zeta; \theta)$. The zeros of $D_1(\zeta; \theta)$ are the only zeros of $D_2(\zeta; \theta)$, which is proved by rearranging equation (22) to:

$$D_2(\zeta; \theta) = -\exp\left(-\frac{ic\zeta T}{c^2 - 1} + i\theta\right) D_1(\zeta; \theta)$$

and following the same proof by induction as above. Thus $D_2(\zeta; \theta)$ is an Evans function for quasi-periodic solutions of the linearised sine-Gordon equation. \square

Corollary 2. The function

$$D_3(\zeta; \theta) = \det \left(\mathbb{H}(T; \zeta) - \exp \left(-\frac{ic\zeta T}{c^2 - 1} - i\theta \right) \mathbb{I} \right)$$

is also an Evans function for quasi-periodic solutions of the linearised sine-Gordon equation.

Proof. Upon expansion we have:

$$\begin{aligned} D_3(\zeta; \theta) &= \det(\mathbb{H}(T; \zeta)) - \exp \left(-\frac{ic\zeta T}{c^2 - 1} - i\theta \right) \text{tr}(\mathbb{H}(T; \zeta)) + \exp \left(-\frac{2ic\zeta T}{c^2 - 1} - 2i\theta \right) \\ &= -\exp \left(-\frac{ic\zeta T}{c^2 - 1} - i\theta \right) \left(\text{tr}(\mathbb{H}(T; \zeta)) - \exp \left(\frac{ic\zeta T}{c^2 - 1} + i\theta \right) - \exp \left(-\frac{ic\zeta T}{c^2 - 1} - i\theta \right) \right) \\ &= -\exp \left(-\frac{ic\zeta T}{c^2 - 1} - i\theta \right) D_2(\zeta; \theta). \end{aligned}$$

□

We can use these new Evans functions to compute the spectrum of the linearised sine-Gordon equation using only the solutions of Hill's equation (14). In figures 1a, 1b and 1d we reproduce the results of [JMMP2013], while figure 1c is a reproduction of a result in [MM2015].

2.1 Krein signatures of the linearised sine-Gordon equation

Making the restriction that $\zeta \in \mathbb{R}$, then $\mathcal{L}_{SG}(\zeta)$ is a self-adjoint, holomorphic family of type (A) with compact resolvent [KM2014, Example 11], meaning that its characteristic values have well-defined Krein signatures. The Evans function $D_2(\zeta; \theta)$ can be turned into a so-called *Evans-Krein function*, used in calculating the Krein signatures of isolated characteristic values ζ_0 . Following Kollar and Miller in [KM2014, Theorem 4.2], an Evans-Krein function $E(\lambda; \mu)$ for an operator pencil $\mathcal{L}(\lambda)$ is an Evans function for the μ -parametrised pencil:

$$\mathcal{K}(\lambda; \mu) := \mathcal{L}(\lambda) - \mu \mathbb{I}.$$

In [KM2014, §4.4], Kollar and Miller prove for an isolated, simple characteristic value λ_0 that:

$$\mu'(\lambda_0) = -\frac{E_\lambda(\lambda_0; 0)}{E_\mu(\lambda_0; 0)}. \quad (24)$$

In our case, we consider the related pencil

$$\mathcal{K}_{SG}(\zeta) := \mathcal{L}_{SG}(\zeta) - \mu \mathbb{I},$$

which has Evans-Krein function:

$$E_{SG}(\zeta; \mu) = \text{tr}(\mathbb{H}(T; \zeta, \mu)) - 2 \cos \left(\frac{c\zeta T}{c^2 - 1} + \theta \right). \quad (25)$$

The monodromy matrix $\mathbb{H}(T; \zeta, \mu)$ is defined as in equation (15), however the related Hill equation is:

$$\left(\frac{\zeta^2}{(c^2 - 1)^2} - \mu \right) q + \left(\partial_z^2 + \frac{\cos(\hat{U})}{c^2 - 1} \right) q = 0. \quad (26)$$

We can use the substitution $v(\zeta, \mu) = \frac{\zeta^2}{(c^2 - 1)^2} - \mu$ to calculate the partial derivatives of E_{SG} :

$$\begin{aligned} \frac{\partial}{\partial \zeta} E_{SG}(\zeta; \mu) &= \frac{\partial v}{\partial \zeta} \frac{\partial}{\partial v} \text{tr}(\mathbb{H}(T; \zeta, \mu)) + \frac{2cT}{c^2 - 1} \sin \left(\frac{c\zeta T}{c^2 - 1} + \theta \right) \\ &= \frac{2\zeta}{(c^2 - 1)^2} \frac{\partial}{\partial v} \text{tr}(\mathbb{H}(T; \zeta, \mu)) + \frac{2cT}{c^2 - 1} \sin \left(\frac{c\zeta T}{c^2 - 1} + \theta \right) \\ \frac{\partial}{\partial \mu} E_{SG}(\zeta; \mu) &= \frac{\partial v}{\partial \mu} \frac{\partial}{\partial v} \text{tr}(\mathbb{H}(T; \zeta, \mu)) \\ &= -\frac{\partial}{\partial v} \text{tr}(\mathbb{H}(T; \zeta, \mu)). \end{aligned}$$

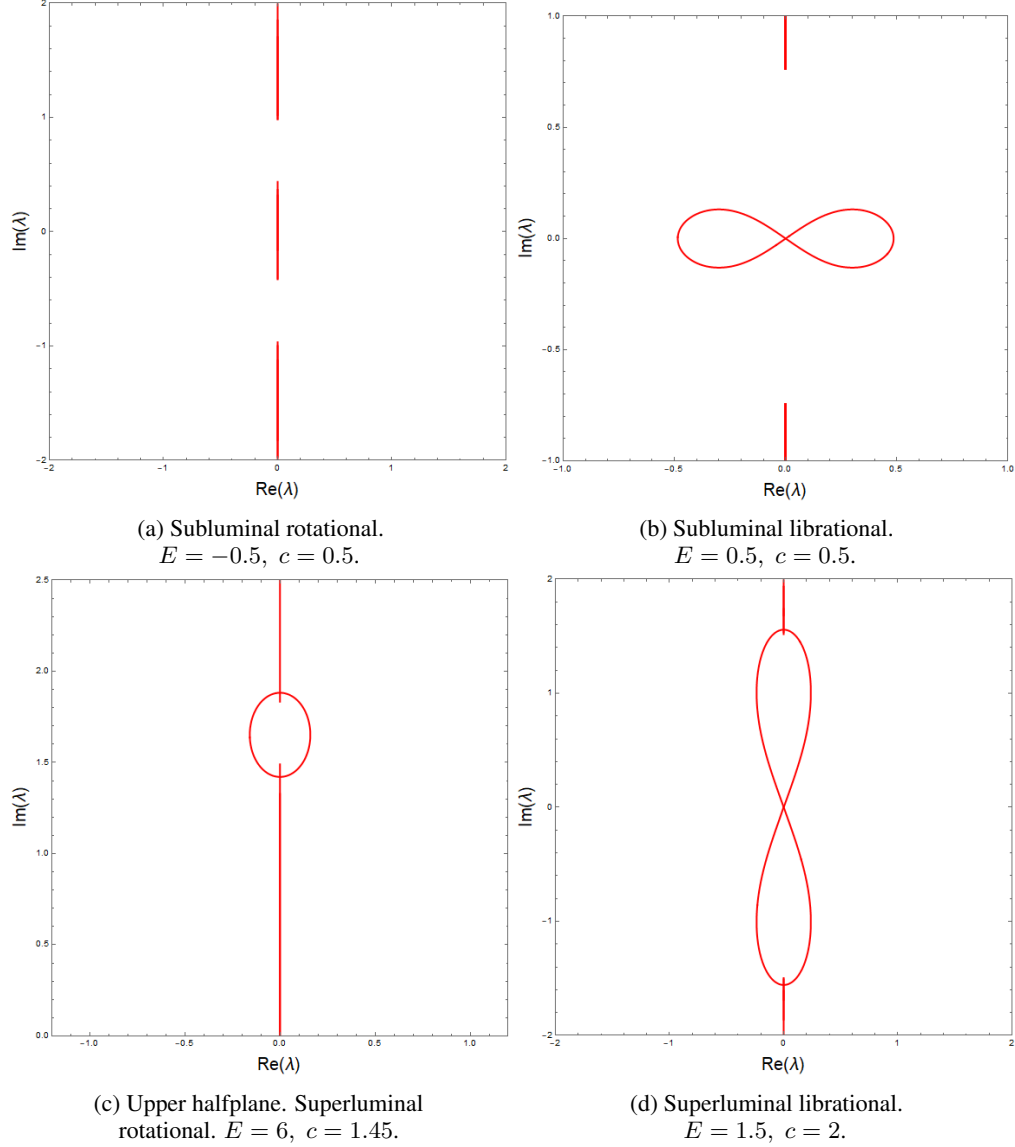


Figure 1: Numerical plots of the spectra σ of various periodic travelling wave solutions to the sine-Gordon equation (1), calculated by finding solutions to the Evans function $D_2(\zeta; \theta)$ with $\zeta \in \mathbb{C}$ and for all $\theta \in [0, 2\pi)$. Recall that $\lambda = i\zeta$.

Using these results with equation (24) we have:

$$\mu'(\zeta_0) = \frac{2\zeta_0}{(c^2 - 1)^2} + \frac{\frac{2cT}{c^2 - 1} \sin\left(\frac{c\zeta_0 T}{c^2 - 1} + \theta\right)}{\frac{\partial}{\partial v} \text{tr}(\mathbb{H}(T; \zeta_0, 0))}. \quad (27)$$

We have from [MW2013, Corollary 2.1, Theorem 2.2] that

$$D_{Hill}(\zeta) := \text{tr}(\mathbb{H}(T; \zeta_0)) - 2$$

is an Evans function for characteristic values of periodic solutions to Hill's equation (14). So we have:

$$\begin{aligned} D'_{Hill}(\zeta_0) &= \frac{\partial v}{\partial \zeta} \frac{\partial}{\partial v} \text{tr}(\mathbb{H}(T; \zeta_0, 0)) \\ \implies \frac{\partial}{\partial v} \text{tr}(\mathbb{H}(T; \zeta_0, 0)) &= \frac{(c^2 - 1)^2}{2\zeta_0} D'_{Hill}(\zeta_0). \end{aligned}$$

We make this substitution in equation (27) because it is easier numerically to compute $D'_{Hill}(\zeta_0)$. Hence we have:

$$\mu'(\zeta_0) = \frac{2\zeta_0}{(c^2 - 1)^2} + \frac{4c\zeta_0 T}{(c^2 - 1)^3} \frac{\sin\left(\frac{c\zeta_0 T}{c^2 - 1} + \theta\right)}{D'_{Hill}(\zeta_0)}. \quad (28)$$

Finally, using equation (8), we compute the Krein signature of simple characteristic values for quasi-periodic solutions of the linearised sine-Gordon equation (5):

$$\kappa(\zeta_0) = \text{sign} \left(\frac{2\zeta_0}{(c^2 - 1)^2} + \frac{4c\zeta_0 T}{(c^2 - 1)^3} \frac{\sin\left(\frac{c\zeta_0 T}{c^2 - 1} + \theta\right)}{D'_{Hill}(\zeta_0)} \right). \quad (29)$$

We pause to note that the situations where $\zeta_0 = 0$ or $D'_{Hill}(\zeta_0) = 0$ pose problems with the above derivation. Firstly, $\zeta_0 = 0$ is never a simple characteristic value of the linearised sine-Gordon equation. We observe that:

$$\begin{aligned} \mathcal{L}_{SG}(0)\hat{U}'(z) &= \hat{U}_{zzz} + \frac{\cos(\hat{U})}{c^2 - 1} \hat{U}_z \\ &= 0, \end{aligned}$$

where the last step follows by differentiating equation (3):

$$(c^2 - 1)\hat{U}_{zz} + \sin(\hat{U}) = 0.$$

The function $\hat{U}'(z)$ is T -periodic, and so it has Floquet exponent $\theta = 0$. Noting that $\text{tr}(\mathbb{H}(T; \zeta))$ is analytic (by [MW2013, Theorem 2.2]) and even in ζ (since Hill's equation (14) is only dependent on ζ^2), then

$$D_2(\zeta; 0) = \text{tr}(\mathbb{H}(T; \zeta)) - 2 \cos\left(\frac{c\zeta T}{c^2 - 1}\right)$$

is analytic and even in ζ . Hence $D_2'(0; 0) = 0$, and so the multiplicity of $\zeta_0 = 0$ is at least 2. Moreover, $\theta = 0$ is the unique Floquet exponent of $\zeta_0 = 0$, since $D_2(0; 0) = 0$ implies that $\text{tr}(\mathbb{H}(T; 0)) = 2$, and hence

$$D_2(0; \theta) = 2 - 2 \cos(\theta) = 0 \iff \theta = 0$$

with $\theta \in [0, 2\pi)$. An alternative proof that $\zeta_0 = 0$ is a characteristic value with even multiplicity for the more general linearised nonlinear Klein-Gordon equation is in [JMMP2014, Lemma 6.2]. As for the values x_0 when

$$D'_{Hill}(x_0) = \frac{\partial}{\partial \zeta} (\text{tr}(\mathbb{H}(T; \zeta))) \Big|_{\zeta=x_0} = 0,$$

we refer to Magnus and Winkler's oscillation theorem [MW2013, Theorem 2.1], which states that

$$D'_{Hill}(x_0) = 0 \iff |\text{tr}(\mathbb{H}(T; x_0))| \geq 2.$$

If $|\text{tr}(\mathbb{H}(T; x_0))| > 2$, then $|D_2(x_0; \theta)| > 0$, so x_0 is not a characteristic value. If:

$$\text{tr}(\mathbb{H}(T; x_0)) = (-1)^j 2$$

for $j = 0, 1$, then there exists

$$\theta_0 = j\pi - \frac{cx_0 T}{c^2 - 1} \pmod{2\pi}$$

such that $D_2(x_0; \theta_0) = D_2'(x_0; \theta_0) = 0$, meaning that x_0 has multiplicity greater than 1. Concretely, our formula in equation (29) is able to calculate the Krein signature of all simple characteristic values of the linearised sine-Gordon equation (5). These results are immediately generalisable to the Klein-Gordon case, where $\cos(\hat{U})$ is replaced with $V''(\hat{U})$, and we will use these results freely in section 3.

The advantage of using $D_2(\zeta; \theta)$ over $D_1(\zeta; \theta)$ when computing Krein signatures is that we do not have to explicitly compute a partial derivative of an Evans-Krein function with respect to μ . Our substitution of $v(\zeta, \mu) = \frac{\zeta^2}{(c^2 - 1)^2} - \mu$ took advantage of the dependence of equation (26) on $\frac{\zeta^2}{(c^2 - 1)^2} - \mu$. This is not possible when computing Krein signatures using $D_1(\zeta; \theta)$, since this function is written in terms of the fundamental solutions to the linearised sine-Gordon equation (5). For the sake of exposition, we can adapt $D_1(\zeta; \theta)$ into an Evans-Krein function $E_1(\zeta; \mu, \theta)$:

$$\begin{aligned} E_1(\zeta; \mu, \theta) &= \det(\mathbb{F}(T; \zeta, \mu) - e^{-i\theta} \mathbb{I}) \\ &= \exp\left(\frac{2icT\zeta}{c^2 - 1}\right) - e^{-i\theta} (\text{tr}(\mathbb{F}(T; \zeta, \mu))) + e^{-2i\theta}, \end{aligned}$$

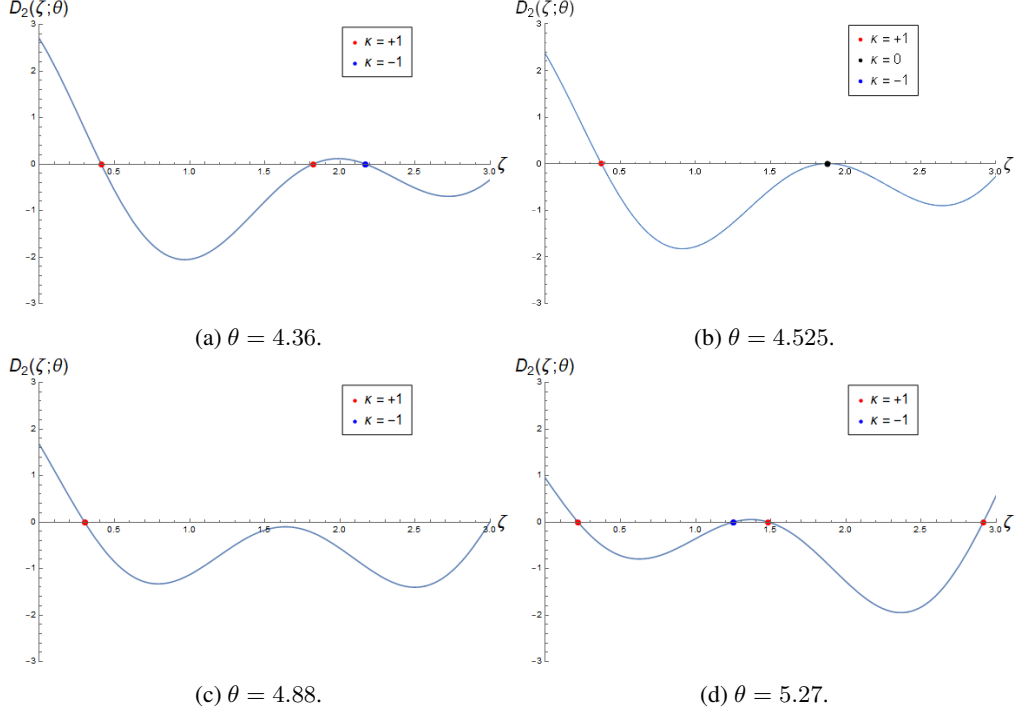


Figure 2: Numerical plots of $D_2(\zeta; \theta)$ showing the Krein signatures of isolated characteristic values of the linearised sine-Gordon equation (5), linearised around the superluminal rotational wave in figure 1c with $E = 6$, $c = 1.45$. A collision of opposite Krein signatures results in a Hamiltonian-Hopf bifurcation. Krein signatures with $\kappa = 1$ are shown online with red dots, while Krein signatures with $\kappa = -1$ are denoted by blue dots. Figure 2b shows the bifurcation point which corresponds to a Krein signature of $\kappa = 0$, denoted by a black dot.

where we have expanded $E_1(\zeta; \mu, \theta)$ as in equation (20). The monodromy matrix $\mathbb{F}(T; \zeta, \mu)$ is made up of fundamental solutions to the related linearised sine-Gordon equation:

$$\left(-\frac{\zeta^2}{c^2-1} - \mu\right)p - \frac{2ic\zeta}{c^2-1}\partial_z p + \left(\partial_z^2 + \frac{\cos(\hat{U})}{c^2-1}\right)p = 0. \quad (30)$$

Since a substitution for ζ and μ is not possible in equation (30), we rely on direct computation of the derivatives of $E_1(\zeta; \mu, \theta)$:

$$\begin{aligned} \frac{\partial}{\partial \zeta} E_1(\zeta; \mu, \theta) &= \frac{2icT}{c^2-1} \exp\left(\frac{2icT\zeta}{c^2-1}\right) - e^{-i\theta} \frac{\partial}{\partial \zeta} (\text{tr}(\mathbb{F}(T; \zeta, \mu))) \\ \frac{\partial}{\partial \mu} E_1(\zeta; \mu, \theta) &= -e^{-i\theta} \frac{\partial}{\partial \mu} (\text{tr}(\mathbb{F}(T; \zeta, \mu))). \end{aligned}$$

Using equation (24), we now have:

$$\mu'(\zeta_0) = -\frac{\frac{2icT}{c^2-1} \exp\left(\frac{2icT\zeta_0}{c^2-1} + i\theta\right)}{\frac{\partial}{\partial \mu} (\text{tr}(\mathbb{F}(T; \zeta_0, 0)))} + \frac{\frac{\partial}{\partial \zeta} (\text{tr}(\mathbb{F}(T; \zeta_0, 0)))}{\frac{\partial}{\partial \mu} (\text{tr}(\mathbb{F}(T; \zeta_0, 0)))}.$$

Using $E_1(\zeta; \mu, \theta)$ to compute $\mu'(\zeta_0)$, we must numerically differentiate

$$\text{tr}(\mathbb{F}(T; \zeta, \mu)) = p_1(T; \zeta, \mu) + p_2'(T; \zeta, \mu)$$

at $(\zeta, \mu) = (\zeta_0, 0)$ with respect to both ζ and μ . However, in equation (28) we were able to find $\mu'(\zeta_0)$ in terms of only a ζ -derivative of $D_{Hill}(\zeta) = q_1(T; \zeta) + q_2(T; \zeta) - 2$ at $\zeta = \zeta_0$. Hence, using our new Evans function $D_2(\zeta; \theta)$ rather than $D_1(\zeta; \theta)$ results in a more elegant calculation of Krein signatures.

In figure 2 we use equation (29) to numerically calculate the Krein signatures of isolated characteristic values in the case of the superluminal rotational wave in figure 1c. As the bifurcation parameter θ is varied, we observe a collision

between two characteristic values of opposite Krein signature, resulting in a Hamiltonian-Hopf bifurcation at $\theta \approx 4.53$ where these two characteristic values enter the complex plane. The two characteristic values then bifurcate back onto the imaginary axis at $\theta \approx 5.16$.

3 The nonlinear Klein-Gordon equation

We now consider the more general nonlinear Klein-Gordon equation:

$$u_{tt} - u_{xx} + V'(u) = 0, \quad (31)$$

where $u(x, t) : \mathbb{R} \times [0, +\infty) \rightarrow \mathbb{R}$ and $V(u) : \mathbb{R} \rightarrow \mathbb{R}$ is a C^2 potential. It is possible to recover the sine-Gordon equation (1) by setting $V(u) = 1 - \cos(u)$. Similar to our derivation of the linearised sine-Gordon equation, we have the linearised nonlinear Klein-Gordon equation:

$$p'' - \frac{2ic\zeta}{c^2 - 1}p' + \left(-\frac{\zeta^2}{c^2 - 1} + \frac{V''(\hat{U})}{c^2 - 1} \right) p = 0, \quad (32)$$

where $\hat{U}(z)$ satisfies

$$(c^2 - 1)\hat{U}_{zz} + V'(\hat{U}) = 0, \quad (33)$$

and has period T . The case when $V(u)$ is a periodic function of u has been studied extensively and we point the reader to [JMMP2014] for a thorough analysis. Provided that $\hat{U}(z)$ is periodic, then $V(\hat{U})$ is also periodic, making available the theory of [JMMP2014] with the caveat that rotational waves will not be observed when V is not periodic. In fact, all the results of section 2 are immediately generalisable to any $V \in C^2$, with the related Hill's equation (14) becoming:

$$q'' + \left(\frac{\zeta^2}{(c^2 - 1)^2} + \frac{V''(\hat{U})}{c^2 - 1} \right) q = 0, \quad (34)$$

and the Evans functions $D_2(\zeta; \theta)$ and $D_3(\zeta; \theta)$ remaining unchanged. The spectrum of the nonlinear Klein-Gordon equation exhibits Hamiltonian symmetry [JMMP2014, Proposition 3.9]. In particular, if $p(z)$ satisfies equation (32) for $\zeta \in \sigma$, then taking complex conjugates implies that $p^*(z)$ satisfies equation (32) for ζ^* , and making the transformation

$$p(z) = e^{\frac{-2ic\zeta}{c^2-1}z} r(z)$$

implies that $r(z)$ satisfies equation (32) for $-\zeta$.

As a point of contrast to the sine-Gordon potential $V(u) = 1 - \cos(u)$, we have chosen to consider the non-periodic potential $V(u) = \frac{1}{4}u^4 - \frac{1}{2}u^2$. As before, we integrate equation (33) once which introduces the energy parameter E :

$$\frac{1}{2}(c^2 - 1)\hat{U}_z^2 + \frac{1}{4}\hat{U}^4 - \frac{1}{2}\hat{U}^2 = E.$$

Figure 3 shows the phase portraits for subluminal and superluminal travelling wave solutions to equation (33). In the subluminal case in figure 3a, we note that the separatrix corresponds to $E = -\frac{1}{4}$, and we have that $-\frac{1}{4} < E < 0$. For superluminal waves in figure 3b, $E > 0$ corresponds to waves outside the homoclinic orbit, while $-\frac{1}{4} < E < 0$ corresponds to waves within one branch of the homoclinic orbit. Given the symmetry of the phase portraits due to the potential $V(u) = \frac{1}{4}u^4 - \frac{1}{2}u^2$ being even in u , there is no difference in the spectra of waves chosen within the left or right branch of the homoclinic orbit for equal values of E . Without loss of generality we chose to consider waves within the right branch of the homoclinic orbit. In figure 4, we numerically compute the spectra of several waves using the Evans function $D_2(\zeta; \theta)$. We chose the waves which produced qualitatively different spectra, however we have not proved that this list is exhaustive. Figure 5 shows the corresponding phase portraits of the waves whose spectra are included in figure 4. In figure 6, we capture two bifurcations on the imaginary axis as θ is varied for the wave corresponding to figure 5b. We observe a phenomenon discussed in [KM2014, §6] where characteristic values of opposite Krein signature pass through each other instead of undergoing a Hamiltonian-Hopf bifurcation. A simple characteristic value of Krein signature $\kappa = -1$ (denoted by a blue dot) bifurcates onto the real axis in the 2nd plot from the right in the bottom row of figure 6. It passes through the simple characteristic values of Krein signature $\kappa = 1$ until it collides with a characteristic value with $\kappa = 1$ at $\zeta \approx 0.8$, bifurcating off the ζ -axis. The ζ values where the bifurcations take place correspond to the values of λ where the spectrum leaves the imaginary axis, as denoted by the grey arrows. We observed the same phenomenon for the wave whose spectrum is plotted in figure 4d.

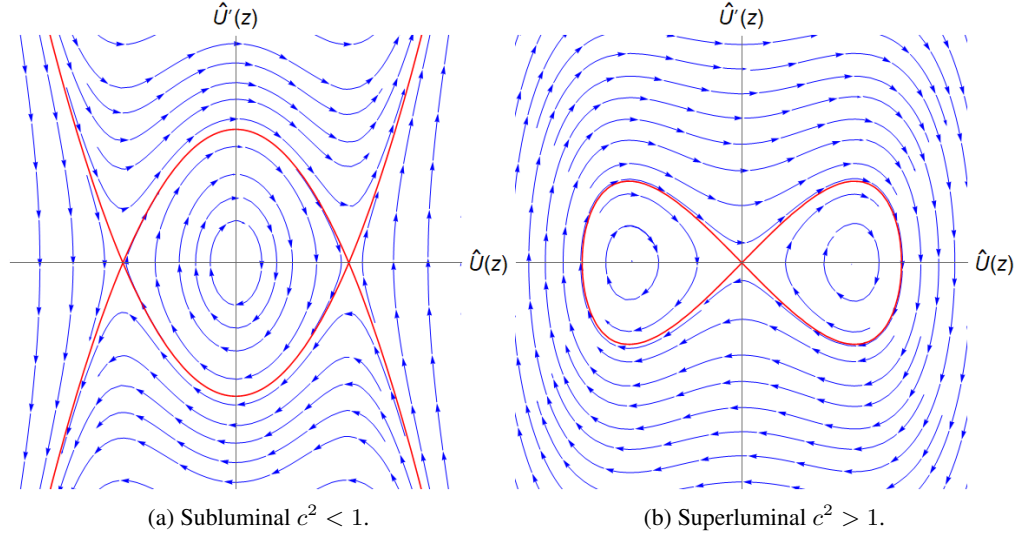


Figure 3: Phase portraits for equation (33) with $V(\hat{U}) = \frac{1}{4}\hat{U}^4 - \frac{1}{2}\hat{U}^2$. In the subluminal case, the librational periodic solutions \hat{U} are within the separatrix (drawn in red). For supraluminal waves, all periodic waves are librational, and these exist both outside the homoclinic orbit (drawn in red) and inside.

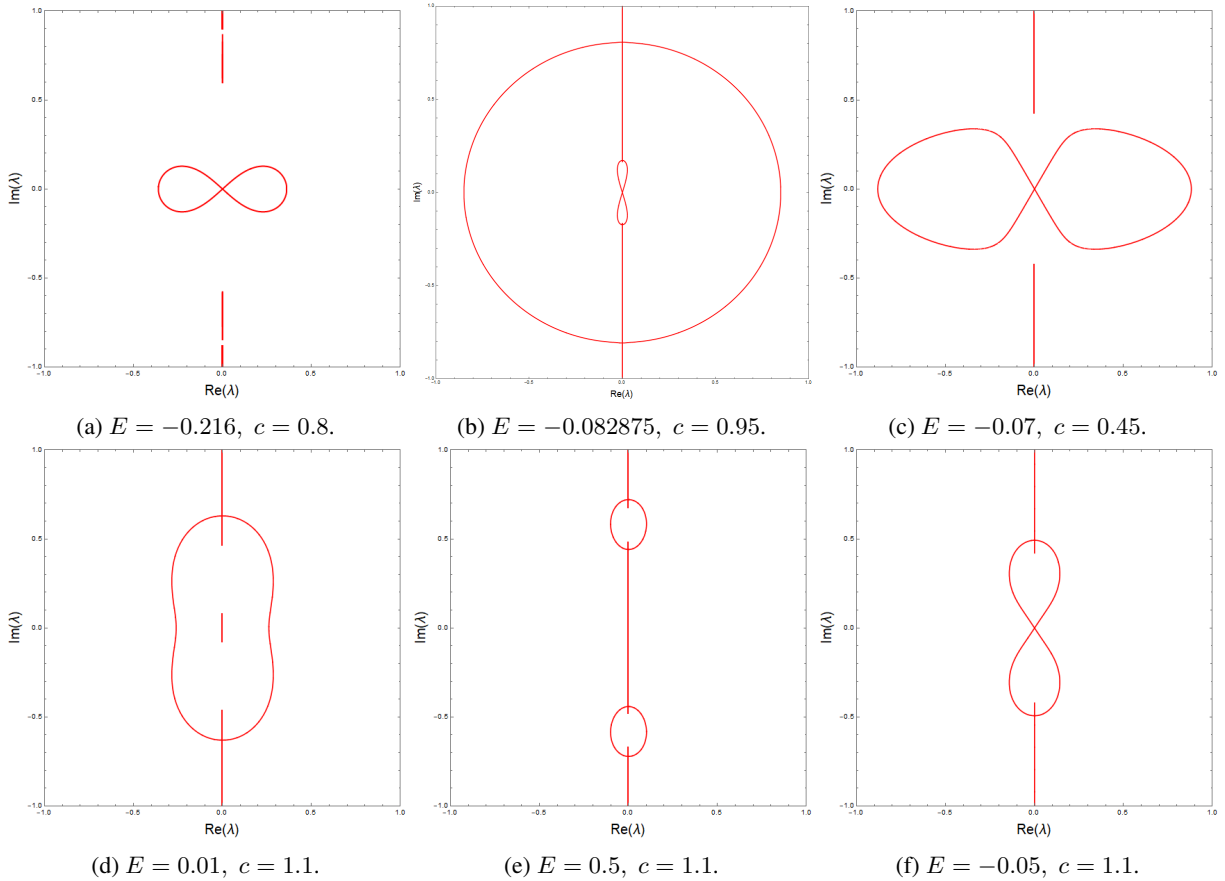


Figure 4: Numerical plots of the spectra σ of various periodic travelling wave solutions to the nonlinear Klein-Gordon equation (31) with potential $V(u) = \frac{u^4}{4} - \frac{u^2}{2}$. Row 1 depicts the spectra for subluminal waves, while row 2 corresponds to supraluminal waves.

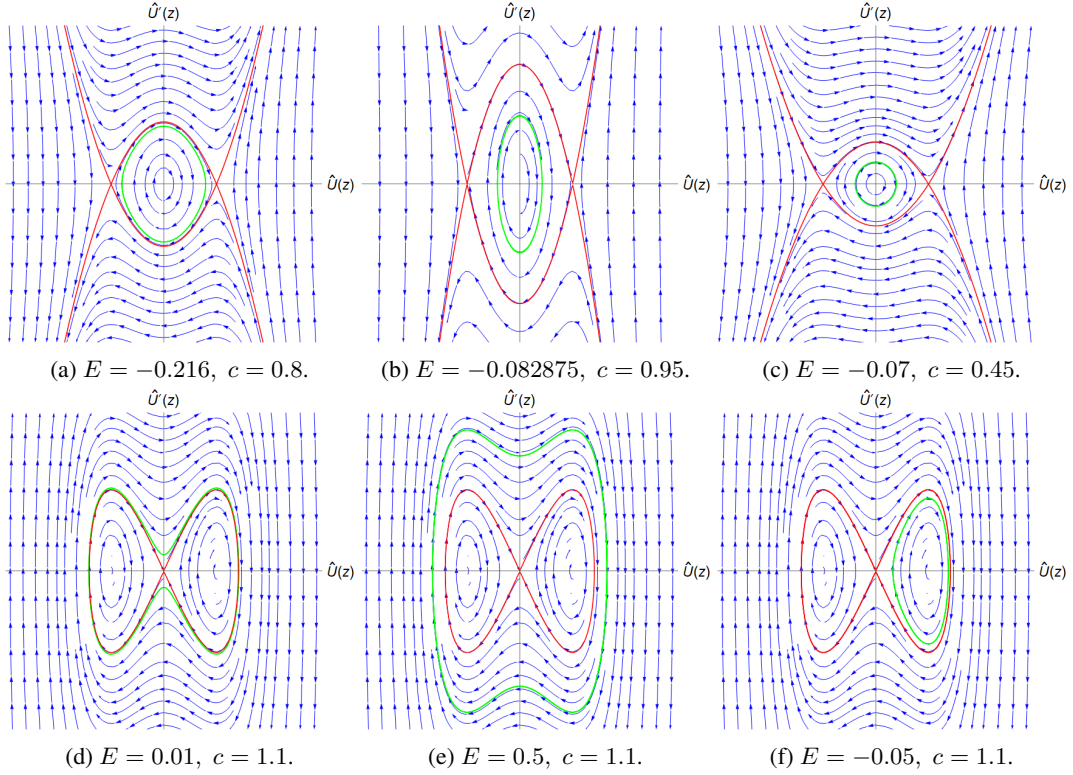


Figure 5: Phase portraits corresponding to the periodic travelling waves in figure 4. The chosen waves are shown in green, while the separatrices are shown in red.

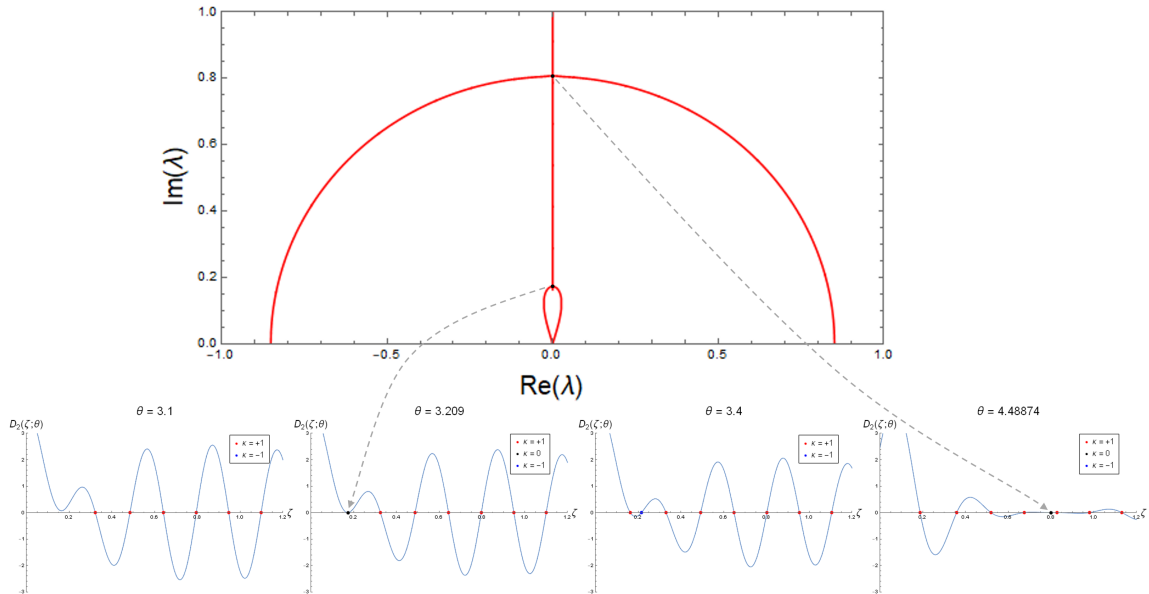


Figure 6: We track the bifurcations in the upper half plane of the spectrum σ in figure 4b as θ is varied. See figure 5b for the original wave. In the bottom row, we plot the Evans function $D_2(\zeta; \theta)$ for the values from left to right: $\theta = 3.1$, $\theta = 3.209$, $\theta = 3.4$, $\theta = 4.48874$. The black dots correspond to Hamiltonian-Hopf bifurcations, and we have labelled these on the spectral diagram in the top row with black dots as well.

4 Discussion and conclusion

In this paper, we apply Floquet theory to results for Hill's equation in [JMMP2013] to construct a new Evans function for quasi-periodic solutions of the linearised sine-Gordon equation. As opposed to the Evans function in [JMMP2014], this new Evans function is readily adaptable to an Evans-Krein function from which we calculate the Krein signatures of simple characteristic values of the linearised sine-Gordon equation. These Krein signatures allow us to track Hamiltonian-Hopf bifurcations in the spectrum of the sine-Gordon equation in terms of the Floquet exponent, which distinguishes our method from [JMMP2013, MM2015]. As a check on the correctness of our methods, we use this new Evans function to numerically compute spectra for different periodic travelling wave solutions of the sine-Gordon equation, replicating the results of [JMMP2013, JMMP2014, MM2015]. Finally, as an example of how to extend our Evans function, we use it to compute the spectrum in a general nonlinear Klein-Gordon equation, producing spectral diagrams and calculating Krein signatures for the potential $V(u) = \frac{1}{4}u^4 - \frac{1}{2}u^2$.

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