1.4 Projective Geometry

Suppose $f(x, y)$ is a polynomial. In this section, we introduce concepts associated with complex curves defined by $f = 0$ in projective space of dimension two.

Definition 1.4.1 (homogeneity)**.** A polynomial $f_k(x, y)$ is called a homogeneous polynomial of degree $k \in \mathbb{N}$ if for all $\lambda \in \mathbb{C}$ it satisfies

$$f_k(\lambda\, x, \lambda\, y) = \lambda^k\, f(x, y). \tag{1.24}$$

Definition 1.4.2 (degree)**.** The *degree* of the algebraic curve $\mathcal{C}$ given by

$$f(x, y) = 0 \tag{1.25}$$

is equal to the degree of f.

Any polynomial can be written as a sum of terms of homogeneous polynomials

$$f(x, y) = f_n(x, y) + f_{n-1}(x, y) + \ldots + f_0, \tag{1.26}$$

where $n \in \mathbb{N}$ is the degree of f and $f_k(x, y)$ is a homogeneous polynomial of degree k, where $k = 0, 1, \ldots, n$.

Let L be a straight line in $\mathbb{C}^2$. Without loss of generality, we assume that it passes through the origin. (If not, we can always transform x, y so that it does so.) That is, we assume L is defined by

$$\begin{cases} x &= \alpha\, t \\ y &= \beta\, t \end{cases} \tag{1.27}$$

for some complex constants α and β.

How many times does L intersect with $\mathcal{C}$? Substituting Equations (1.27) into the equation for the curve, we get

$$f_n(\alpha, \beta)\, t^n + f_{n-1}(\alpha, \beta)\, t^{n-1} + \ldots + f_0 = 0 \tag{1.28}$$

If $f_n(\alpha, \beta) \neq 0$, this equation has exactly n roots in $\mathbb{C}^2$. But, if

$$f_n(\alpha, \beta) = f_{n-1}(\alpha, \beta) = \ldots = f_{m+1}(\alpha, \beta) = 0 \tag{1.29}$$

with $f_m(\alpha, \beta) \neq 0$ for some $0 < m < n$, then there are only m roots in $\mathbb{C}^2$. Where are the remaining roots?

Consider $t = 1/s$. Then Equation (1.28) becomes

$$f_n(\alpha, \beta) + f_{n-1}(\alpha, \beta)\, s + \ldots + f_0\, s^n = 0 \tag{1.30}$$

If Equation (1.29) holds, then we have

$$f_m(\alpha, \beta)\, s^{n-m} + \ldots + f_0\, s^n = 0$$

so, $s = 0$ is a multiple root with multiplicity $n - m$. These intersection points lie at infinity in the original variables (x, y).

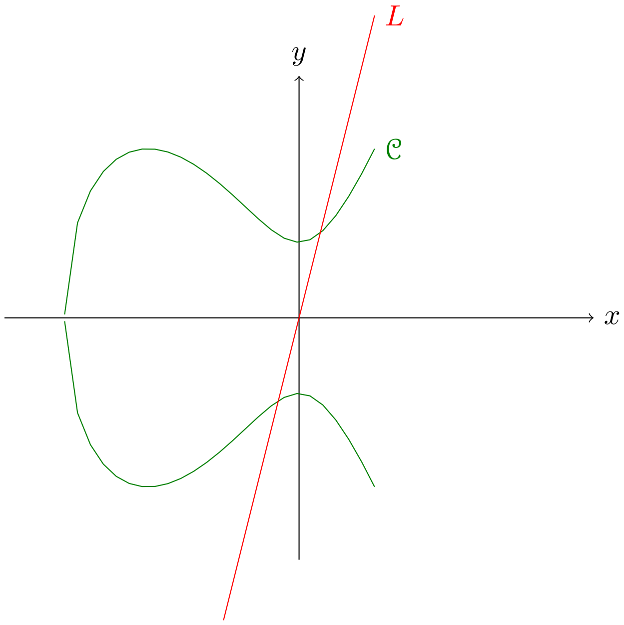

Figure 1.7: Intersections of a curve and a line

Example 1.4.1. Consider a curve $\mathcal{C}$ given by $f = 0$, where $f(x, y) = y^2 + x^2\,y - x^3 + 2\,x$. Then

$$f(x, y) = f_3(x, y) + f_2(x, y) + f_1(x, y),$$

where

$$f_3(x, y) = x^2\,y - \,x^3,\;\; f_2(x, y) = y^2, f_1(x, y) = 2\,x.$$

The line L and the curve $\mathcal{C}$ intersect where

$$\alpha^2\,(\beta - \,\alpha)\,t^3 + \beta^2\,t^2 + 2\,\alpha\,t = 0$$

1. If $\alpha \neq 0$ and $\beta \neq \alpha$, we get three roots: $t = 0$ and t_1, t_2 given by roots of the quadratic $\alpha^2\,(\beta - \,\alpha)\,t^2 + \beta^2\,t + 2\,\alpha = 0$.

2. On the other hand, if $\alpha = \beta$, $\beta \neq 0$, there are only two roots in the finite domain: $t = 0$ and $t_1 = -2/\alpha$. The remaining root lies at infinity. That is, the line $y = x$ intersects $\mathcal{C}$ twice at the origin (i.e., with multiplicity two) and once at infinity.

3. In the case $\beta = 0$, with non-zero α (i.e., the y-axis), we get three finite intersections: the origin and the points $t = \pm\sqrt{2}/\alpha$.

4. Finally, in the case $\alpha = 0$, with non-zero β (i.e., L is the y-axis), we get only one finite root, namely $t = 0$ with multiplicity two. The third intersection point lies at infinity.

To account for and describe roots that lie at infinity, we use coordinates that allow us to describe a line at infinity.

Definition 1.4.3 (homogeneous coordinates)**.** A point $(x, y) \in \mathbb{C}^2$ is said to be represented by *homogeneous coordinates* $[u, v, w]$, when $x = \frac{u}{w}$, $y = \frac{v}{w}$. When the homogeneous coordinates satisfy an equivalence $\sim$ under non-zero multiplication:

$$[u, v, w] = [\lambda\, u, \lambda\, v, \lambda\, w], \quad \forall \lambda \in \mathbb{C} \backslash \{0\}$$

we denote the corresponding quotient space as $\mathbb{CP}^2$, or more succinctly $\mathbb{P}^2$, where

$$\mathbb{CP}^2 = \left(\mathbb{C}^3 \backslash \{[0, 0, 0\}\right) \Big/ {\sim}$$

A point in $\mathbb{P}^2$ is then said to have *affine* coordinates (x, y) or equivalently *homogeneous* coordinates $[u, v, w]$.

Definition 1.4.4 (points at infinity)**.** The point at infinity in the direction u/v is given by $w = 0$. The set of points at infinity in each direction is called the *line at infinity* and denoted L_∞.

If we had started with one-complex-dimension, with affine coordinate $x \in \mathbb{C}$, the equivalent homogeneous coordinates would be $[x, 1] = [u/v, 1] = [u, v]$ and the corresponding projective space would be denoted by $\mathbb{CP}^1$ or $\mathbb{P}^1$, where

$$\mathbb{P}^1 = \left(\mathbb{C}^2 \backslash\ \{[0, 0\}\right) \Big/ {\sim}$$

Note that the direct product $\mathbb{P}^1 \times \mathbb{P}^1$ provides another method of describing a point in two-dimensional complex space. In this case, we have homogeneous coordinates

$$(x, y) \sim \big([x, 1], [y, 1]\big) = \big([u/v, 1], [w/z, 1]\big) = \big([u, v], [w, z]\big) \tag{1.31}$$

In homogeneous coordinates in $\mathbb{P}^2$, a polynomial $f(x, y)$ becomes $F(u, v, w)$ where

$$\begin{aligned} F(u, v, w) &= f_n(u, v) + f_{n-1}(u, v)\, w + \ldots + f_0\, w^n \\ &= w^n\, f(u/w, v/w) \end{aligned}$$

To describe points on a curve, we can use either affine coordinates or homogeneous coordinates. However, it is important to keep in mind that more than one coordinate chart may be needed to describe all possible points on the curve. Below, we use the following three coordinate charts:

$$[x, y, 1] \quad =: \quad [w_{011}, w_{012}, 1] \tag{1.32}$$

$$\left[1, \frac{y}{x}, \frac{1}{x}\right] \quad =: \quad [1, w_{022}, w_{021}] \tag{1.33}$$

$$\left[\frac{x}{y}, 1, \frac{1}{y}\right] \quad =: \quad [w_{031}, 1, w_{032}] \tag{1.34}$$

Consider the Weierstrass curve (1.4) again, now in $\mathbb{P}^2$. In these charts, the equation for the curve becomes respectively

$$\begin{aligned} w_{012}^2 &= 4\,w_{011}^3 - g_2\,w_{011} - g_3 \\ w_{021}\,w_{022}^2 &= 4 - g_2\,w_{021}^2 - g_3 w_{021}^3 \\ w_{032} &= 4\,w_{031} - g_2\,w_{031}\,w_{032}^2 - g_3\,w_{032}^3 \end{aligned} \tag{1.35}$$

On the other hand, the ODE (1.1) or its equivalent first-order system becomes

$$\begin{aligned} &\begin{cases} w_{012}' &= 6\,w_{011}^2 - \frac{g_2}{2} \\ w_{011}' &= w_{012} \end{cases} \\ &\begin{cases} w_{021}' &= -\,w_{021}\,w_{022} \\ w_{022}' &= \dfrac{6}{w_{021}} - \frac{g_2}{2}\,w_{021} - w_{022}^2 \end{cases} \\ &\begin{cases} w_{031}' &= 1 - 6\,\dfrac{w_{031}^3}{w_{032}} + \frac{g_2}{2}\,w_{031}\,w_{032} \\ w_{032}' &= 6\,w_{031}^2 + \frac{g_2}{2}\,w_{032}^2 \end{cases} \end{aligned} \tag{1.36}$$

Note that the right side of the first equation in the system (1.36) in the $(0,3)$ chart is not well defined when $w_{031} = 0$, $w_{032} = 0$ simultaneously.

Consider the equation for the curve in the same chart. Rewriting the pencil of curves given by Equation (1.35) as

$$g_3\,w_{032}^3 = 4\,w_{031} - g_2\,w_{031}\,w_{032}^2 - \,w_{032} \tag{1.37}$$

it is easy to see that an intersection point of all of these curves (which are each given by choosing a different value of g_3) occurs only if we have

$$w_{032}^3 = 0 \quad \text{and} \quad 4\,w_{031} - g_2\,w_{031}\,w_{032}^2 - w_{032} = 0$$

which implies $(w_{031}, w_{032}) = (0,0)$. This is a *base point* of this pencil. Note that this point lies at infinity in the original coordinates.

That is, the problematic point where the definition of a solution of the system (1.36) fails is equivalent to a base point of the pencil of curves that arises as an integral of the system.

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