Counting eigenvalues in Hamiltonian systems via the Maslov index

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Set-up and main result.

Eigenvalue problem:

$$N \begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \end{pmatrix}, \qquad \begin{pmatrix} u(0) \\ v(0) \end{pmatrix} = \begin{pmatrix} u(\ell) \\ v(\ell) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1}$$

where

$$N = \begin{pmatrix} 0 & L_- \\ -L_+ & 0 \end{pmatrix}, \quad \begin{cases} L_+ = \partial_{xx} + g(x), \\ L_- = \partial_{xx} + h(x), \end{cases} \qquad g, h \in C^2([0, \ell]; \mathbb{R}).$$

Define:

$$\begin{aligned} P &:= \# \;\; \text{positive eigenvalues of } \; L_+, \\ Q &:= \# \;\; \text{positive eigenvalues of } \; L_-, \\ n_+(N) &:= \# \;\; \text{positive real eigenvalues of } \; N, \end{aligned}$$

Then we have the lower bound:

$$n_+(N) \geq |P - Q - \mathfrak{c}| \tag{2}$$

where $\mathfrak{c} \in \{-1, 0, 1\}$ is the contribution to the Maslov index from the "corner" of the Maslov box.

Motivating example for the Maslov index: Sturm-Liouville theory

Consider the eigenvalue problem

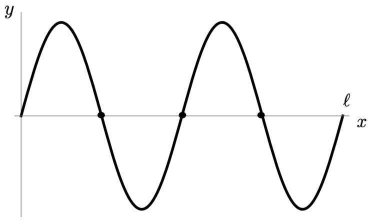
$$y'' + q(x)y = \lambda y, \qquad y(0) = y(\ell) = 0. \tag{3}$$

Sturm-Liouville theory:

- Eigenvalues λ_n of (3) are real, discrete, simple, and satisfy

  $$\lambda_1 > \lambda_2 > \lambda_3 > \dots \lambda_n > \dots \to -\infty$$

- Eigenfunction y_n for nth eigenvalue has $n-1$ zeros on $(0,\ell)$.



Second statement is actually a statement about **oscillations in phase space** (yy'-plane).

Example:

EVP: $y'' + q(x)y = \lambda y \qquad y(0) = y(\ell) = 0$

Define the polar angle in the phase plane: $\tan\theta(x;\lambda) = \dfrac{y'(x;\lambda)}{y(x;\lambda)}$

Initial condition: $y(0) = 0 \implies \theta(0;\lambda) = \frac{\pi}{2}.$

Observations:

Eigenvalue $\lambda = \lambda^*$ when $y(\ell) = 0$, i.e.

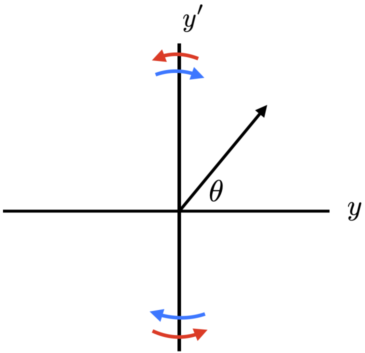
$$\theta(\ell;\lambda^*) = \frac{\pi}{2} + n\pi, \quad n \in \mathbb{Z}.$$

Fix $\lambda = \lambda^*$. Can show:

$$\frac{\partial\theta}{\partial x}(x;\lambda^*)\Big|_{\theta=\frac{\pi}{2}+n\pi} < 0$$

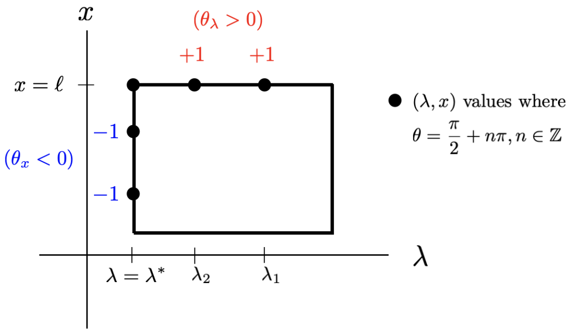
Fix $x = \ell$. Can show:

$$\frac{\partial\theta}{\partial\lambda}(\ell;\lambda)\Big|_{\theta=\frac{\pi}{2}+n\pi} > 0$$



Motivating example for the Maslov index: Sturm-Liouville theory

We interpret this oscillation in phase space with the following picture:



"Box theorem": the signatures of the points on this box sum to zero!

These ideas are generalisable to Hamiltonian systems **via the Maslov index**.

Motivating example for the Maslov index: Sturm-Liouville theory

Yet another interpretation is offered by the **monotonicity of the eigenvalue curves**:

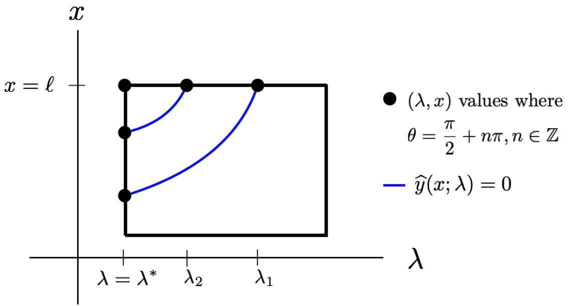


Figure: Locus of points in λ, s plane where $\widehat{y}(x;\lambda) = 0$ (where $\widehat{y}(0;\lambda) = 0$)

Can show $x'(\lambda) > 0$ using the I.F.T. and the original ODE

$\implies$ # {crossings on left}= #{crossings on top}

The Maslov index: framework

A symplectic form on $\mathbb{R}^{2n}$ is a nondegenerate, skew-symmetric bilinear form

$$\omega : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \longrightarrow \mathbb{R}, \qquad \omega(x,y) = \langle Jx, y \rangle_{\mathbb{R}^{2n}}, \qquad J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

The *Lagrangian Grassmannian* is the set of all *Lagrangian subspaces* of $\mathbb{R}^{2n}$,

$$\mathcal{L}(n) = \{\Lambda \subset \mathbb{R}^{2n} : \dim \Lambda = n, \quad \omega(x,y) = 0 \ \ \forall x, y \in \Lambda\}.$$

The Maslov index can be thought of as a **winding number** for loops in $\mathcal{L}(n)$.

In practice we compute it by **counting signed intersections** of our path with a **codimension one submanifold** of $\mathcal{L}(n)$:

$$\mathcal{T}(\Lambda_0) := \{\Lambda \in \mathcal{L}(n) : \Lambda \cap \Lambda_0 \neq \{0\}\}$$

(the *train* of a fixed Lagrangian plane Λ_0).

The Maslov index: framework

Consider a path $\Lambda : [a, b] \longrightarrow \mathcal{L}(n)$, and fix $\Lambda_0 \in \mathcal{L}(n)$.

- A *crossing* is a value $t = t_0$ s.t. $\Lambda(t_0) \in \mathcal{T}(\Lambda_0)$

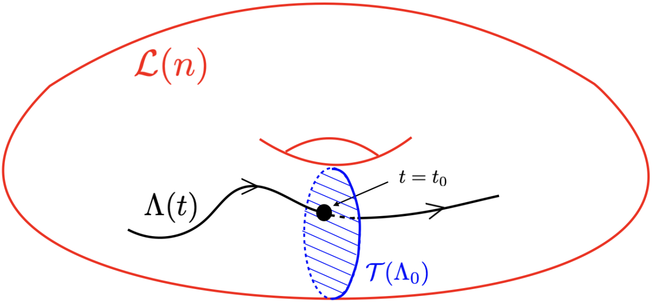


Figure: Shematic of a path $\Lambda(t)$ in the Lagrangian Grassmannian $\mathcal{L}(n)$ intersecting the train $\mathcal{T}(\Lambda_0)$ at $t = t_0$.

- The Maslov index is a **signed count** of the crossings, with the signature being determined by that of a certain quadratic form.

Application to eigenvalue problem at hand.

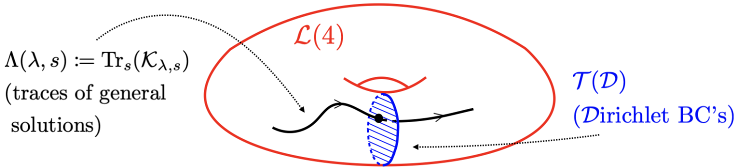
Restrict problem to $[0, s\ell]$:

$$N\mathbf{u} = \lambda \mathbf{u}, \quad \mathbf{u}(0) = \mathbf{u}(s\ell) = 0, \quad x \in [0, s\ell].$$

General solutions: $\mathcal{K}_{\lambda,s} := \{\mathbf{u} \in H^2(0, s\ell) : N\mathbf{u} = \lambda \mathbf{u}\}$ **(no BC's)**

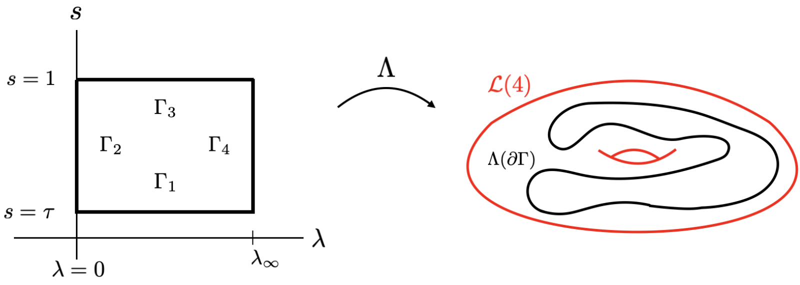
Trace of $\mathbf{u} \in H^2(0, s\ell)$:

$$\mathrm{Tr}_s\, \mathbf{u} := \big(u(0), v(0), u(s\ell), v(s\ell), -u'(0), v'(0), u'(s\ell), -v'(s\ell)\big)^\top \in \mathbb{R}^8$$



The Maslov box

Consider the following rectangle in the λs-plane with image in $\mathcal{L}(4)$ under Λ:

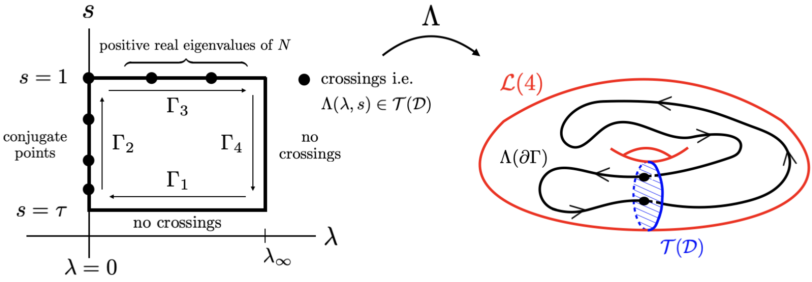


Let Γ be the solid box, so that $\partial\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$.

Our **Lagrangian path** is the **image** in $\mathcal{L}(4)$ of $\partial\Gamma$; i.e. $\Lambda : \partial\Gamma \to \mathcal{L}(4)$.

The Maslov box

Now mark the intersections of this path with the train $\mathcal{T}(\mathcal{D})$.



Topological properties of Maslov index imply

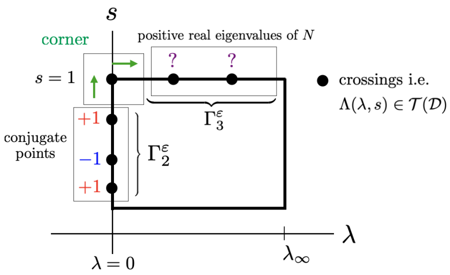
$$\text{Mas}(\Lambda, \mathcal{D}; \partial\Gamma) = \text{Mas}(\Lambda, \mathcal{D}; \Gamma_1) + \text{Mas}(\Lambda, \mathcal{D}; \Gamma_2) + \text{Mas}(\Lambda, \mathcal{D}; \Gamma_3) + \text{Mas}(\Lambda, \mathcal{D}; \Gamma_4) = 0.$$

No crossings on $\Gamma_1, \Gamma_4 \implies \text{Mas}(\Lambda, \mathcal{D}; \Gamma_1) = \text{Mas}(\Lambda, \mathcal{D}; \Gamma_4) = 0$. Thus

$$\text{Mas}(\Lambda, \mathcal{D}; \Gamma_2) + \text{Mas}(\Lambda, \mathcal{D}; \Gamma_3) = 0.$$

The Maslov box

Now assign signature to each crossing and sum!



- $\mathrm{Mas}(\Lambda, \mathcal{D}; \Gamma_2^\varepsilon) = +Q - P$, where

  $Q = \#$ positive eigenvalues of L_-

  $P = \#$ positive eigenvalues of L_+

- Along Γ_3^ε: signatures may offset each other; therefore

$$n_+(N) \geq |\,\mathrm{Mas}(\Lambda, \mathcal{D}; \Gamma_3^\varepsilon)|$$

- Contribution from corner is $\mathfrak{c} := \mathrm{Mas}(\Lambda, \mathcal{D}; \text{corner})$.

Therefore: $\mathrm{Mas}(\Lambda, \mathcal{D}; \Gamma_2) + \mathrm{Mas}(\Lambda, \mathcal{D}; \Gamma_3) = 0$

$\Longrightarrow$ $\mathrm{Mas}(\Lambda, \mathcal{D}; \Gamma_2^\varepsilon) + \mathfrak{c} + \mathrm{Mas}(\Lambda, \mathcal{D}; \Gamma_3^\varepsilon) = 0$

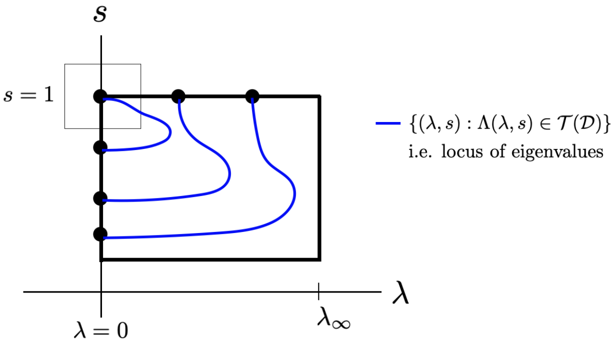
$\Longrightarrow$ $\mathrm{Mas}(\Lambda, \mathcal{D}; \Gamma_3^\varepsilon) = -\mathrm{Mas}(\Lambda, \mathcal{D}; \Gamma_2^\varepsilon) - \mathfrak{c}$

$\Longrightarrow$ $n_+(N) \geq |\,\mathrm{Mas}(\Lambda, \mathcal{D}; \Gamma_3^\varepsilon)| = |P - Q - \mathfrak{c}|$ $\square$

Computing $\mathfrak{c}$

The contribution $\mathfrak{c}$ is **irregular**, since the associated quadratic form is degenerate.

This corresponds to our 'box' being tangential to the (flat) eigenvalue curve at $\lambda = 0, s = 1$:



We will use a homotopy argument to compute $\mathfrak{c}$, which hinges on knowing the **concavity of the eigenvalue curves**...

Computing $\mathfrak{c}$

Theorem (Cox, Curran, Latushkin, Marangell)

Let $s = s(\lambda)$ be the eigenvalue curve through $(\lambda, s) = (0, 1)$.

If $0 \in \mathrm{Spec}(L_-) \setminus \mathrm{Spec}(L_+)$ with $L_- v = 0$, then

$$\mathrm{sign}\, \ddot{s}(0) = \mathrm{sign} \int_0^\ell \widehat{u}\, v\, dx,$$

where $-L_+ \widehat{u} = v$. Note $\mathbf{u} = \begin{pmatrix} 0 \\ v \end{pmatrix} \in \ker(N)$ and $\widehat{\mathbf{u}} = \begin{pmatrix} \widehat{u} \\ 0 \end{pmatrix} \in \ker(N^2) \setminus \ker(N)$.

If $0 \in \mathrm{Spec}(L_+) \setminus \mathrm{Spec}(L_-)$ with $L_+ u = 0$, then

$$\mathrm{sign}\, \ddot{s}(0) = -\,\mathrm{sign} \int_0^\ell \widehat{v}\, u\, dx,$$

where $L_- \widehat{v} = u$. Note $\mathbf{u} = \begin{pmatrix} u \\ 0 \end{pmatrix} \in \ker(N)$ and $\widehat{\mathbf{u}} = \begin{pmatrix} 0 \\ \widehat{v} \end{pmatrix} \in \ker(N^2) \setminus \ker(N)$.

Computing $\mathfrak{c}$

Homotoping the top left corner of the Maslov box:

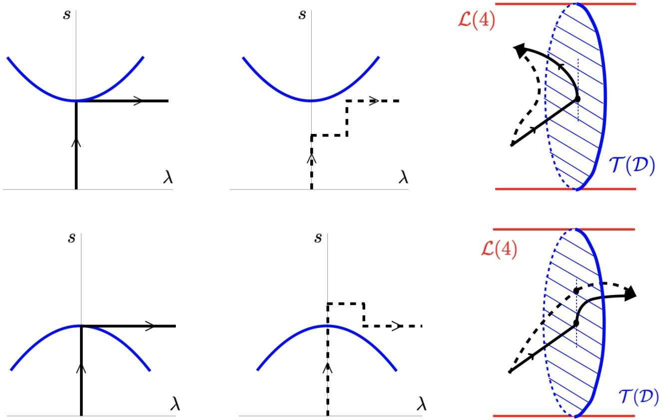


Figure: Blow-up of the crossing at $(\lambda, s) = (0, 1)$, with the (blue) eigenvalue curve, Maslov box (solid black) and homotoped path (dashed) passing through it. Images of black and dashed paths in $\mathcal{L}(4)$ on the right.

Computing $\mathfrak{c}$

Theorem (Cox, Curran, Latushkin, Marangell)

Let $s = s(\lambda)$ be the eigenvalue curve through $(\lambda, s) = (0, 1)$.

If $0 \in \operatorname{Spec}(L_-) \setminus \operatorname{Spec}(L_+)$ *then*

$$\mathfrak{c} = \begin{cases} 0 & \operatorname{sign} \ddot{s}(0) > 0 \\ +1 & \operatorname{sign} \ddot{s}(0) < 0. \end{cases}$$

If $0 \in \operatorname{Spec}(L_+) \setminus \operatorname{Spec}(L_-)$ *then*

$$\mathfrak{c} = \begin{cases} 0 & \operatorname{sign} \ddot{s}(0) > 0 \\ -1 & \operatorname{sign} \ddot{s}(0) < 0. \end{cases}$$

Application: The Vakhitov-Kolokolov criterion

Nonlinear Schrödinger (NLS) equation on a compact interval,

$$\mathrm{i}\psi_t = \psi_{xx} + f\left(|\psi|^2\right)\psi, \qquad \psi(x,t) : [0,\ell] \times [0,\infty) \longrightarrow \mathbb{C} \tag{4}$$

Linearising (4) about a standing wave solution

$$\widehat{\psi}(x,t) = e^{\mathrm{i}\beta t}\phi(x), \qquad \phi(x) \in \mathbb{R}, \qquad \beta \in \mathbb{R},$$

using a complex perturbation

$$\psi(x,t) = \widehat{\psi}(x,t) + \varepsilon(u(x,t) + \mathrm{i}v(x,t))$$

leads to the linearised dynamics in u, v:

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = N \begin{pmatrix} u \\ v \end{pmatrix}$$

where

$$N = \begin{pmatrix} 0 & L_- \\ -L_+ & 0 \end{pmatrix}, \quad \begin{cases} L_- = \partial_{xx} + f(\phi^2) + \beta, \\ L_+ = \partial_{xx} + 2f'(\phi^2)\phi^2 + f(\phi^2) + \beta. \end{cases}$$

Application: The Vakhitov-Kolokolov criterion

Known result:

NLS equation on the **real line**,

$$i\psi_t = \psi_{xx} + f\left(|\psi|^2\right)\psi, \qquad \psi(x,t) : \mathbb{R} \times [0,\infty) \longrightarrow \mathbb{C}$$

Standing wave:

$$\widehat{\psi}(x,t) = e^{i\beta t}\phi(x), \qquad \phi \in L^2(\mathbb{R};\mathbb{R}), \qquad \beta \in \mathbb{R}$$

Theorem (VK criterion)

If $P = 1$ and $Q = 0$ then:

$$\frac{\partial}{\partial\beta}\int_{-\infty}^{\infty} \phi^2(x;\beta)dx > 0 \implies n_+(N) = 1$$

$$\implies \textit{standing wave } \widehat{\psi} \textit{ spectrally unstable}$$

$$\frac{\partial}{\partial\beta}\int_{-\infty}^{\infty} \phi^2(x;\beta)dx < 0 \implies \mathrm{Spec}(N) \subset i\mathbb{R}$$

$$\implies \textit{standing wave } \widehat{\psi} \textit{ spectrally stable}$$

Application: The Vakhitov-Kolokolov criterion

Analogous result for NLS on compact interval:

Concavity of the eigenvalue curve through the top left corner provides an (in)stability criterion!

Lemma

If $P = 0$ *or* $Q = 0$ *then* $\mathrm{Spec}(N) \subset \mathbb{R} \cup i\mathbb{R}$ *and* $n_+(N) = |P - Q - \mathfrak{c}|$.

Theorem (Cox, Curran, Latushkin, Marangell)

For standing waves where $0 \in \mathrm{Spec}(L_-) \setminus \mathrm{Spec}(L_+)$ *and* $P = 1, Q = 0$*:*

$$\begin{aligned}
\operatorname{sign} \ddot{s}(\lambda)|_{(\lambda,s)=(0,1)} > 0 &\implies n_+(N) = 1 \\
&\implies \widehat{\psi} \text{ spectrally unstable} \\
\operatorname{sign} \ddot{s}(\lambda)|_{(\lambda,s)=(0,1)} < 0 &\implies n_+(N) = 0 \implies \mathrm{Spec}(N) \subset i\mathbb{R} \\
&\implies \widehat{\psi} \text{ spectrally stable}
\end{aligned}$$

A similar statement holds when $0 \in \mathrm{Spec}(L_+) \setminus \mathrm{Spec}(L_-)$ *and* $P = 0, Q = 1$.

The Vakhitov-Kolokolov criterion

If $P + Q = 1$ then $\exists$ exactly **one** conjugate point on the left side of the Maslov box (excluding $(\lambda, s) = (0, 1)$). Thus,

$$\text{sign}\, \ddot{s}(\lambda)|_{(\lambda,s)=(0,1)} > 0 \implies n_+(N) = 1 \implies \text{instability}$$
$$\text{sign}\, \ddot{s}(\lambda)|_{(\lambda,s)=(0,1)} < 0 \implies n_+(N) = 0 \implies \text{stability}$$

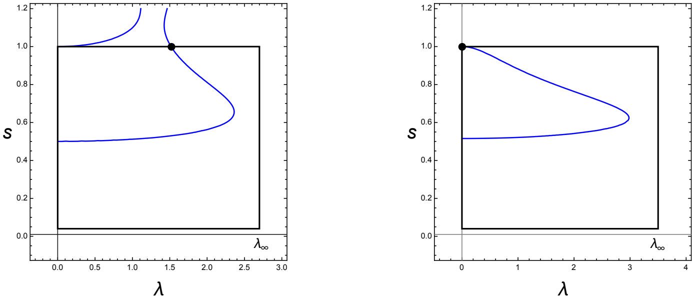
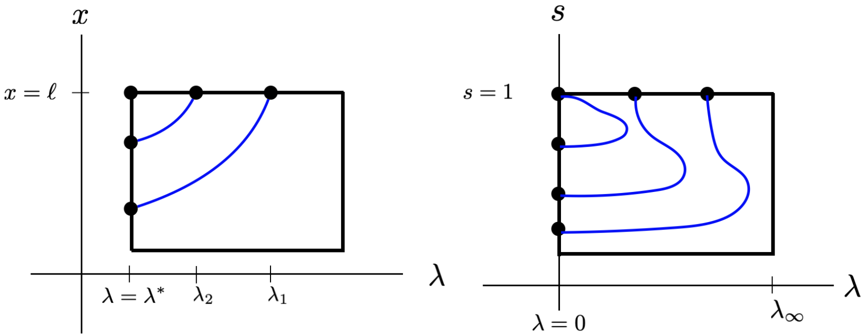


Figure: Two scenarios when $P + Q = 1$. Left: $n_+(N) = 1$. Right: $n_+(N) = 0$.

Thank you.



x
$x = \ell$
λ
$\lambda = \lambda^*$
λ_2
λ_1
s
$s = 1$
λ
$\lambda = 0$
λ_∞