1.3 Local Asymptotics of a discrete Painlevé equation

A similar problem arises in the analysis of solutions of discrete Painlevé equations. Consider the following discrete (or difference) equation

$$\overline{x}\,\underline{x} = b_3\,b_4\,\frac{(x-b_1\,t)\,(x-b_2\,t)}{(x-b_3\,t)\,(x-b_4\,t)} \tag{1.21}$$

where b_j, $j = 1\ldots,4$ are parameters, x is a function of t and we use the notation $\overline{x} = x(q\,t)$, $\underline{x} = x(t/q)$. This equation is often referred to as (symmetric) $\mathrm{qP_{III}}$ because its continuum limit is the differential equation known as the third Painlevé equation

$$y_{ss} = \frac{y_s^2}{y} - \frac{1}{s}\,y + \frac{1}{s}\,(\alpha\,y^2+\beta) + \gamma\,y^2 + \frac{\delta}{y}$$

where y is a function of s and α, β, γ, δ are constant parameters.

If we study Equation (1.21) in the limit $|t| \to \infty$ simultaneously as $|b_j| \to \infty$, we find

$$\overline{w}\,\underline{w} = c\,d\,\frac{(w-a)\,(w-b)}{(w-c)\,(w-d)} \tag{1.22}$$

where we have assumed $b_1\,t^{1/2} \to a$, $b_2\,t^{1/2} \to b$, $b_3\,t^{-1/2} \to c$, $b_4\,t^{-1/2} \to d$m with a, b, c, d being constants to leading order as $|t| \to \infty$.

Consider the bi-quadratic function

$$\begin{aligned} f(X,Y) \;\; := \;\; & \frac{1}{X\,Y}\Big(X^2\,Y^2 - (c+d)\,X\,Y\,(X+Y) + c\,d\,(X^2+Y^2) \\ & -c\,d\,(a+b)\,(X+Y) + a\,b\,c\,d\Big) \end{aligned} \tag{1.23}$$

Exercise 1.3.1. *Show by considering* $f(\overline{w},w) - f(w,\underline{w})$ *that* $f(X,Y)$ *is an invariant of the autonomous equation* (1.22).

Note that the level curves of $f(X,Y)$ are symmetric under the interchange of X and Y and intersect when $(X,Y) = (0,a)$, $(X,Y) = (0,b)$, $(X,Y) = (\infty,c)$ and $(X,Y) = (\infty,c)$. These two observations provide eight points of intersection for the level curves. Once again, the limiting solutions of Equation (1.21) parametrize these curves and we need to know how to continue these solutions through such base points.