Pricing European Barrier Options

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Abstract

A new method is described to price barrier options which incorporate a constant rebate. The method exploits the symmetries and properties of elementary solutions of the Black-Scholes partial differential equation. The rebate and non-rebate terms obtained agree with other published solutions, but are obtained without recourse to a single transformation or integration. The complexity of the solution methods previously published are shown to be completely unnecessary.

1 Introduction

Barrier options are a class of exotic options which were first priced by Merton (1973). There are two fundamentally different ways of pricing barrier options. They are the expectations method and the differential equation method. The former has been worked out in detail by Rubinstein and Reiner (1991) and also Rich (1994). The expectations approach requires the determination of the risk-neutral densities of the underlying asset price as it breaches the barrier from above and below. If rebates apply then the first exit time densities through the barrier are also required. Barrier option prices are then obtained, in the usual way, by integrating the discounted barrier option pay-off functions over the calculated densities. These densities are difficult to work out and require repeated use of the reflection principle (Harrison (1985)). Considering the complexities encountered in the expectations approach, it is remarkable that closed form solutions for all barrier option types are in fact obtained.

The differential equation method has not been as widely published. A brief discussion can be found in Wilmott *et al* (1993). The basic idea is that all barrier options satisfy the Black-Scholes partial differential equation but with different domains, expiry conditions and boundary conditions. In principle, these pde's can be transformed to the diffusion equation and solved by the method of images. Once again the analysis is complex and also requires the evaluation of complicated integrals, but the same closed form solutions are obtained. The method of images for the pde solution is of course related to the reflection principle of the expectations solution.

Ritchken (1995) has investigated computational aspects of barrier option pricing using binomial and trinomial lattices.

The pde method will be adopted here to show that a direct and simple analysis leads to the closed form solutions without the need for any density calculations, pde transformations or even any integrations. The method employs symmetry properties of the Black-Scholes (BS) pde and requires little more than the well-known basic European vanilla option solutions.

There are eight types of barrier option: the down-and-out; down-and-in; up-and-out; up-and-in; each being of either the call-type (right to buy) or put-type (right to sell). Knock-out options may pay a rebate if and when the asset price hits the barrier; knock-in options will pay the rebate at expiry only if the asset price fails to hit the barrier. If the asset price hits the barrier before expiry, then the knock-in is converted to a vanilla option of the corresponding type. Down-options infer that the initial asset price is greater than the barrier price; up-options infer that the initial asset price is below the barrier price.

1.1 Parameter Definitions

The following parameters and variables are defined in the usual Black-Scholes framework, in which the asset price is assumed to follow standard geometric Brownian motion with constant volatility. The rebate is also assumed to be a fixed constant.

$$
\begin{aligned}
x &= \text{underlying asset price} \\
t &= \text{time remaining to expiry} \\
y(x,t) &= \text{general barrier option price} \\
r &= \text{risk-free interest rate} \\
\sigma &= \text{asset price volatility} \\
\alpha &= \frac{2r}{\sigma^2} - 1 \\
a &= \text{strike price} \\
b &= \text{asset barrier price} \\
R &= \text{rebate} \\
\mathcal{L}y &= y_t + ry - rxy_x - \tfrac{1}{2}\sigma^2 x^2 y_{xx} \quad \text{(BS-operator)} \\
C_a(x,t) &= \text{vanilla call option price; satisfies } \mathcal{L}C_a = 0; \quad C_a(x,0) = (x-a)^+ \\
P_a(x,t) &= \text{vanilla put option price; satisfies } \mathcal{L}P_a = 0; \quad P_a(x,0) = (a-x)^+ \\
Y_a(x,t) &= \text{denotes either vanilla option price; } C_a \text{ for a call or } P_a \text{ for a put}
\end{aligned}
$$

2 PDE's for Barrier Options

All barrier option prices satisfy the BS-pde: $\mathcal{L}y = 0$. They differ only in their active domain (AD), expiry condition (EC) and boundary condition (BC). Actually the expiry condition is an initial condition for the pde, but the former name will be used to remind the reader of its financial significance.

Let $\mathcal{A}_+$ and $\mathcal{A}_-$ denote the domains $x > b$ and $x < b$ respectively. Then the active domain is defined to be the region in which the barrier option has not been converted (into a rebate for the knock-outs, or a vanilla option for the knock-ins). Thus down-barrier options have the active domain $\mathcal{A}_+$, whereas up-barrier options have active domain $\mathcal{A}_-$. In addition, out-barrier options will pay the rebate R if $x = b$ and the standard pay-off $Y_a(x,0)$ if it survives to expiry. On the other hand, in-barrier options will pay the rebate R only at expiry and will convert to the vanilla option value $Y_a(x,t)$ when $x = b$. In summary:

Table 1

Barrier	AD	EC ($t = 0$)	BC ($x = b$)
Down-Out	$\mathcal{A}_+$	$Y_a(x,0)$	R
Up-Out	$\mathcal{A}_-$	$Y_a(x,0)$	R
Down-In	$\mathcal{A}_+$	R	$Y_a(b,t)$
Up-In	$\mathcal{A}_-$	R	$Y_a(b,t)$

Theorem 1 *Since $\mathcal{L}$ is a linear operator all barrier options prices can be decomposed into two components:*

$$Y(x,t) = Y(rebate) + Y(non\text{-}rebate). \tag{1}$$

Theorem 1 follows from the representation given in Table-2 below:

Table 2

Barrier Option	AD	Rebate Term EC	Rebate Term BC	Non-Rebate Term EC	Non-Rebate Term BC
Down-Out	$\mathcal{A}_+$	0	R	$Y_a(x,0)$	0
Up-Out	$\mathcal{A}_-$	0	R	$Y_a(x,0)$	0
Down-In	$\mathcal{A}_+$	R	0	0	$Y_a(b,t)$
Up-In	$\mathcal{A}_-$	R	0	0	$Y_a(b,t)$

It is therefore possible (and highly desirable) to solve for the rebate and non-rebate terms separately.

3 Solutions of $\mathcal{L}y = 0$

We develop in this section two classes of solutions of the BS-pde. The first, termed *elementary solutions*, are just the building blocks of vanilla option solutions. The second, termed *image solutions*, provide the means for solving the barrier option prices without recourse to complicated integrations.

3.1 Elementary Solutions

Let $\xi > 0$ be a parameter (later it will denote either the strike price a or the barrier price b) and define

$$z_\xi = \frac{\log(x/\xi) + (r + \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}; \quad z'_\xi = z_\xi - \sigma\sqrt{t} \tag{2}$$

We then have the four elementary solutions of the BS-pde, which together with their expiry values are:

$$\left.\begin{array}{rclrcl} U_\xi(x,t) &=& x\mathcal{N}(z_\xi); & U_\xi(x,0) &=& x\theta(x-\xi) \\ \bar{U}_\xi(x,t) &=& x\mathcal{N}(-z_\xi); & \bar{U}_\xi(x,0) &=& x\theta(\xi-x) \\ V_\xi(x,t) &=& e^{-rt}\mathcal{N}(z'_\xi); & V_\xi(x,0) &=& \theta(x-\xi) \\ \bar{V}_\xi(x,t) &=& e^{-rt}\mathcal{N}(-z'_\xi); & \bar{V}_\xi(x,0) &=& \theta(\xi-x) \end{array}\right\} \tag{3}$$

where $\mathcal{N}(z)$ is the usual normal distribution function and $\theta(x)$ denotes the unit step-function (1 if $x > 0$; 0 if $x < 0$).

Theorem 2 *For any* $\xi > 0$

$$\left.\begin{array}{rcl} U_\xi(x,t) + \bar{U}_\xi(x,t) & = & x \\ V_\xi(x,t) + \bar{V}_\xi(x,t) & = & e^{-rt} \end{array}\right\} \tag{4}$$

The proof follows from the symmetry property $\mathcal{N}(-z) = 1 - \mathcal{N}(z)$.

3.2 Image Solutions

The following definition and theorem are fundamental to the solution method to be decribed. In that sense, they are the main results of this paper.

Definition 1 *Let* $u(x,t)$ *be any solution of the BS-pde. Then the image of* $u(x,t)$ *relative to the barrier* $x = b$ *is defined to be the function*

$$u^*(x,t) = \left(\frac{b}{x}\right)^\alpha u\left(\frac{b^2}{x}, t\right). \tag{5}$$

Theorem 3 *If $u(x,t)$ and $u^*(x,t)$ are defined as above the following are true:*

1. $(u^*)^* = u$
2. *If* $\mathcal{L}u = 0$ *then* $\mathcal{L}u^* = 0$ *for any* $b > 0$
3. $u = u^*$ *when* $x = b$
4. $y(x,t) = u - u^*$ *satisfies* $\mathcal{L}y = 0$ *and the BC* $y(b,t) = 0$
5. *If the active domain of* u *is* $\mathcal{A}_\pm$ *then the active domain of* u^* *is* $\mathcal{A}_\mp$

The proof of Theorem 3 is elementary.

It follows from this theorem, that the four basic solutions $(U, \bar{U}, V, \bar{V})_\xi$ give rise to four new image solutions $(U^*, \bar{U}^*, V^*, \bar{V}^*)_\xi$ with the BC's:

$$U_\xi^* = U_\xi;\ \bar{U}_\xi^* = \bar{U}_\xi;\ V_\xi^* = V_\xi;\ \bar{V}_\xi^* = \bar{V}_\xi \quad \text{when } x = b.$$

It transpires that all barrier option prices, regardless of type, can be expressed analytically in terms of these eight fundamental solutions of the BS-pde.

The table below summarises the expiry values at $t = 0$ of all eight solutions for the case $\xi = b$:

Table 3

Basic Solution	EC$(t=0)$ $\mathcal{A}_+(x > b)$	EC$(t=0)$ $\mathcal{A}_-(x < b)$
U_b	x	0
$\bar{U}_b$	0	x
U_b^*	0	$\gamma(x)$
$\bar{U}_b^*$	$\gamma(x)$	0
V_b	1	0
$\bar{V}_b$	0	1
V_b^*	0	$\delta(x)$
$\bar{V}_b^*$	$\delta(x)$	0

where

$$\delta(x) = (b/x)^\alpha \qquad \text{and} \qquad \gamma(x) = (b^2/x)\delta(x). \tag{6}$$

Two symmetry properties of the BS-solutions, stated next, help considerably in barrier option evaluations.

Theorem 4 *For the out-rebate terms only:*

$$Y^*_{DO}(x,t) = Y_{UO}(x,t); \qquad Y^*_{UO}(x,t) = Y_{DO}(x,t) \tag{7}$$

and for the in-non-rebate terms only:

$$Y^*_{DI}(x,t) = Y_{UI}(x,t); \qquad Y^*_{UI}(x,t) = Y_{DI}(x,t) \tag{8}$$

The proof follows directly from the pde, and its EC and BC for each of the terms given in Table 2 and the properties of the image solution stated in Theorem 3.

4 The Rebate Terms

The complete set of rebate terms for barrier option prices (valid for both call and puts) are:

$$\left.\begin{array}{rcll} Y_{DO}(x,t) &=& \dfrac{R}{b}\left[U^*_b(x,t) + \bar{U}_b(x,t)\right] & \text{in } \mathcal{A}_+ \\ Y_{UO}(x,t) &=& \dfrac{R}{b}\left[U_b(x,t) + \bar{U}^*_b(x,t)\right] & \text{in } \mathcal{A}_- \\ Y_{DI}(x,t) &=& R\left[V_b(x,t) - V^*_b(x,t)\right] & \text{in } \mathcal{A}_+ \\ Y_{UI}(x,t) &=& R\left[\bar{V}_b(x,t) - \bar{V}^*_b(x,t)\right] & \text{in } \mathcal{A}_- \end{array}\right\} \tag{9}$$

Observe that from Table 3, the barrier options satisfy the correct expiry conditions at $t = 0$ and from Theorems 2 and 3.4, reduce to the correct boundary values at $x = b$.

5 The Non-Rebate Terms

The evaluation of these terms is a little more complicated than for the corresponding rebate terms. First we define four new solutions which are just special linear combinations of the eight fundamental solutions described in previous sections. Let

$$\left.\begin{array}{rcr} C_\xi(x,t) &=& U_\xi(x,t) - aV_\xi(x,t) \\ P_\xi(x,t) &=& -\bar{U}_\xi(x,t) + a\bar{V}_\xi(x,t) \\ C^*_\xi(x,t) &=& U^*_\xi(x,t) - aV^*_\xi(x,t) \\ P^*_\xi(x,t) &=& -\bar{U}^*_\xi(x,t) + a\bar{V}^*_\xi(x,t) \end{array}\right\} \tag{10}$$

The last pair are image solutions of the first pair and the symbols C and P indicate that they are call-like and put-like solutions. Indeed, it is clear that when $\xi = a$ we get the standard vanilla call and put option values $C_a(x,t)$ and $P_a(x,t)$ for strike price a.

The table of expiry values for these new solutions when $\xi = (a,b)$ is given below:

Table 4

	EC($t=0$) in $\mathcal{A}_+(x>b)$		EC($t=0$) in $\mathcal{A}_-(x<b)$	
	$b<a$	$b>a$	$b<a$	$b>a$
C_a	$(x-a)^+$	$(x-a)^{(+)}$	0	$(x-a)^+$
C_b	$(x-a)$	$(x-a)^{(+)}$	0	0
C_a^*	0	$\lambda_-(x)$	$\lambda_-(x)$	$\lambda(x)$
C_b^*	0	0	$\lambda(x)$	$\lambda(x)$
P_a	$(a-x)^+$	0	$(a-x)^{(+)}$	$(a-x)^+$
P_b	0	0	$(a-x)^{(+)}$	$(a-x)$
P_a^*	$-\lambda(x)$	$-\lambda_+(x)$	$-\lambda_+(x)$	0
P_b^*	$-\lambda(x)$	$-\lambda(x)$	0	0

In this table we have defined:

$$\lambda(x) = \gamma(x) - a\delta(x) = (b/x)^\alpha (b^2/x - a) \tag{11}$$

and

$$\lambda_-(x) = \lambda(x)\theta(b^2/a - x); \qquad \lambda_+(x) = \lambda(x)\theta(x - b^2/a). \tag{12}$$

We have also introduced the notation: $(x-a)^{(+)}$ to mean that $(x-a)^+ = (x-a)$ in the domain indicated (i.e. some subset of $x > a$).

The pricing of the non-rebate terms is considerably simplified with the following:

Theorem 5 *For the non-rebate terms only we have the identities:*

$$\left.\begin{array}{rcl} Y_{DO}(x,t) + Y_{DI}(x,t) &=& Y_a(x,t) \;\; in \;\; \mathcal{A}_+ \\ Y_{UO}(x,t) + Y_{UI}(x,t) &=& Y_a(x,t) \;\; in \;\; \mathcal{A}_- \\ Y_{DO}(x,t) + Y_{UI}^*(x,t) &=& Y_a(x,t) \;\; in \;\; \mathcal{A}_+ \\ Y_{UO}(x,t) + Y_{DI}^*(x,t) &=& Y_a(x,t) \;\; in \;\; \mathcal{A}_- \end{array}\right\} \tag{13}$$

The proof follows immediately by checking the active domains, expiry and boundary conditions for the solutions on the left hand side of the equations and showing that they reduce to the standard conditions for a vanilla option with strike price a.

The first two identities in Theorem 5 are well known. But the other two appear not to have been noticed before and also derive from the second part of Theorem 4. Their importance lies in the observation that given any one of the four basic barrier option prices, the other three are immediately determined. The results of Theorem 5 are valid for both call-barrier and put-barrier options

The following results are now obtained for the call and put non-rebate terms:

Table 5

Barrier Option	AD	Case $b < a$	Case $b > a$
$C_{DO}(x,t)$	$\mathcal{A}_+$	$C_a - C_a^*$	$C_b - C_b^*$
$C_{UO}(x,t)$	$\mathcal{A}_-$	0	$C_a - C_a^* - (C_b - C_b^*)$
$C_{DI}(x,t)$	$\mathcal{A}_+$	C_a^*	$C_a - (C_b - C_b^*)$
$C_{UI}(x,t)$	$\mathcal{A}_-$	C_a	$C_a^* + C_b - C_b^*$
$P_{DO}(x,t)$	$\mathcal{A}_+$	$P_a - P_a^* - (P_b - P_b^*)$	0
$P_{UO}(x,t)$	$\mathcal{A}_-$	$P_b - P_b^*$	$P_a - P_a^*$
$P_{DI}(x,t)$	$\mathcal{A}_+$	$P_a^* + P_b - P_b^*$	P_a
$P_{UI}(x,t)$	$\mathcal{A}_-$	$P_a - (P_b - P_b^*)$	P_a^*

It is quite remarkable that these solutions can be obtained in such a simple and direct fashion. This simplicity has also made the symmetry in the solutions considerably more transparent than has hitherto been published. To appreciate this last point, one only needs to compare the solutions in Table 5 with those of Rubinstein and Reiner (1991) and Rich (1994). Yet after tedious calculations, they are found to be identical.

One final point to make is the observation that not all the terms appearing in Table 5 are independent. In fact the relations:

$$\left.\begin{array}{rcl} C_a - P_a & = & C_b - P_b \\ C_a^* - P_a^* & = & C_b^* - P_b^* \end{array}\right\} \quad (14)$$

are easily derived and correspond to put-call parity relations. These relations are in fact required to demonstrate the equivalence of the solutions obtained here with those of the above mentioned works.

6 References

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