BLOCKS OF AFFINE AND CYCLOTOMIC HECKE ALGEBRAS

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ABSTRACT. This paper classifies the blocks of the affine Hecke algebras of type A and the blocks of the cyclotomic Hecke algebras of type $G(r,1,n)$ over an arbitrary algebraically closed field. Rather than working with the Hecke algebras directly we work instead with the cyclotomic Schur algebras. The advantage of these algebras is that the cyclotomic Jantzen sum formula gives an easy combinatorial characterization of the blocks of the cyclotomic Schur algebras. We obtain an explicit description of the blocks by analyzing the combinatorics of 'Jantzen equivalence'.

We remark that a proof of the classification of the blocks of the cyclotomic Hecke algebras was announced in 1999. Unfortunately, Cox has discovered that this previous proof is incomplete.

1. INTRODUCTION

The affine Hecke algebras arise naturally in representation theory of reductive p–adic groups as well as having applications to the representation theory of semisimple algebraic groups in positive characteristic and to quantum groups at roots of unity. These algebras can be defined geometrically using the K–theory of the Steinberg variety. This leads to explicit formulae for the decomposition numbers in terms of Kazhdan–Lusztig polynomials when q is a complex root of unity; see [**10**, Theorem 8.6.23].

This paper is concerned with the affine Hecke algebra of the general linear group $\mathscr{H}_n^{\mathrm{aff}}$, which is also known as the extended affine Hecke algebra of type A_{n-1}. Let $\mathbb{F}$ be a field. Then, using the Bernstein presentation, $\mathscr{H}_n^{\mathrm{aff}}$ can be written as a twisted tensor product $\mathscr{H}_q(\mathfrak{S}_n)\otimes\mathbb{F}[X_1^{\pm},\dots,X_n^{\pm}]$ of the Iwahori–Hecke algebra $\mathscr{H}_q(\mathfrak{S}_n)$ of the symmetric group and the Laurent polynomial ring $\mathbb{F}[X_1^{\pm},\dots,X_n^{\pm}]$.

If A is an algebra then two simple A–modules D and D' belong to same **block** if there exist simple A–modules $D=D_1,D_2,\dots,D_k=D'$ such that either $\operatorname{Ext}^1_A(D_i,D_{i+1})\neq 0$ or $\operatorname{Ext}^1_A(D_{i+1},D_i)\neq 0$, for $1\le i<k$. More generally, two A–modules M and N belong to the same block if all of their composition factors belong to the same block.

By a well–known theorem of Bernstein [**22**, Prop. 3.11], the centre of $\mathscr{H}_n^{\mathrm{aff}}$ is the set $\mathbb{F}[X_1^{\pm},\dots,X_n^{\pm}]^{\mathfrak{S}_n}$ of symmetric Laurent polynomials in $X_1,\dots,X_n$. It is not difficult to show that any $\mathscr{H}_n^{\mathrm{aff}}$–module decomposes as an $\mathscr{H}_n^{\mathrm{aff}}$–module into a direct sum of generalized eigenspaces for the central characters of $\mathscr{H}_n^{\mathrm{aff}}$. As $\mathscr{H}_n^{\mathrm{aff}}$ is finite dimensional over its centre every irreducible $\mathscr{H}_n^{\mathrm{aff}}$–module is finite dimensional and, in particular, has a central character by Schur's Lemma. It follows that any two $\mathscr{H}_n^{\mathrm{aff}}$–modules which are in the same block have the same central character.

Theorem A. *Suppose that $\mathbb{F}$ is algebraically closed and that $q\neq 1$. Let D and D' be two $\mathscr{H}_n^{\mathrm{aff}}$–modules. Then D and D' belong to the same block if and only if they have the same central character.*

The centre of $\mathscr{H}_n^{\mathrm{aff}}$ is well understood, however, as far as we know, Theorem A is new. When $q\neq 1$ the simple $\mathscr{H}_n^{\mathrm{aff}}$–modules have been classified in terms of 'content functions' on aperiodic multisegments; see [**4**, Theorem B] for a precise statement. As a consequence the values of the central characters on the simple $\mathscr{H}_n^{\mathrm{aff}}$–modules are easy to compute.

To prove Theorem A we do not work with the affine Hecke itself, but rather with certain natural quotients of $\mathscr{H}_n^{\mathrm{aff}}$ which are known as the Ariki–Koike algebras, or the cyclotomic Hecke algebras of

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type $G(r,1,n)$. These algebras appear first in the work of Cherednik [**9**], however, their properties were first systematically studied by Ariki and Koike [**3**] and Broué and Malle [**7**]. Apart from being interesting in their own right, these algebras are central to the conjectures of Broué, Malle and Michel [**6**] which attempt to understand Broué's abelian defect group conjecture for the finite groups of Lie type.

If $\mathbf{Q} = (Q_1,\dots,Q_r) \in (\mathbb{F}^\times)^r$ then the Ariki–Koike algebra $\mathscr{H}_{r,n}(q,\mathbf{Q})$ is the quotient algebra $\mathscr{H}_n^{\mathrm{aff}}/\langle (X_1-Q_1)\dots(X_1-Q_r)\rangle$. Consequently, every irreducible $\mathscr{H}_{r,n}(q,\mathbf{Q})$–module can be considered as an irreducible $\mathscr{H}_n^{\mathrm{aff}}$–module. Conversely, by quotienting out by the characteristic polynomial of X_1, every irreducible $\mathscr{H}_n^{\mathrm{aff}}$–module is an irreducible module for some Ariki–Koike algebra. Deep results of Ariki [**2**] and Grojnowski [**17**] show that the module categories of the affine Hecke algebras and the Ariki–Koike algebras are intimately intertwined.

The natural surjection $\mathscr{H}_n^{\mathrm{aff}} \longrightarrow \mathscr{H}_{r,n}$ shows that if D and D' are in the same block as $\mathscr{H}_{r,n}$–modules then they are in the same block as $\mathscr{H}_n^{\mathrm{aff}}$–modules. The second result of this paper shows that the blocks of the Ariki–Koike algebras are determined by the affine Hecke algebra.

Theorem B. *Suppose that $\mathbb{F}$ is an algebraically closed field and that $q \neq 1$. Let D and D' be irreducible modules for the Ariki–Koike algebra $\mathscr{H}_{r,n}(q,\mathbf{Q})$. Then D and D' belong to the same block as $\mathscr{H}_{r,n}(q,\mathbf{Q})$–modules if and only if they belong to the same block as $\mathscr{H}_n^{\mathrm{aff}}$–modules.*

We also classify the blocks of the Ariki–Koike algebras when $q=1$ and when some of the parameters $Q_1,\dots,Q_r$ are zero. Motivated by Theorem A we can give an explicit combinatorial criterion for two $\mathscr{H}_{r,n}(q,\mathbf{Q})$–modules to belong to the same block, and it is this statement that we actually prove. See Theorem 2.11 for the precise statement. With this in hand, we then deduce Theorem A from Theorem B.

Observe that the Theorem B is equivalent to the following property of the blocks of $\mathscr{H}_n^{\mathrm{aff}}$.

Corollary. *Suppose that $q \neq 1$ and let D and D' be two simple $\mathscr{H}_{r,n}(q,\mathbf{Q})$–modules. Then D and D' belong to the same block as $\mathscr{H}_n^{\mathrm{aff}}$–modules if and only if there exist simple $\mathscr{H}_{r,n}(q,\mathbf{Q})$–modules $D = D_1, D_2, \dots, D_k = D'$ such that either*

$$\operatorname{Ext}^1_{\mathscr{H}_n^{\mathrm{aff}}}(D_i, D_{i+1}) \neq 0 \quad \textit{or} \quad \operatorname{Ext}^1_{\mathscr{H}_n^{\mathrm{aff}}}(D_{i+1}, D_i) \neq 0,$$

for $1 \le i < k$.

In 1999 Grojnowski [**18**] announced a proof of Theorem B. Using an ingenious argument, what Grojnowski actually proves is that

$$\operatorname{Ext}^1_{\mathscr{H}_n^{\mathrm{aff}}}(D,D') = \operatorname{Ext}^1_{\mathscr{H}_{r,n}(q,\mathbf{Q})}(D,D')$$

whenever $D \neq D'$ are simple $\mathscr{H}_{r,n}(q,\mathbf{Q})$–modules. Unfortunately, as Anton Cox [**11**] has pointed out, this is not enough to classify the blocks of the Ariki–Koike algebras. For example, it could happen that there are no $\mathscr{H}_n^{\mathrm{aff}}$–module extensions between different $\mathscr{H}_{r,n}(q,\mathbf{Q})$–modules which belong to the same block as $\mathscr{H}_n^{\mathrm{aff}}$–modules. We note that Grojnowski's result does not follow from Theorem B.

Lusztig [**22**] introduced a graded, or degenerate, Hecke algebra for each affine Hecke algebra. Brundan [**8**] has shown that the centre of the degenerate affine Hecke algebra maps onto the centre of the degenerate cyclotomic Hecke algebras. This gives a classification of the blocks of the degenerate cyclotomic and affine Hecke algebras analogous to our Theorems A and B above. It should be possible to use the arguments from this paper to classify the blocks of the degenerate cyclotomic Hecke algebras of type $G(r,1,n)$ and the associated degenerate cyclotomic Schur algebras. All of the combinatorics that we use goes through without change, however, it is necessary to check that arguments of [**21**] can be adapted to prove a sum formula for the Jantzen filtrations of the degenerate cyclotomic Schur algebras. This should be routine (cf. [**5**, §6]), however, we have not checked the details.

The outline of this paper is as follows. In the next section we introduce the Ariki–Koike algebras and the cyclotomic q–Schur algebras. Using the representation theory of these two algebras, we reduce the proof of Theorem B to a purely combinatorial problem of showing that two equivalence relations on the set of multipartitions coincide (Theorem 2.11). The first of these equivalence relations comes from the cyclotomic Jantzen sum formula [**21**], and the second equivalence relation is the combinatorial criterion which classified the central characters the affine Hecke algebras. Assuming Theorem 2.11 we prove Theorem A and Theorem B at the end of section 2. In section 3 we develop the combinatorial machinery needed to show that our two equivalence relations on the set of multipartitions coincide when $q \neq 1$ and when the parameters $Q_1, \dots, Q_r$ are non–zero. Here we are greatly aided by recent work of Fayers [**14, 15**] on the 'core block' of a multipartition. Finally, in section 4 we consider the blocks of the Ariki–Koike algebras with 'exceptional' parameters; that is, those algebras with $q = 1$ or with some of the parameters $Q_1, \dots, Q_r$ being zero. Quite surprisingly, the algebras with 'exceptional parameters' have only a single block (unless $q = 1$ and $r = 1$).

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2. Cyclotomic Hecke algebras and Schur algebras

This section begins by introducing the cyclotomic Hecke algebras and Schur algebras. We then reduce the proof of Theorem B to a purely combinatorial statement which amounts to showing that two equivalence relations on the set of multipartitions coincide.

2.1. **Ariki–Koike algebras.** Let $\mathbb{F}$ be a field of characteristic $p \in \{2, 3, \dots\} \cup \{\infty\}$ and fix positive integers n and r. Suppose that $q, Q_1, \dots Q_r$ are elements of $\mathbb{F}$ such that q is invertible and let $\mathbf{Q} = (Q_1, \dots, Q_r)$. The **Ariki–Koike algebra** $\mathscr{H}_{r,n} = \mathscr{H}_{r,n}(q, \mathbf{Q})$ is the unital associative $\mathbb{F}$–algebra with generators $T_0, T_1, \dots, T_{n-1}$ and relations

$$\begin{aligned}
(T_i + q)(T_i - 1) &= 0, && 1 \le i \le n-1,\\
(T_0 - Q_1)\dots(T_0 - Q_r) &= 0, &&\\
T_iT_j &= T_jT_i, && 0 \le i < j-1 \le n-2,\\
T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1}, && 1 \le i \le n-2,\\
T_0T_1T_0T_1 &= T_1T_0T_1T_0. &&
\end{aligned}$$

Define $e \ge 2$ to be minimal such that $1 + q + \ldots + q^{e-1} = 0 \in \mathbb{F}$. Then $e \in \{2, 3, \ldots\} \cup \{\infty\}$. Note that $e = p$ if and only if $q = 1$. If $e \neq p$ and p is finite then p does not divide e.

Recall that a partition $\lambda = (\lambda_1, \lambda_2, \dots)$ of n is a weakly decreasing sequence of non–negative integers which sum to $|\lambda| = n$. An r–multipartition of n, or more simply a multipartition, is an ordered r–tuple $\boldsymbol{\lambda} = \big(\lambda^{(1)}, \dots, \lambda^{(r)}\big)$ of partitions with $|\boldsymbol{\lambda}| = |\lambda^{(1)}| + \dots + |\lambda^{(r)}| = n$. Let $\Lambda^+_{r,n}$ be the set of multipartitions of n. We regard a partition as a multipartition with one component, so any subsequent definition concerning multipartitions specializes to a corresponding definition for partitions.

The set of multipartitions is naturally ordered by **dominance** where $\boldsymbol{\lambda} \trianglerighteq \boldsymbol{\mu}$ if

$$\sum_{t=1}^{s-1} |\lambda^{(t)}| + \sum_{j=1}^{i} \lambda^{(s)}_j \ge \sum_{t=1}^{s-1} |\mu^{(t)}| + \sum_{j=1}^{i} \mu^{(s)}_j$$

for $s = 1, 2, \dots, r$ and all $i \ge 1$. We write $\boldsymbol{\lambda} \triangleright \boldsymbol{\mu}$ if $\boldsymbol{\lambda} \trianglerighteq \boldsymbol{\mu}$ and $\boldsymbol{\lambda} \neq \boldsymbol{\mu}$.

The Ariki–Koike algebra $\mathscr{H}_{r,n}$ is a cellular algebra [**12, 16**]. The cell modules of $\mathscr{H}_{r,n}$ are indexed by the multipartitions of n. The cell module indexed by the multipartition $\boldsymbol{\lambda}$ is the **Specht module** $S(\boldsymbol{\lambda})$. By the theory of cellular algebras [**16, 23**], there is an $\mathscr{H}_{r,n}$–invariant bilinear form $\langle\ ,\ \rangle_{\boldsymbol{\lambda}}$ on the Specht module $S(\boldsymbol{\lambda})$, so the radical $\operatorname{rad} S(\boldsymbol{\lambda}) = \{\, x \in S(\boldsymbol{\lambda}) \mid \langle x, y\rangle_{\boldsymbol{\lambda}} = 0 \text{ for all } y \in S(\boldsymbol{\lambda}) \,\}$ is an $\mathscr{H}_{r,n}$–submodule of $S(\boldsymbol{\lambda})$. Set $D(\boldsymbol{\lambda}) = S(\boldsymbol{\lambda})/\operatorname{rad} S(\boldsymbol{\lambda})$. Then the non–zero $D(\boldsymbol{\lambda})$ give a complete set of pairwise non–isomorphic simple $\mathscr{H}_{r,n}$–modules.

The theory of cellular algebras gives us the following fact which is vital for this paper because it allows us work with Specht modules rather than with the simple $\mathscr{H}_{r,n}$–modules.

2.1. **Lemma** (Graham–Lehrer [**16**, 3.9.8], [**23**, Cor. 2.2])**.** *Suppose that $\boldsymbol{\lambda}$ is a multipartition. Then all of the composition factors of $S(\boldsymbol{\lambda})$ belong to the same block.*

Equivalently, if we decompose $\mathscr{H}_{r,n}$ into a direct sum of indecomposable subalgebras then exactly one of these subalgebras has a non–zero action on $S(\boldsymbol{\lambda})$. Thus we can talk of the block of $\mathscr{H}_{r,n}$ which contains the Specht module $S(\boldsymbol{\lambda})$.

2.2. **Cyclotomic q–Schur algebras.** Rather than working with Specht modules to classify the blocks we want to work with Weyl modules. To this end let $\{\, L_1^{a_1}\dots L_n^{a_n}T_w \mid 0\le a_i<r \text{ and } w\in\mathfrak{S}_n\,\}$ be the standard basis of $\mathscr{H}_{r,n}$ [**3**, Prop. 3.4]. That is, $L_1 = T_0$ and $L_{i+1} = q^{1-i}T_iL_iT_i$, for $1 < i < n$, and if $w \in \mathfrak{S}_n$ then $T_w = T_{i_1}\dots T_{i_k}$ whenever $w = (i_1, i_1 + 1)\dots(i_k, i_k + 1)$ with k minimal (so this is a reduced expression of w). For each multipartition $\boldsymbol{\lambda}$ define

$$m_{\boldsymbol{\lambda}} = \prod_{s=1}^{r}\prod_{k=1}^{|\boldsymbol{\lambda}^{(1)}|+\dots+|\boldsymbol{\lambda}^{(s-1)}|}(L_k - Q_s)\cdot\sum_{w\in\mathfrak{S}_{\boldsymbol{\lambda}}}T_w,$$

where $\mathfrak{S}_{\boldsymbol{\lambda}} = \mathfrak{S}_{\boldsymbol{\lambda}^{(1)}}\times\dots\times\mathfrak{S}_{\boldsymbol{\lambda}^{(r)}}$ is the parabolic subgroup of $\mathfrak{S}_n$ associated to $\boldsymbol{\lambda}$. The **cyclotomic q–Schur algebra** is the endomorphism algebra

$$\mathscr{S}_{r,n} = \mathscr{S}_{r,n}(q,\mathbf{Q}) = \operatorname{End}_{\mathscr{H}_{r,n}}\Big(\bigoplus_{\boldsymbol{\lambda}\in\Lambda^+_{r,n}} m_{\boldsymbol{\lambda}}\mathscr{H}_{r,n}\Big).$$

We remark this variant of the cyclotomic q–Schur algebra is Morita equivalent to the one of the algebras introduced in [**12**]. The representation theory of $\mathscr{S}_{r,n}$ is discussed in [**24**].

The cyclotomic q–Schur algebra $\mathscr{S}_{r,n}$ is a quasi–hereditary cellular algebra. The cell modules of $\mathscr{S}_{r,n}$ are the **Weyl modules** $\Delta(\boldsymbol{\lambda})$, for $\boldsymbol{\lambda}\in\Lambda^+_{r,n}$. For each $\boldsymbol{\lambda}\in\Lambda^+_{r,n}$, there is a non–zero simple module $L(\boldsymbol{\lambda}) = \Delta(\boldsymbol{\lambda})/\operatorname{rad}\Delta(\boldsymbol{\lambda})$. Just as with Lemma 2.1, the theory of cellular algebras tells us the following.

2.2. **Lemma** (Graham–Lehrer [**16**, 3.9.8], [**23**, Cor. 2.2])**.** *Suppose that $\boldsymbol{\lambda}$ is a multipartition. Then all of the composition factors of $\Delta(\boldsymbol{\lambda})$ belong to the same block.*

The next result shows that in order to classify the blocks of $\mathscr{H}_{r,n}$ it is enough to consider the blocks of $\mathscr{S}_{r,n}$. In fact, this is an easy consequence of double centralizer theory.

2.3. **Proposition.** *Let $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ be multipartitions of n. Then $S(\boldsymbol{\lambda})$ and $S(\boldsymbol{\mu})$ are in the same block as $\mathscr{H}_{r,n}$–modules if and only if $\Delta(\boldsymbol{\lambda})$ and $\Delta(\boldsymbol{\mu})$ are in the same block as $\mathscr{S}_{r,n}$–modules.*

Proof. By a standard Schur functor argument [**21**, Prop. 2.17], if $D(\boldsymbol{\nu})\ne 0$ then $[S(\boldsymbol{\lambda}){:}D(\boldsymbol{\nu})] = [\Delta(\boldsymbol{\lambda}){:}L(\boldsymbol{\nu})]$. Therefore, if $S(\boldsymbol{\lambda})$ and $S(\boldsymbol{\mu})$ are in the same block then $\Delta(\boldsymbol{\lambda})$ and $\Delta(\boldsymbol{\mu})$ are in the same block. Note that this implies that $\mathscr{S}_{r,n}$ cannot have more blocks (that is, indecomposable subalgebras) than $\mathscr{H}_{r,n}$.

To prove the converse let $M = \bigoplus_{\boldsymbol{\lambda}\in\Lambda^+_{r,n}} m_{\boldsymbol{\lambda}}\mathscr{H}_{r,n}$ and suppose that $\mathscr{H}_{r,n} = B_1\oplus\dots\oplus B_k$ is the unique decomposition of $\mathscr{H}_{r,n}$ into blocks (that is, indecomposable subalgebras). Then

$$M = M\mathscr{H}_{r,n} = MB_1 + \dots + MB_k.$$

In fact, this sum is direct because, by definition, $MB_i \cap MB_j = \emptyset$ if $i \neq j$, and $MB_i \neq 0$ since $\mathscr{H}_{r,n}$ is a submodule of M. Therefore,

$$\begin{aligned}\mathscr{S}_{r,n} &= \mathrm{End}_{\mathscr{H}_{r,n}}(M) = \mathrm{End}_{\mathscr{H}_{r,n}}\big(MB_1 \oplus \cdots \oplus MB_k\big)\\ &= \bigoplus_{1\le i,j\le k} \mathrm{Hom}_{\mathscr{H}_{r,n}}(MB_i, MB_j) = \bigoplus_{i=1}^{k} \mathrm{End}_{\mathscr{H}_{r,n}}(MB_i),\end{aligned}$$

where the last equality follows because B_i and B_j have no common irreducible constituents if $i \neq j$. Consequently, $\mathscr{S}_{r,n}$ has at least as many blocks as $\mathscr{H}_{r,n}$.

Combining the last two paragraphs proves the proposition. □

Thus, to prove Theorem B it suffices to determine when two Weyl modules are in the same block. The advantage of working with Weyl modules is shown in Lemma 2.4 below. Before we can state this result we need some notation.

If A is an algebra let $K_0(A)$ be the Grothendieck group of finite dimensional A–modules and if M is a A–module let $[M]$ be its image in $K_0(A)$. In particular, the Grothendieck group $K_0(\mathscr{S}_{r,n})$ of $\mathscr{S}_{r,n}$ is the free $\mathbb{Z}$–module with basis $\{\, [L(\boldsymbol{\lambda})] \mid \boldsymbol{\lambda} \in \Lambda^+_{r,n}\,\}$. The images $\{\, [\Delta(\boldsymbol{\lambda})] \mid \boldsymbol{\lambda} \in \Lambda^+_{r,n}\,\}$ of the Weyl modules give a second basis of $K_0(\mathscr{S}_{r,n})$ since $[\Delta(\boldsymbol{\lambda}){:}L(\boldsymbol{\lambda})] = 1$ and $[\Delta(\boldsymbol{\lambda}){:}L(\boldsymbol{\mu})] > 0$ only if $\boldsymbol{\lambda} \trianglerighteq \boldsymbol{\mu}$, for all $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \Lambda^+_{r,n}$ (see [**16**]). Hence, we have the following.

2.4. **Lemma.** *Suppose that $a_{\boldsymbol{\lambda}} \in \mathbb{Z}$. Then $\sum_{\boldsymbol{\lambda}} a_{\boldsymbol{\lambda}}[\Delta(\boldsymbol{\lambda})] = 0$ in $K_0(\mathscr{S}_{r,n})$ if and only if $a_{\boldsymbol{\lambda}} = 0$ for all $\boldsymbol{\lambda} \in \Lambda^+_{r,n}$.*

Note that, in general, there exist non–zero integers $a_{\boldsymbol{\lambda}} \in \mathbb{Z}$ such that $\sum_{\boldsymbol{\lambda}} a_{\boldsymbol{\lambda}}[S(\boldsymbol{\lambda})] = 0$. This follows because $K_0(\mathscr{H}_{r,n})$ is a free $\mathbb{Z}$–module of rank $L = \#\{\, \boldsymbol{\lambda} \in \Lambda^+_{r,n} \mid D(\boldsymbol{\lambda}) \neq 0\,\}$ and $L = \#\Lambda^+_{r,n}$ if (and only if) $\mathscr{H}_{r,n}$ is semisimple.

2.3. **The cyclotomic Jantzen sum formula.** The next step is to recall (a special case of) the machinery of the cyclotomic Jantzen sum formula [**21**]. Let t be an indeterminate over $\mathbb{F}$ and let $\mathcal{O} = \mathbb{F}[t,t^{-1}]_\pi$ be the localization of $\mathbb{F}[t,t^{-1}]$ at the prime ideal $\pi = \langle t-1\rangle$. Let $\mathscr{S}_{\mathcal{O}} = \mathscr{S}_{\mathcal{O}}(qt, \mathbf{X})$ be the cyclotomic Schur algebra over $\mathcal{O}$ with parameters qt and $\mathbf{X} = (X_1,\dots,X_r)$ where

$$X_a = \begin{cases} Q_a t^{na}, & \text{if } Q_a \neq 0,\\ (t-1)t^{na}, & \text{if } Q_a = 0.\end{cases}$$

Consider $\mathbb{F}$ as an $\mathcal{O}$–module by letting t act on $\mathbb{F}$ as multiplication by 1. Then $\mathscr{S}_{r,n} \cong \mathscr{S}_{\mathcal{O}} \otimes_{\mathcal{O}} \mathbb{F}$, since $\mathscr{S}_{\mathcal{O}}$ is free as an $\mathcal{O}$–module by [**12**, Theorem 6.6]. The algebra $\mathscr{S}_{\mathcal{O}} \otimes_{\mathcal{O}} \mathbb{F}(t)$ is split semisimple by Schur–Weyl duality [**24**, Theorem 5.3] and Ariki's criterion for the semisimplicity for $\mathscr{H}_{r,n}$ [**1**]. Thus we are in the general setting considered in [**21**, §4].

Let ν_π be the π–adic evaluation map on $\mathcal{O}^\times$; thus, $\nu_\pi(f(t)) = k$ if $k \geq 0$ is maximal such that $(t-1)^k$ divides $f(t) \in \mathbb{F}[t,t^{-1}]$. Let $\Delta_{\mathcal{O}}(\boldsymbol{\lambda})$ be the Weyl module of $\mathscr{S}_{\mathcal{O}}$ indexed by the multipartition $\boldsymbol{\lambda} \in \Lambda^+_{r,n}$. Recall that $\Delta_{\mathcal{O}}(\boldsymbol{\lambda})$ carries a bilinear form $\langle\ ,\ \rangle_{\boldsymbol{\lambda}}$ by the general theory of cellular algebras. For each integer $i \geq 0$ define

$$\Delta_{\mathcal{O}}(\boldsymbol{\lambda})_i = \{\, x \in \Delta_{\mathcal{O}}(\boldsymbol{\lambda}) \mid \nu_\pi(\langle x,y\rangle) \geq i \text{ for all } y \in \Delta_{\mathcal{O}}(\boldsymbol{\lambda})\,\}.$$

Finally, let $\Delta(\boldsymbol{\lambda})_i = \big(\Delta_{\mathcal{O}}(\boldsymbol{\lambda})_i + \pi\Delta_{\mathcal{O}}(\boldsymbol{\lambda})\big)/\pi\Delta_{\mathcal{O}}(\boldsymbol{\lambda})$. Then

$$\Delta(\boldsymbol{\lambda}) = \Delta(\boldsymbol{\lambda})_0 \supset \Delta(\boldsymbol{\lambda})_1 \supseteq \Delta(\boldsymbol{\lambda})_2 \supseteq \dots$$

is a **Jantzen filtration** of the $\mathscr{S}_{r,n}$–module $\Delta(\boldsymbol{\lambda})$. Then $\Delta(\boldsymbol{\lambda})_k = 0$ for $k \gg 0$ since $\Delta(\boldsymbol{\lambda})$ is finite dimensional.

To describe the Jantzen filtration of $\Delta(\boldsymbol{\lambda})$ we need some combinatorics. The **diagram** of a multipartition $\boldsymbol{\lambda}$ is the set $[\boldsymbol{\lambda}] = \{\,(i,j,a) \mid 1 \le j \le \lambda_i^{(a)} \text{ and } 1 \le a \le r\,\}$. A **node** is any ordered triple (i,j,a) in $\mathbb{N}\times\mathbb{N}\times\{1,\dots,r\}$. For example, the elements of $[\boldsymbol{\lambda}]$ are nodes.

Each node $x = (i,j,a) \in [\boldsymbol{\lambda}]$ determines a **rim hook**

$$r_x^{\boldsymbol{\lambda}} = \{\,(k,l,a) \in [\boldsymbol{\lambda}] \mid k \ge i, l \ge j \text{ and } (k+1,l+1,a) \notin [\boldsymbol{\lambda}]\,\}.$$

We say that $r_x^{\boldsymbol{\lambda}}$ is a h**–rim hook** if $h = |r_x^{\boldsymbol{\lambda}}|$. Let i' be maximal such that $(i',j,a) \in [\boldsymbol{\lambda}]$; so i' is the length of column j of $\lambda^{(a)}$. Then $f_x^{\boldsymbol{\lambda}} = (i',j,a) \in [\boldsymbol{\lambda}]$ is the **foot** of $r_x^{\boldsymbol{\lambda}}$ and $r_x^{\boldsymbol{\lambda}}$ has **leg length** $\ell\ell(r_x^{\boldsymbol{\lambda}}) = i' - i$. If $x \in [\boldsymbol{\lambda}]$ let $\boldsymbol{\lambda}\backslash r_x^{\boldsymbol{\lambda}}$ be the multipartition with diagram $[\boldsymbol{\lambda}]\backslash r_x^{\boldsymbol{\lambda}}$. We say that $\boldsymbol{\lambda}\backslash r_x^{\boldsymbol{\lambda}}$ is the multipartition obtained by **unwrapping** the rim hook $r_x^{\boldsymbol{\lambda}}$ from $\boldsymbol{\lambda}$, and that $\boldsymbol{\lambda}$ is the multipartition obtained from $\boldsymbol{\lambda}\backslash r_x^{\boldsymbol{\lambda}}$ by **wrapping** on the rim hook $r_x^{\boldsymbol{\lambda}}$.

Define the $\mathcal{O}$–residue of the node $x = (i,j,a)$ to be $\operatorname{res}_{\mathcal{O}}(x) = (qt)^{j-i}X_a = q^{j-i}Q_a t^{na+j-i}$.

2.5. **Definition.** Suppose that $\boldsymbol{\lambda} = (\lambda^{(1)},\dots,\lambda^{(r)})$ and $\boldsymbol{\mu} = (\mu^{(1)},\dots,\mu^{(r)})$ are multipartitions of n. The **Jantzen coefficient** is the integer

$$J_{\boldsymbol{\lambda\mu}} = \begin{cases} \displaystyle\sum_{x\in[\boldsymbol{\lambda}]}\ \sum_{\substack{y\in[\boldsymbol{\mu}]\\ [\boldsymbol{\mu}]\backslash r_y^{\boldsymbol{\mu}} = [\boldsymbol{\lambda}]\backslash r_x^{\boldsymbol{\lambda}}}} (-1)^{\ell\ell(r_x^{\boldsymbol{\lambda}})+\ell\ell(r_y^{\boldsymbol{\mu}})}\nu_\pi\big(\operatorname{res}_{\mathcal{O}}(f_x^{\boldsymbol{\lambda}}) - \operatorname{res}_{\mathcal{O}}(f_y^{\boldsymbol{\mu}})\big), & \text{if } \boldsymbol{\lambda} \rhd \boldsymbol{\mu},\\ 0, & \text{otherwise.}\end{cases}$$

The Jantzen coefficient $J_{\boldsymbol{\lambda\mu}}$ depends on the choices of $\mathbb{F}$, q and $\mathbf{Q}$; in fact, $J_{\boldsymbol{\lambda\mu}}$ depends only on p, e and $\mathbf{Q}$. By definition $J_{\boldsymbol{\lambda\mu}}$ is an integer which is determined by the combinatorics of multipartitions. The definition of $J_{\boldsymbol{\lambda\mu}}$ is reasonably involved, however, it turns out that these integers are computable. In sections 3 and 4 we determine $J_{\boldsymbol{\lambda\mu}}$ explicitly.

2.6. **Theorem** (James and Mathas [**21**], Theorem 4.3)**.** *Suppose that $\boldsymbol{\lambda}$ is a multipartition of n. Then*

$$\sum_{i>0}[\Delta(\boldsymbol{\lambda})_i] = \sum_{\boldsymbol{\mu}\in\Lambda_{r,n}^+} J_{\boldsymbol{\lambda\mu}}[\Delta(\boldsymbol{\mu})]$$

in $K_0(\mathscr{S}_{r,n})$.

For multipartitions $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ in $\Lambda_{r,n}^+$ let $d_{\boldsymbol{\lambda\mu}} = [\Delta(\boldsymbol{\lambda}){:}L(\boldsymbol{\mu})]$ be the number of composition factors of $\Delta(\boldsymbol{\lambda})$ which are isomorphic to $L(\boldsymbol{\mu})$. Define

$$J'_{\boldsymbol{\lambda\mu}} = \sum_{\substack{\boldsymbol{\nu}\in\Lambda_{r,n}^+\\ \boldsymbol{\lambda}\rhd\boldsymbol{\nu}\trianglerighteq\boldsymbol{\mu}}} J_{\boldsymbol{\lambda\nu}}d_{\boldsymbol{\nu\mu}}.$$

By Theorem 2.6, $J'_{\boldsymbol{\lambda\mu}}$ is the composition multiplicity of the simple module $L(\boldsymbol{\mu})$ in $\bigoplus_{i>0}\Delta(\boldsymbol{\lambda})_i$. Therefore, $J'_{\boldsymbol{\lambda\mu}} \ge 0$, for all $\boldsymbol{\lambda},\boldsymbol{\mu} \in \Lambda_{r,n}^+$. As $\Delta(\boldsymbol{\lambda})_1 = \operatorname{rad}\Delta(\boldsymbol{\lambda})$ we obtain the following.

2.7. **Corollary.** *Suppose that $\boldsymbol{\lambda} \ne \boldsymbol{\mu}$ are multipartitions of n. Then $d_{\boldsymbol{\lambda\mu}} \le J'_{\boldsymbol{\lambda\mu}}$ and, moreover, $d_{\boldsymbol{\lambda\mu}} \ne 0$ if and only if $J'_{\boldsymbol{\lambda\mu}} \ne 0$.*

We now use Theorem 2.6 to classify the blocks of $\mathscr{S}_{r,n}$.

2.8. **Definition.** Suppose that $\boldsymbol{\lambda},\boldsymbol{\mu} \in \Lambda_{r,n}^+$. Then $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are **Jantzen equivalent**, and we write $\boldsymbol{\lambda} \sim_J \boldsymbol{\mu}$, if there exists a sequence of multipartitions $\boldsymbol{\lambda}_0 = \boldsymbol{\lambda}, \boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_k = \boldsymbol{\mu}$ such that either

$$J_{\boldsymbol{\lambda}_i\boldsymbol{\lambda}_{i+1}} \ne 0 \qquad \text{or} \qquad J_{\boldsymbol{\lambda}_{i+1}\boldsymbol{\lambda}_i} \ne 0,$$

for $1 \le i \le k$.

Jantzen equivalence gives us our first combinatorial characterization of the blocks of $\mathscr{S}_{r,n}$.

2.9. **Proposition.** *Suppose that $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \Lambda^+_{r,n}$. Then $\Delta(\boldsymbol{\lambda})$ and $\Delta(\boldsymbol{\mu})$ belong to the same block as $\mathscr{S}_{r,n}$–modules if and only if $\boldsymbol{\lambda} \sim_J \boldsymbol{\mu}$.*

Proof. We first show that $\Delta(\boldsymbol{\lambda})$ and $\Delta(\boldsymbol{\nu})$ belong to the same block whenever $\boldsymbol{\lambda} \sim_J \boldsymbol{\nu}$. By definition $\Delta(\boldsymbol{\lambda})_i$ is a submodule of $\Delta(\boldsymbol{\lambda})$ for all i, so all of the composition factors of $\sum_{i>0} \Delta(\boldsymbol{\lambda})_i$ belong to the same block by Lemma 2.2. Consequently, all of the composition factors of the virtual module $\sum_{\boldsymbol{\mu}} J_{\boldsymbol{\lambda\mu}}[\Delta(\boldsymbol{\mu})]$ belong to the same block. Let Λ' be the set of multipartitions $\boldsymbol{\mu}$ such that $\Delta(\boldsymbol{\mu})$ is not in the same block as $\Delta(\boldsymbol{\lambda})$. Then we have $\sum_{\boldsymbol{\mu}\in\Lambda'} J_{\boldsymbol{\lambda\mu}}[\Delta(\boldsymbol{\mu})] = 0$. Hence, $J_{\boldsymbol{\lambda\mu}} = 0$ whenever $\boldsymbol{\mu} \in \Lambda'$ by Lemma 2.4. It follows that $\Delta(\boldsymbol{\lambda})$ and $\Delta(\boldsymbol{\mu})$ belong to the same block whenever $\boldsymbol{\lambda} \sim_J \boldsymbol{\mu}$.

To prove the converse it is sufficient to show that $\boldsymbol{\lambda} \sim_J \boldsymbol{\mu}$ whenever $d_{\boldsymbol{\lambda\mu}} \neq 0$. Hence, by Corollary 2.7 we must show that $\boldsymbol{\lambda} \sim_J \boldsymbol{\mu}$ whenever $J'_{\boldsymbol{\lambda\mu}} \neq 0$. However, if $J'_{\boldsymbol{\lambda\mu}} \neq 0$ then we can find a multipartition $\boldsymbol{\nu}_1$ such that $J_{\boldsymbol{\lambda\nu}_1} \neq 0$, $d_{\boldsymbol{\nu}_1\boldsymbol{\mu}} \neq 0$ and $\boldsymbol{\lambda} \rhd \boldsymbol{\nu}_1 \trianglerighteq \boldsymbol{\mu}$. Consequently, $\boldsymbol{\lambda} \sim_J \boldsymbol{\nu}_1$. If $\boldsymbol{\nu}_1 \neq \boldsymbol{\mu}$ then $J'_{\boldsymbol{\nu}_1\boldsymbol{\mu}} \neq 0$ by Corollary 2.7 since $d_{\boldsymbol{\nu}_1\boldsymbol{\mu}} \neq 0$. Therefore, we can find a multipartition $\boldsymbol{\nu}_2$ such that $J_{\boldsymbol{\nu}_1\boldsymbol{\nu}_2} \neq 0$, $d_{\boldsymbol{\nu}_2\boldsymbol{\mu}} \neq 0$ and $\boldsymbol{\nu}_1 \rhd \boldsymbol{\nu}_2 \trianglerighteq \boldsymbol{\mu}$. Continuing in this way we can find multipartitions $\boldsymbol{\nu}_0 = \boldsymbol{\lambda}, \boldsymbol{\nu}_1, \dots, \boldsymbol{\nu}_k = \boldsymbol{\mu}$ such that $J_{\boldsymbol{\nu}_{i-1}\boldsymbol{\nu}_i} \neq 0$, $d_{\boldsymbol{\nu}_i\boldsymbol{\mu}} \neq 0$, for $0 < i < k$, and $\boldsymbol{\lambda} \rhd \boldsymbol{\nu}_1 \rhd \cdots \rhd \boldsymbol{\nu}_k = \boldsymbol{\mu}$. Note that we must have $\boldsymbol{\nu}_k = \boldsymbol{\mu}$ for some k since $\Lambda^+_{r,n}$ is finite. Therefore, $\boldsymbol{\lambda} \sim_J \boldsymbol{\nu}_1 \sim_J \cdots \sim_J \boldsymbol{\nu}_k = \boldsymbol{\mu}$ as required. □

Remark. Without using the cyclotomic q–Schur algebras it is not clear that Jantzen equivalence determines the blocks of $\mathscr{H}_{r,n}$. Applying the Schur functor to Theorem 2.6 gives an analogous description of the Jantzen filtration of the Specht modules: $\sum_{i>0}[S(\boldsymbol{\lambda})_i] = \sum_{\boldsymbol{\mu}} J_{\boldsymbol{\lambda\mu}}[S(\boldsymbol{\mu})]$. The problem is that, *a priori*, the composition factors of $\bigoplus_{\boldsymbol{\mu}} J_{\boldsymbol{\lambda\mu}} S(\boldsymbol{\mu})$ could belong to different blocks because the analogue of Lemma 2.4 fails for Specht modules.

Remark. The argument of Proposition 2.9 is completely generic. It shows that the blocks of any quasi–hereditary algebra are determined by the Jantzen coefficients once a sum formula for the Jantzen filtrations of its standard modules is known.

2.4. **A second combinatorial characterization of the blocks.** Proposition 2.9 completely determines the blocks of $\mathscr{S}_{r,n}$, and hence the blocks of $\mathscr{H}_{r,n}$. Unfortunately, it is not obvious when two multipartitions are Jantzen equivalant.

The **residue** of the node $x = (i, j, a)$ is

$$\operatorname{res}(x) = \begin{cases} q^{j-i}Q_a, & \text{if } q \neq 1 \text{ and } Q_a \neq 0, \\ (\overline{j-i}, Q_a), & \text{if } q = 1 \text{ and } Q_a \neq Q_b \text{ for } b \neq a, \\ Q_a, & \text{otherwise,} \end{cases}$$

where $\overline{z} = z \pmod{p}$ for $z \in \mathbb{Z}$ (if $p = \infty$ we set $\overline{z} = z$). Let

$$\operatorname{Res}(\Lambda^+_{\mathrm{r,n}}) = \{\, \operatorname{res}(x) \mid x \in [\boldsymbol{\lambda}] \text{ for some } \boldsymbol{\lambda} \in \Lambda^+_{r,n} \,\}$$

be the set of all possible residues. For any multipartition $\boldsymbol{\lambda} \in \Lambda^+_{r,n}$ and $f \in \operatorname{Res}(\Lambda^+_{\mathrm{r,n}})$ define

$$C_f(\boldsymbol{\lambda}) = \#\{\, x \in [\boldsymbol{\lambda}] \mid \operatorname{res}(x) = f \,\}.$$

We can now define our second combinatorial equivalence relation on $\Lambda^+_{r,n}$.

2.10. **Definition.** Suppose that $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are multipartitions. Then $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are **residue equivalent**, and we write $\boldsymbol{\lambda} \sim_C \boldsymbol{\mu}$, if $C_f(\boldsymbol{\lambda}) = C_f(\boldsymbol{\mu})$ for all $f \in \operatorname{Res}(\Lambda^+_{\mathrm{r,n}})$.

It is easy to determine if two multipartitions are residue equivalent, so the next result gives an effective characterization of the blocks of the algebras $\mathscr{H}_{r,n}$ and $\mathscr{S}_{r,n}$.

2.11. **Theorem.** *Suppose that $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are multipartitions of n. Then the following are equivalent.*

a) *$S(\boldsymbol{\lambda})$ and $S(\boldsymbol{\mu})$ belong to the same block as $\mathscr{H}_n(\mathbf{Q})$–modules.*
b) *$\Delta(\boldsymbol{\lambda})$ and $\Delta(\boldsymbol{\mu})$ belong to the same block as $\mathscr{S}_{r,n}(\mathbf{Q})$–modules.*
c) *$\boldsymbol{\lambda} \sim_J \boldsymbol{\mu}$.*
d) *$\boldsymbol{\lambda} \sim_C \boldsymbol{\mu}$.*

By Proposition 2.3 and Proposition 2.9, (a), (b) and (c) are equivalent. Therefore, to prove the theorem it is enough to prove that $\boldsymbol{\lambda} \sim_J \boldsymbol{\mu}$ if and only if $\boldsymbol{\lambda} \sim_C \boldsymbol{\mu}$. The proof of this fact is given in sections 3 and 4. It turns out that, combinatorially, these equivalence relations depend very much on whether or not $q = 1$ and whether or not some of the parameters $Q_1, \dots, Q_r$ are zero. The following result allows us to treat these cases separately.

2.12. **Theorem** (Dipper and Mathas [**13**], Theorem 1.5 and Corollary 5.7)**.**
Suppose that $\mathbf{Q} = \mathbf{Q}_1 \coprod \mathbf{Q}_2 \coprod \cdots \coprod \mathbf{Q}_\kappa$ is a partition of $\mathbf{Q}$ such that $q^c Q_a \in \mathbf{Q}_\alpha$ only if $Q_a \in \mathbf{Q}_\alpha$, for $c \in \mathbb{Z}$, $1 \le a \le r$ and $1 \le \alpha \le \kappa$. Set $r_0 = 0$ and $r_i = |\mathbf{Q}_\alpha|$, for $1 \le \alpha \le \kappa$. Then $\mathscr{S}_{r,n}(\mathbf{Q})$ is Morita equivalent to

$$\bigoplus_{\substack{n_1,\dots,n_\kappa \ge 0\\ n_1+\dots+n_\kappa = n}} \mathscr{S}_{r_1,n_1}(\mathbf{Q}_1) \boxtimes \mathscr{S}_{r_2,n_2}(\mathbf{Q}_2) \boxtimes \cdots \boxtimes \mathscr{S}_{r_\kappa,n_\kappa}(\mathbf{Q}_\kappa).$$

Moreover, the Morita equivalence is induced by the map $\Delta(\boldsymbol{\lambda}) \mapsto \Delta(\boldsymbol{\lambda}_1) \boxtimes \cdots \boxtimes \Delta(\boldsymbol{\lambda}_\kappa)$, where $\boldsymbol{\lambda}_\alpha = (\lambda^{(r_{\alpha-1}+1)}, \dots, \lambda^{(r_\alpha)})$, for $1 \le \alpha \le \kappa$ and $\boldsymbol{\lambda} \in \Lambda^+_{r,n}$.

There is an analogous result for the Ariki–Koike algebra $\mathscr{H}_{r,n}$; see [**13**, Theorem 1.1].

Theorem 2.12 says that the blocks of $\mathscr{H}_{r,n}(\mathbf{Q})$ and $\mathscr{S}_{r,n}(\mathbf{Q})$ depend only on the q–orbits of the parameters and, further, that it is enough to consider the case where $\mathbf{Q}$ is contained in a single q–orbit. Hence, by rescaling T_0 we can assume that the parameters $Q_1, \dots, Q_r$ are all powers of q. That is, we can assume that there exist integers $c_1, \dots, c_r$ such that $Q_a = q^{c_a}$, for $1 \le a \le r$. Consequently, to prove Theorem 2.11 we are reduced to considering the following five cases:

(2.13) Case 1. $q \ne 1$ and $Q_a = q^{c_a}$, for $1 \le a \le r$.
Case 2. $r = 1$, $q = 1$ and $Q_1 = 1$.
Case 3. $r > 1$, $q = 1$ and $Q_1 = \dots = Q_r = 1$.
Case 4. $r > 1$, $q = 1$ and $Q_1 = \dots = Q_r = 0$.
Case 5. $r > 1$, $q \ne 1$ and $Q_1 = \dots = Q_r = 0$.

The proof of Theorem 2.11 for case 1 is given in section 3. Cases 2–5 are considered in section 4 using similar, but easier, arguments. Given a node $x = (i, j, a)$ note that $\operatorname{res}(x) = q^{j-i} Q_a$ in case 1, $\operatorname{res}(x) = (\overline{j - i}, 1)$ in case 2 and $\operatorname{res}(x) = Q_a$ in the other three cases.

We treat all of these cases separately because the underlying combinatorics is different. Fayers has pointed out that the Ariki–Koike algebras in cases 3 and 4 are isomorphic via the algebra homomorphism determined by $T_0 \mapsto (T_0 - 1)$ and $T_i \mapsto T_i$, for $1 \le i < n$, so we do not really need to consider case 4 (we deal with Cases 3–5 simultaneously).

2.5. **The blocks of the affine Hecke algebra.** Assuming Theorem 2.11 we now prove Theorem A and Theorem B from the introduction.

As the centre $Z(\mathscr{H}_n^{\mathrm{aff}})$ of $\mathscr{H}_n^{\mathrm{aff}}$ is the set of symmetric Laurent polynomials in $X_1, \dots, X_n$, the central characters of $\mathscr{H}_n^{\mathrm{aff}}$ are indexed by $\mathfrak{S}_n$–orbits of $(\mathbb{F}^\times)^n$. More precisely, if $\gamma \in (\mathbb{F}^\times)^n / \mathfrak{S}_n$ then the central character χ_γ is given by evaluation at γ.

By Lemma 2.1, all of the composition factors of the Specht module $S(\boldsymbol{\lambda})$ belong to the same block as $\mathscr{H}_{r,n}$–modules. Therefore, all of the composition factors of $S(\boldsymbol{\lambda})$ belong to the same block an $\mathscr{H}_n^{\mathrm{aff}}$–modules. We need to know the central characters of the Specht modules.

2.14. **Lemma.** *Suppose that $q \neq 1$ and that $D(\boldsymbol{\lambda}) \neq 0$, for some multipartition $\boldsymbol{\lambda} \in \Lambda^+_{r,n}$. Then $f(X) \in \mathbb{Z}(\mathscr{H}_n^{\mathit{aff}})$ acts on $D(\boldsymbol{\lambda})$ as multiplication by $f(\gamma)$, where $\gamma = \big(\mathrm{res}(x_1), \mathrm{res}(x_2), \dots, \mathrm{res}(x_n)\big)$ and $[\boldsymbol{\lambda}] = \{x_1, \dots, x_n\}$ (in any order).*

Proof. As all of the composition factors of $S(\boldsymbol{\lambda})$ belong to the same block as $D(\boldsymbol{\lambda})$, $f(X)$ acts on $S(\boldsymbol{\lambda})$ and on $D(\boldsymbol{\lambda})$ as multiplication by the same scalar. By [**21**, Prop. 3.7] this scalar is given by evaluating the polynomial $f(X)$ at $\big(\mathrm{res}(x_1), \mathrm{res}(x_2), \dots, \mathrm{res}(x_n)\big)$. □

2.15. **Theorem** (Theorem A)**.** *Suppose that $q \neq 1$ and that $\mathbb{F}$ is algebraically closed. Then two simple $\mathscr{H}_n^{\mathit{aff}}$–modules D and D' belong to the same block if and only if they have the same central character.*

Proof. Any two simple modules in the same block have the same central character. Conversely, suppose that D and D' are simple $\mathscr{H}_n^{\mathrm{aff}}$–modules which have the same central character. Let $(X_1 - Q_1) \dots (X_1 - Q_s)$ and $(X_1 - Q_{s+1}) \dots (X_1 - Q_r)$, respectively, be the minimal polynomials for X_1 acting on D and D'. (Note that $Q_1, \dots, Q_r$ are non–zero since $X_1, \dots, X_n$ are invertible.) Then D' and D' are both simple modules for the Ariki–Koike algebra $\mathscr{H}_{r,n}$ with parameters $Q_1, \dots, Q_r$. Therefore, $D \cong D(\boldsymbol{\lambda})$ and $D' \cong D(\boldsymbol{\mu})$ for some multipartitions $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \Lambda^+_{r,n}$. By assumption, D and D' have the same central characters. The central character of $D(\boldsymbol{\lambda})$ is uniquely determined by the multiset of residues $\{\,\mathrm{res}(x) \mid x \in [\boldsymbol{\lambda}]\,\}$ by Lemma 2.14. Similarly, the central character of $D(\boldsymbol{\mu})$ is determined by the multiset $\{\,\mathrm{res}(x) \mid x \in [\boldsymbol{\mu}]\,\}$. Hence, $C_f(\boldsymbol{\lambda}) = C_f(\boldsymbol{\mu})$, for all $f \in \mathrm{Res}(\Lambda^+_{\mathrm{r,n}})$. Therefore, $\boldsymbol{\lambda} \sim_C \boldsymbol{\mu}$, so $D \cong D(\boldsymbol{\lambda})$ and $D' = D(\boldsymbol{\mu})$ are in the same block as $\mathscr{H}_{r,n}$–modules by Theorem 2.11. Hence, D and D' are in the same block as $\mathscr{H}_n^{\mathrm{aff}}$–modules. □

Using Theorems 2.11 and 2.15 we obtain a more descriptive version of Theorem B.

2.16. **Corollary** (Theorem B)**.** *Suppose that $\mathbb{F}$ is an algebraically closed field, $q \neq 1$ and that the parameters $Q_1, \dots, Q_r$ are non–zero. Let $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ be multipartitions in $\Lambda^+_{r,n}$ with $D(\boldsymbol{\lambda}) \neq 0$ and $D(\boldsymbol{\mu}) \neq 0$. Then the following are equivalent:*

a) *$D(\boldsymbol{\lambda})$ and $D(\boldsymbol{\mu})$ belong to the same block as $\mathscr{H}_{r,n}$–modules.*
b) *$D(\boldsymbol{\lambda})$ and $D(\boldsymbol{\mu})$ belong to the same block as $\mathscr{H}_n^{\mathit{aff}}$–modules.*
c) *$D(\boldsymbol{\lambda})$ and $D(\boldsymbol{\mu})$ have the same central character as $\mathscr{H}_n^{\mathit{aff}}$–modules.*
d) *$\boldsymbol{\lambda} \sim_C \boldsymbol{\mu}$.*

3. COMBINATORICS

In this section, we prove $\boldsymbol{\lambda} \sim_J \boldsymbol{\mu}$ if and only if $\boldsymbol{\lambda} \sim_C \boldsymbol{\mu}$, for $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \Lambda^+_{r,n}$ in the cases when $q \neq 1$ and all of the parameters $Q_1, \dots, Q_r$ are powers of q. This is Case 1 of (2.13). The basic idea is that we want to reduce the comparision of the Jantzen and residue equivalence relations to the case where the multipartitions $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are both 'cores'. The complication is that, unlike for partitions (the case $r = 1$), we do not have a good notion of 'core' for multipartitions when $r > 1$. We circumvent this difficulty using ideas of Fayers [**14, 15**].

As we are assuming that the parameters $Q_1, \dots, Q_r$ are all powers of q, there exist integers $c_1, \dots, c_r$ such that $Q_a = q^{c_a}$, for $1 \leq a \leq r$. The sequence $\mathbf{c} = (c_1, \dots, c_r)$ is called the **multi–charge** of $\mathbf{Q}$.

Now that $\mathbf{Q}$ is contained in a single q–orbit, we redefine the **residue** of a node $x = (i, j, a)$ to be

$$\mathrm{res}(x) = (j - i + c_a) \pmod{e}.$$

Therefore, $\{\,\mathrm{res}(x) \mid x \in [\boldsymbol{\lambda}]$ for some $\boldsymbol{\lambda} \in \Lambda^+_{r,n}\,\} \subseteq \mathbb{Z}/e\mathbb{Z}$.

For $\boldsymbol{\lambda} \in \Lambda^+_{r,n}$ and $f \in \mathbb{Z}/e\mathbb{Z}$ put $C_f(\boldsymbol{\lambda}) = \#\{\, x \in [\boldsymbol{\lambda}] \mid \mathrm{res}(x) = f\,\}$. It is straightforward to check that with these new conventions $\boldsymbol{\lambda} \sim_C \boldsymbol{\mu}$ if and only if $C_f(\boldsymbol{\lambda}) = C_f(\boldsymbol{\mu})$, for all $f \in \mathbb{Z}/e\mathbb{Z}$.

3.1. **Abacuses.** Abacuses first appeared in the work of Gordon James [**19**] and have since been used extensively in the modular representation theory of the symmetric groups and related algebras. An e–**abacus** is an abacus with e vertical runners, which are infinite in both directions. If e is finite then we label the runners $0, 1, \dots, e-1$ from left to right and position $z \in \mathbb{Z}$ on the abacus is the bead position in row x on runner y, where $z = xe+y$ and $0 \le y < e$. If $e = \infty$ then we label the runners $\dots, -1, 0, 1, \dots$ and position z on the abacus is the bead position in row 0 on runner z.

Let $\boldsymbol{\lambda} \in \Lambda^+_{r,n}$ be a multipartition and recall that we have fixed a sequence of integers $\mathbf{c} = (c_1, \dots, c_r)$. Fix a with $1 \le a \le r$. The β**–numbers** of the partition $\lambda^{(a)}$ is the set of integers $B_a = \{\beta^a_1, \beta^a_2, \dots\}$, where

$$\beta^a_i = \lambda^{(a)}_i - i + c_a,$$

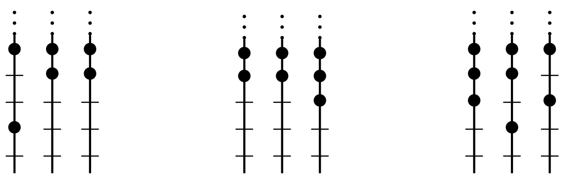
for $i \ge 0$. The e**–abacus display** of $\lambda^{(a)}$ is the e–abacus with a bead at position β^a_i, for $i \ge 1$. The e–abacus display of $\boldsymbol{\lambda}$ is the ordered r–tuple of abacuses for the partitions $\lambda^{(1)}, \dots, \lambda^{(r)}$.

It is easy to check that a multipartition is uniquely determined by its abacus display and, conversely, that every abacus display corresponds to a unique multipartition.

3.1. **Example.** Suppose that $e = 3$, $r = 3$ and $\mathbf{c} = (0, 1, 2)$. Let $\boldsymbol{\lambda} = \big((4,1,1), (2), (3,2,1)\big)$. Then

$$B_1 = \{3, -1, -2, -4, -5, \dots\}, \quad B_2 = \{2, -1, -2, \dots\}, \quad B_3 = \{4, 2, 0, -2, -3, \dots\}$$

and the abacus display for $\boldsymbol{\lambda}$ is given by



Let λ be a partition and suppose that $B = \{\beta_1, \beta_2, \dots\}$ is the set of β–numbers for λ. Then the e–abacus for λ has beads at positions β_i, for $i \ge 0$. If $\beta_i + h \notin B$ then **moving** the bead at position β_i to the *right* h positions gives a new abacus display with beads at positions $\{\beta_1, \beta_2, \dots, \beta_{i-1}, \beta_i+h, \beta_{i+1}, \dots\}$. Similarly, if $\beta_i - h \notin B$ then **moving** this bead h positions to the *left* creates a new abacus display with beads at positions $\{\beta_1, \beta_2, \dots, \beta_{i-1}, \beta_i - h, \beta_{i+1}, \dots\}$. The conditions $\beta_i \pm h \notin B$ are needed to ensure that the abacus display for λ does not already have a bead at the new position. Note that with these conventions moving a bead on runner 0 one position to the left moves the bead to a position on runner $e-1$ in the preceding row. Similarly, moving a bead on runner $e-1$ to the right moves a bead to a position on runner 0 in the next row. We also talk of moving beads in the abacus displays of multipartitions.

Recasting the above discussion in terms of the abacus we have the following well–known result which goes back to Littlewood and James.

3.2. **Lemma.** *Suppose that λ is a partition. Then moving a bead to the right h positions from runner f to runner f' corresponds to wrapping an h–rim hook with foot residue f onto λ. Similarly, moving a bead h positions to the left, from runner f to runner f' corresponds to unwrapping an h–rim hook from λ with foot residue f.*

That increasing a beta number by h corresponds to wrapping on an h–rim hook is proved in [**23**, Lemma 5.26]. The remaining claim about residues follows easily from our definitions. As a consequence we obtain the following.

3.3. **Corollary.** *Suppose that λ is a partition and $f \in \mathbb{Z}/e\mathbb{Z}$, where $e < \infty$. Then*

a) *Moving a bead down one row on a runner corresponds to wrapping an e–rim hook onto $[\lambda]$. If this bead is on runner f then the rim hook has foot residue f.*

b) *Moving a bead up one row on a runner corresponds to unwrapping an e–rim hook from $[\lambda]$. If this bead is on runner f the rim hook has foot residue f.*
c) *Moving the lowest bead on runner f down one row corresponds to wrapping on an e–hook with foot residue f. Consequently, we can add an e–hook with foot residue f to any partition.*

Suppose that λ is a partition. The e**–core** of λ is the partition $\overline{\lambda}$ whose e–abacus display is obtained from the e–abacus display for λ by moving all beads as high as possible on their runners, that is, successively removing all e–hooks from the diagram of λ. If $e=\infty$ then the e–core of λ is λ itself. Define the e**–weight** of the partition, $\mathrm{w}(\lambda)$, to be the number of e–hooks that we remove in order to construct $\overline{\lambda}$.

3.2. **Jantzen equivalence.** In order to prove Theorem 2.11 we first simplify the formula for $J_{\boldsymbol{\lambda}\boldsymbol{\mu}}$. Let $\boldsymbol{\lambda}$ be a multipartition and recall that if $x\in[\boldsymbol{\lambda}]$ then $r_x^{\boldsymbol{\lambda}}\subseteq[\boldsymbol{\lambda}]$ is the associated rim hook. To ease notation we let $h_x^{\boldsymbol{\lambda}}=|r_x^{\boldsymbol{\lambda}}|$ be the **hook length** of $r_x^{\boldsymbol{\lambda}}$. Before we start the proof of Theorem 2.11 we simplify the formula for $J_{\boldsymbol{\lambda}\boldsymbol{\mu}}$.

Recall that $\mathbb{F}$ is a field of characteristic p. Define $\nu_p:\mathbb{N}\longrightarrow\mathbb{N}$ to be the map

$$\nu_p(h)=\begin{cases}p^k, & \text{if } p \text{ is finite,}\\ 1, & \text{if } p=\infty.\end{cases}$$

where $k\ge 0$ is maximal such that p^k divides h.

If $\sigma=(\sigma_1,\sigma_2,\dots)$ is a partition let $\sigma'=(\sigma'_1,\sigma'_2,\dots)$ be its conjugate. Then $\sigma'_i=c$ if c is maximal such that $(c,i)\in[\sigma]$. (So σ'_i is the length of column i of $[\sigma]$.) For any integer $h\in\mathbb{Z}$ let $[h]_t=\frac{t^h-1}{t-1}\in\mathbb{F}[t,t^{-1}]$.

3.4. **Lemma.** *Suppose that $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are multipartitions of n and that $[\boldsymbol{\lambda}]\backslash r_x^{\boldsymbol{\lambda}}=[\boldsymbol{\mu}]\backslash r_y^{\boldsymbol{\mu}}$, for some nodes $x=(i,j,a)\in[\boldsymbol{\lambda}]$ and $y=(k,l,b)\in[\boldsymbol{\mu}]$. Then $\nu_\pi\big(\operatorname{res}_{\mathcal{O}}(f_x^{\boldsymbol{\lambda}})-\operatorname{res}_{\mathcal{O}}(f_y^{\boldsymbol{\mu}})\big)\neq 0$ if only if $\operatorname{res}(f_x^{\boldsymbol{\lambda}})=\operatorname{res}(f_y^{\boldsymbol{\mu}})$, in which case*

$$\nu_\pi\big(\operatorname{res}_{\mathcal{O}}(f_x^{\boldsymbol{\lambda}})-\operatorname{res}_{\mathcal{O}}(f_y^{\boldsymbol{\mu}})\big)=\nu_p\big(n(a-b)+j-\lambda_i^{(a)\prime}-l+\mu_k^{(b)\prime}\big).$$

Proof. Let $i'=\lambda_i^{(a)\prime}$ and $k'=\mu_k^{(b)\prime}$ so that $f_x^{\boldsymbol{\lambda}}=(i',j,a)$ and $f_y^{\boldsymbol{\mu}}=(k',l,b)$. Then

$$\begin{aligned}\operatorname{res}_{\mathcal{O}}(f_x^{\boldsymbol{\lambda}})-\operatorname{res}_{\mathcal{O}}(f_y^{\boldsymbol{\mu}})&=q^{j-i'}Q_at^{na+j-i'}-q^{l-k'}Q_bt^{nb+l-k'}\\ &=q^{l-k'}Q_bt^{nb+l-k'}\big(q^{j-i'-l+k'}Q_aQ_b^{-1}t^{n(a-b)+j-i'-l+k'}-1\big).\end{aligned}$$

Therefore, $\nu_\pi(\operatorname{res}_{\mathcal{O}}(x)-\operatorname{res}_{\mathcal{O}}(y))\neq 0$ if and only if $q^{j-i'-l+k'}Q_aQ_b^{-1}=1$, which is if and only if $\operatorname{res}(f_x^{\boldsymbol{\lambda}})=q^{j-i'}Q_a=q^{l-k'}Q_b=\operatorname{res}(f_y^{\boldsymbol{\mu}})$. Now suppose that $\operatorname{res}(f_x^{\boldsymbol{\lambda}})=\operatorname{res}(f_y^{\boldsymbol{\mu}})$ and let $h=n(a-b)+j-i'-l+k'$. Then

$$\nu_\pi\big(\operatorname{res}_{\mathcal{O}}(f_x^{\boldsymbol{\lambda}})-\operatorname{res}_{\mathcal{O}}(f_y^{\boldsymbol{\mu}})\big)=\nu_\pi(t^{n(a-b)+j-i'-l+k'}-1)=1+\nu_\pi([h]_t).$$

If $p=\infty$ then $(t-1)$ does not divide $[h]_t$, so that $\nu_\pi(\operatorname{res}_{\mathcal{O}}(x)-\operatorname{res}_{\mathcal{O}}(y))=1=\nu_p(h)$. If p is finite then write $h=p^kh'$, where $p\nmid h'$. Then

$$[h]_t=[p^kh']_t=[p^k]_t[h']_{t^{p^k}}=(t-1)^{p^k-1}[h']_t^{p^k}.$$

Now, $t-1$ does not divide $[h']_t$ since $p\nmid h'$. Therefore, $\nu_\pi([h]_t)=\nu_p(h)-1$ and the result follows. □

We can now prove that (c) $\Longrightarrow$ (d) in Theorem 2.11.

3.5. **Corollary.** *Suppose that $\boldsymbol{\lambda}\sim_J\boldsymbol{\mu}$, where $\boldsymbol{\lambda},\boldsymbol{\mu}\in\Lambda_{r,n}^+$. Then $\boldsymbol{\lambda}\sim_C\boldsymbol{\mu}$.*

Proof. By the Lemma and Definition 2.5, $J_{\boldsymbol{\lambda}\boldsymbol{\mu}}$ is non–zero only if there exist nodes $x\in[\boldsymbol{\lambda}]$ and $y\in[\boldsymbol{\mu}]$ such that $[\boldsymbol{\lambda}]\backslash r_x^{\boldsymbol{\lambda}}=[\boldsymbol{\mu}]\backslash r_y^{\boldsymbol{\mu}}$ and $\operatorname{res}(f_x^{\boldsymbol{\lambda}})=\operatorname{res}(f_y^{\boldsymbol{\mu}})$. These two conditions imply that $C_f(\boldsymbol{\lambda})=C_f(\boldsymbol{\mu})$, for all $f\in\mathbb{Z}/e\mathbb{Z}$, so $\boldsymbol{\lambda}\sim_C\boldsymbol{\mu}$. □

Establishing the reverse implication takes considerably more effort. We start by explicitly describing the Jantzen coefficients.

3.6. **Proposition.** *Let $\boldsymbol{\lambda}=(\lambda^{(1)},\dots,\lambda^{(r)})$ and $\boldsymbol{\mu}=(\mu^{(1)},\dots,\mu^{(r)})$ be multipartitions in $\Lambda^+_{r,n}$.*

a) *Suppose that there exist integers $a<b$ such that $\lambda^{(c)}=\mu^{(c)}$, for $c\ne a,b$. Then $J_{\boldsymbol{\lambda}\boldsymbol{\mu}}\ne 0$ only if there exist nodes $x=(i,j,a)\in[\boldsymbol{\lambda}]$ and $y=(k,l,b)\in[\boldsymbol{\mu}]$ such that $\operatorname{res}(f^{\boldsymbol{\lambda}}_x)=\operatorname{res}(f^{\boldsymbol{\mu}}_y)$ and $[\boldsymbol{\lambda}]\backslash r^{\boldsymbol{\lambda}}_x=[\boldsymbol{\mu}]\backslash r^{\boldsymbol{\mu}}_y$. In this case*
$$J_{\boldsymbol{\lambda}\boldsymbol{\mu}}=(-1)^{\ell\ell(r^{\boldsymbol{\lambda}}_x)+\ell\ell(r^{\boldsymbol{\mu}}_y)}\nu_p\big(n(a-b)+j-\lambda^{(a)\prime}_i-l+\mu^{(b)\prime}_k\big).$$

b) *Suppose that e is finite and for some integer a we have $\lambda^{(c)}=\mu^{(c)}$, for $c\ne a$. Then $J_{\boldsymbol{\lambda}\boldsymbol{\mu}}\ne0$ only if there exist nodes $x=(i,j,a),(i,m,a)\in[\boldsymbol{\lambda}]$ such that $m<j$, $e\mid h^{\boldsymbol{\lambda}}_{(i,m,a)}$ and $\boldsymbol{\mu}$ is obtained by wrapping a rim hook of length $h^{\boldsymbol{\lambda}}_x$ onto $\boldsymbol{\lambda}\backslash r^{\boldsymbol{\lambda}}_x$ with its highest node in column m. In this case*
$$J_{\boldsymbol{\lambda}\boldsymbol{\mu}}=\begin{cases}(-1)^{\ell\ell(r^{\boldsymbol{\lambda}}_x)+\ell\ell(r^{\boldsymbol{\mu}}_y)}\nu_p(h^{\boldsymbol{\lambda}}_{(i,m,a)}), & \text{if } e\nmid h^{\boldsymbol{\lambda}}_{(i,j,a)},\\[1ex] (-1)^{\ell\ell(r^{\boldsymbol{\lambda}}_x)+\ell\ell(r^{\boldsymbol{\mu}}_y)}\Big(\nu_p(h^{\boldsymbol{\lambda}}_{(i,m,a)})-\nu_p(h^{\boldsymbol{\lambda}}_{(i,j,a)})\Big), & \text{if } e\mid h^{\boldsymbol{\lambda}}_{(i,j,a)},\end{cases}$$
where the node $y\in[\boldsymbol{\mu}]$ is determined by $[\boldsymbol{\mu}]\backslash r^{\boldsymbol{\mu}}_y=[\boldsymbol{\lambda}]\backslash r^{\boldsymbol{\lambda}}_x$.

c) *In all other cases, $J_{\boldsymbol{\lambda}\boldsymbol{\mu}}=0$.*

Proof. Suppose that $J_{\boldsymbol{\lambda}\boldsymbol{\mu}}\ne0$. Then $\boldsymbol{\lambda}\rhd\boldsymbol{\mu}$ by Definition 2.5 and $\operatorname{res}(f^{\boldsymbol{\lambda}}_x)=\operatorname{res}(f^{\boldsymbol{\mu}}_y)$ by Lemma 3.4. Furthermore, there exist nodes $x=(i,j,a)\in[\boldsymbol{\lambda}]$ and $y=(k,l,b)\in[\boldsymbol{\mu}]$ such that $[\boldsymbol{\lambda}]\backslash r^{\boldsymbol{\lambda}}_x=[\boldsymbol{\mu}]\backslash r^{\boldsymbol{\mu}}_y$. Consequently, $\lambda^{(c)}\ne\mu^{(c)}$ for at most two values of c. Therefore, we may assume that we have integers $1\le a\le b\le r$ such that $\lambda^{(c)}=\mu^{(c)}$, for $c\ne a,b$.

If $a\ne b$ then the nodes x and y are uniquely determined because $r^{\boldsymbol{\lambda}}_x=[\lambda^{(a)}]\backslash[\mu^{(a)}]$ and $r^{\boldsymbol{\mu}}_y=[\mu^{(b)}]\backslash[\lambda^{(b)}]$. Note that $a<b$ since $\boldsymbol{\lambda}\rhd\boldsymbol{\mu}$. Therefore, we are in the situation considered in part (a). The formula for $J_{\boldsymbol{\lambda}\boldsymbol{\mu}}$ now follows directly from Definition 2.5 and Lemma 3.4.

Now assume that $a=b$. If $e=\infty$ then $\operatorname{res}(f^{\boldsymbol{\lambda}}_x)=\operatorname{res}(f^{\boldsymbol{\mu}}_y)$ if and only if $x=y$ since $h^{\boldsymbol{\lambda}}_x=h^{\boldsymbol{\mu}}_y$. This forces $\boldsymbol{\lambda}=\boldsymbol{\mu}$, which is not possible since $\boldsymbol{\lambda}\rhd\boldsymbol{\mu}$. Hence, e must be finite. Let $x=(i,j,a)$ and $y=(k,l,a)$ and observe that $l<j$ if and only if $\boldsymbol{\lambda}\rhd\boldsymbol{\mu}$, so we may assume that $l<j$. By Lemma 3.2 the abacus display for $\mu^{(a)}$ is obtained from the abacus display for $\lambda^{(b)}$ by moving one bead $h^{\boldsymbol{\lambda}}_x$ positions to the left *from* runner $\operatorname{res}(f^{\boldsymbol{\lambda}}_x)$, and other bead $h^{\boldsymbol{\lambda}}_x$ positions to the right *to* runner $\operatorname{res}(f^{\boldsymbol{\lambda}}_x)$.

Case 1. $e\nmid h^{\boldsymbol{\lambda}}_{(i,j,a)}$: By Lemma 3.2 and the remarks above, the beads on the abacus displays of $\lambda^{(a)}$ and $\mu^{(a)}$ are being moved between different runners. Therefore, the nodes $x=(i,j,a)\in[\boldsymbol{\lambda}]$ and $y=(k,l,a)\in[\boldsymbol{\mu}]$ are uniquely determined by the conditions $\operatorname{res}(f^{\boldsymbol{\lambda}}_x)=\operatorname{res}(f^{\boldsymbol{\mu}}_y)$ and $[\boldsymbol{\lambda}]\backslash r^{\boldsymbol{\lambda}}_x=[\boldsymbol{\mu}]\backslash r^{\boldsymbol{\mu}}_y$. Let $m=\mu^{(a)}_k$. Then $h^{\boldsymbol{\lambda}}_{(i,m,a)}=(j-\lambda^{(a)\prime}_i)-(l-\mu^{(a)\prime}_k)$ is the 'axial distance' from $f^{\boldsymbol{\lambda}}_x$ to $f^{\boldsymbol{\mu}}_y$, so that $e\mid h^{\boldsymbol{\lambda}}_{(i,m,a)}$. (In fact, $h^{\boldsymbol{\lambda}}_{(i,m,a)}$ is the axial distance between the corresponding 'hand nodes', but this distance is, of course, the same. Note also that, since $\operatorname{res}(f^{\boldsymbol{\lambda}}_x)=\operatorname{res}(f^{\boldsymbol{\mu}}_y)$, we have that $e\mid h^{\boldsymbol{\lambda}}_{(i,m,a)}$.) Hence, $J_{\boldsymbol{\lambda}\boldsymbol{\mu}}=(-1)^{\ell\ell(r^{\boldsymbol{\lambda}}_x)+\ell\ell(r^{\boldsymbol{\mu}}_y)}\nu_p\big(h^{\boldsymbol{\lambda}}_{(i,m,a)}\big)$ by Definition 2.5 and Lemma 3.4.

Case 2. $e\mid h^{\boldsymbol{\lambda}}_{(i,j,a)}$: Since $h^{\boldsymbol{\lambda}}_x\equiv0\pmod e$ unwrapping $r^{\boldsymbol{\lambda}}_x$ from $\boldsymbol{\lambda}$ and wrapping $r^{\boldsymbol{\mu}}_y$ back onto $\boldsymbol{\lambda}\backslash r^{\boldsymbol{\lambda}}_x$ corresponds to moving one bead on runner $\operatorname{res}(f^{\boldsymbol{\lambda}}_x)$ up $\frac1e h^{\boldsymbol{\lambda}}_x$ rows and another bead on runner $\operatorname{res}(f^{\boldsymbol{\lambda}}_x)$ down $\frac1e h^{\boldsymbol{\lambda}}_x$ rows. If in the abacus display for $\boldsymbol{\lambda}$ these beads were moved from rows $r_1>r_2$ to rows r'_1 and r'_2, respectively, then the abacus display for $\boldsymbol{\mu}$ can also be obtained from abacus display for $\boldsymbol{\lambda}$ by moving the bead in row r_1 to row r'_2 and moving the bead in row r_2 to row r'_1. That is, there exist nodes $x'\ne x$ and $y'\ne y$ such that we can obtain $\boldsymbol{\mu}$ by unwrapping $r^{\boldsymbol{\lambda}}_{x'}$ from $\boldsymbol{\lambda}$ and wrapping $r^{\boldsymbol{\mu}}_{y'}$ back onto $\boldsymbol{\lambda}\backslash r^{\boldsymbol{\lambda}}_{x'}$. By Lemma 3.2 there are no other ways of obtaining $\boldsymbol{\mu}$ by unwrapping a rim hook from $\boldsymbol{\lambda}$

and wrapping it back on again. Since $\boldsymbol\lambda \rhd \boldsymbol\mu$ we can choose the nodes $x = (i,j,a)$ and $y = (k,l,a)$ above so that $r_1 > r_1' > r_2' > r_2$. Then $x' = (i,m,a)$, where $m = \mu_k^{(a)}$, and $y' = (\lambda_j^{(a)'}, l, a)$. Further, $\ell\ell(r_x^{\boldsymbol\lambda}) + \ell\ell(r_y^{\boldsymbol\mu}) = \lambda_j^{(a)'} - i + \mu_l^{(a)'} - k$ and $\ell\ell(r_{x'}^{\boldsymbol\lambda}) + \ell\ell(r_{y'}^{\boldsymbol\mu}) = \lambda_m^{(a)'} - i + \mu_l^{(a)'} - \lambda_j^{(a)'}$. But by construction, $k = \lambda_m^{(a)'} + 1$ so $\ell\ell(r_x^{\boldsymbol\lambda}) + \ell\ell(r_y^{\boldsymbol\mu})$ and $\ell\ell(r_{x'}^{\boldsymbol\lambda}) + \ell\ell(r_{y'}^{\boldsymbol\mu})$ have opposite parities. The axial distance from $f_x^{\boldsymbol\lambda}$ to $f_y^{\boldsymbol\mu}$ is $h_{(i,m,a)}^{\boldsymbol\lambda}$ (where $e \mid h_{(i,m,a)}^{\boldsymbol\lambda}$ since $\mathrm{res}(f_x^{\boldsymbol\lambda}) = \mathrm{res}(f_y^{\boldsymbol\mu})$) and the axial distance from $f_{x'}^{\boldsymbol\lambda}$ to $f_{y'}^{\boldsymbol\mu}$ is $h_{(i,j,a)}^{\boldsymbol\lambda}$. Therefore,

$$J_{\boldsymbol\lambda\boldsymbol\mu} = (-1)^{\ell\ell(r_x^{\boldsymbol\lambda}) + \ell\ell(r_y^{\boldsymbol\mu})}\Big(\nu_p(h_{(i,m,a)}^{\boldsymbol\lambda}) - \nu_p(h_{(i,j,a)}^{\boldsymbol\lambda})\Big)$$

as required.

We have now exhausted all of the cases where $J_{\boldsymbol\lambda\boldsymbol\mu}$ is non–zero, so the Proposition is proved. □

3.3. **Residue equivalence.** We are now ready to start proving that $\boldsymbol\lambda \sim_J \boldsymbol\mu$ whenever $\boldsymbol\lambda \sim_C \boldsymbol\mu$.

We say that a rim hook of $\boldsymbol\lambda$ is vertical if it is contained within a single column of $[\boldsymbol\lambda]$.

3.7. **Proposition.** *Suppose that $\boldsymbol\lambda, \boldsymbol\mu \in \Lambda_{r,n}^+$ and that there is an integer a, with $1 \le a \le r$, such that $\overline{\lambda^{(a)}} = \overline{\mu^{(a)}}$ and $\lambda^{(c)} = \mu^{(c)}$, for $c \ne a$. Then $\boldsymbol\lambda \sim_J \boldsymbol\mu$.*

Proof. If $e = \infty$ then $\overline{\lambda^{(a)}} = \overline{\mu^{(a)}}$ if and only if $\lambda^{(a)} = \mu^{(a)}$ so there is nothing to prove. Assume that e is finite. Let $w_a = \mathrm{w}(\lambda^{(a)})$. If $w_a = 0$ then $\overline{\lambda^{(a)}} = \lambda^{(a)}$ so that $\boldsymbol\lambda = \boldsymbol\mu$ and there is nothing to prove. So we can assume that $w_a > 0$.

Let $\Lambda_a(\boldsymbol\lambda) = \{\, \boldsymbol\mu \in \Lambda_{r,n}^+ \mid \overline{\mu^{(a)}} = \overline{\lambda^{(a)}}$ and $\mu^{(c)} = \lambda^{(c)}$ when $c \ne a\,\}$ and let $\boldsymbol\rho$ be the multipartition in $\Lambda_a(\boldsymbol\lambda)$ where $\rho^{(a)}$ is the partition obtained by wrapping w_a vertical e–hooks onto the first column of the e–core of $\lambda^{(a)}$. Then $\boldsymbol\mu \trianglerighteq \boldsymbol\rho$ for all $\boldsymbol\mu \in \Lambda_a(\boldsymbol\lambda)$. To prove the Lemma it is enough to show that $\boldsymbol\mu \sim_J \boldsymbol\rho$, for all $\boldsymbol\mu \in \Lambda_a(\boldsymbol\lambda)$. By induction on dominance we may assume that $\boldsymbol\mu \sim_J \boldsymbol\rho$ whenever $\boldsymbol\mu \in \Lambda_a(\boldsymbol\lambda)$ and $\boldsymbol\lambda \rhd \boldsymbol\mu$. If $J_{\boldsymbol\lambda\boldsymbol\mu} \ne 0$ for some $\boldsymbol\mu \in \Lambda_a(\boldsymbol\lambda)$ then $\boldsymbol\lambda \sim_J \boldsymbol\mu$. As $\boldsymbol\lambda \rhd \boldsymbol\mu$, we have that $\boldsymbol\mu \sim_J \boldsymbol\rho$ by induction, so that $\boldsymbol\lambda \sim_J \boldsymbol\mu \sim_J \boldsymbol\rho$.

It remains to consider the case when $\boldsymbol\lambda \rhd \boldsymbol\rho$ and $J_{\boldsymbol\lambda\boldsymbol\mu} = 0$ for all $\boldsymbol\mu \in \Lambda_a(\boldsymbol\lambda)$. By Lemma 3.6 (b),

$$\nu_p(h_{(i,m,a)}^{\boldsymbol\lambda}) = \nu_p(h_{(i,j,a)}^{\boldsymbol\lambda}), \qquad \text{for all } (i,m,a), (i,j,a) \in [\boldsymbol\lambda].$$

This is precisely the condition for the Weyl module $\Delta(\lambda^{(a)})$ to be irreducible (in the case $r = 1$). The conjugates of these partitions are described explicitly in [**20**, Theorem 4.19]. For us the most important properties of these partitions is that all of e–hooks which can be unwrapped from $\lambda^{(a)}$ when constructing its e–core $\rho^{(a)}$ are vertical, ν_p is constant on the rows of $[\lambda^{(a)}]$, and $\rho_i^{(a)'} \equiv \rho_{i-1}^{(a)'} - 1 \pmod e$ whenever $\lambda_i^{(a)'} \ne \rho_i^{(a)'}$. Since $w_a > 0$ we can find a (unique) node $(i,j,a) \in [\boldsymbol\lambda]$ such that $h_{(i,j,a)}^{\boldsymbol\lambda} \equiv 0 \pmod e$ and $h_{(i',j',a)}^{\boldsymbol\lambda} \not\equiv 0 \pmod e$, for all $(i',j',a) \in [\boldsymbol\lambda]$ with $(i',j') \ne (i,j)$, $i' \le i$ and $j' \ge j$. Let $\boldsymbol\nu$ be the multipartition obtained by unwrapping $r_{(i,j,a)}^{\boldsymbol\lambda}$ from $[\boldsymbol\lambda]$ and wrapping it back on to the end of the first row of $[\boldsymbol\lambda]\backslash r_{(i,j,a)}^{\boldsymbol\lambda}$. Similarly, let $\boldsymbol\mu$ be the multipartition obtained by unwrapping this same hook from $\boldsymbol\lambda$ and wrapping it back on to the end of the first column of $[\boldsymbol\lambda]\backslash r_{(i,j,a)}^{\boldsymbol\lambda}$. Therefore, $J_{\boldsymbol\nu\boldsymbol\lambda} \ne 0$ and $J_{\boldsymbol\nu\boldsymbol\mu} \ne 0$, by Lemma 3.6 (b), so that $\boldsymbol\lambda \sim_J \boldsymbol\nu \sim_J \boldsymbol\mu$. Note that $\boldsymbol\lambda \rhd \boldsymbol\rho$ implies that $j > 1$, so that $\boldsymbol\lambda \rhd \boldsymbol\mu$. Consequently, $\boldsymbol\lambda \sim_J \boldsymbol\rho$ by induction. □

Recall that the e–cores of the partitions of n completely determine the blocks when $r = 1$. We have the following imperfect generalization when $r > 1$.

3.8. **Definition.** Suppose that $\boldsymbol\lambda = \big(\lambda^{(1)}, \dots, \lambda^{(r)}\big)$ is a multipartition. Then the e**–multicore** of $\boldsymbol\lambda$ is the multipartition $\overline{\boldsymbol\lambda} = \big(\overline{\lambda}^{(1)}, \dots, \overline{\lambda}^{(r)}\big)$. We abuse notation and say that $\boldsymbol\lambda$ is a multicore if $\boldsymbol\lambda = \overline{\boldsymbol\lambda}$.

By Corollary 3.3 (a), the e–multicore $\overline{\boldsymbol{\lambda}}$ of $\boldsymbol{\lambda}$ is obtained from $\boldsymbol{\lambda}$ by sequentially unwrapping all e–rim hooks from the diagram of $\boldsymbol{\lambda}$, in any order. Note that if $e = \infty$ then every multipartition is an e–multicore.

Mimicking the representation theory of the symmetric groups, define $\mathrm{w_e}(\boldsymbol{\lambda})$ to be the number of e–hooks that have to be unwrapped from $\boldsymbol{\lambda}$ to construct $\overline{\boldsymbol{\lambda}}$. If e is finite then $\mathrm{w_e} = \frac{1}{e}(|\boldsymbol{\lambda}| - |\overline{\boldsymbol{\lambda}}|)$, whereas $\mathrm{w}_\infty(\boldsymbol{\lambda}) = 0$. Now define

$$\mathrm{W_e}(\boldsymbol{\lambda}) = \max\{\mathrm{w_e}(\boldsymbol{\mu}) \mid \boldsymbol{\mu} \sim_C \boldsymbol{\lambda}\}.$$

Note that while $\mathrm{W_e}(\boldsymbol{\lambda})$ is well defined, it is not immediately clear how to compute it.

3.9. **Lemma.** *Suppose that $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \Lambda^+_{r,n}$ and that $\overline{\boldsymbol{\lambda}} = \overline{\boldsymbol{\mu}}$. Then $\boldsymbol{\lambda} \sim_J \boldsymbol{\mu}$.*

Proof. Suppose first that $|\lambda^{(a)}| = |\mu^{(a)}|$, for $1 \le a \le r$. Then $\boldsymbol{\lambda} \sim_J \boldsymbol{\mu}$ by successive applications of Proposition 3.7. If this is not the case then by successively unwrapping e–hooks from one component of $\boldsymbol{\lambda}$ and wrapping them back onto a different component without changing their foot residue we can obtain another multipartition $\boldsymbol{\nu}$ such that $|\nu^{(a)}| = |\mu^{(a)}|$, for $1 \le a \le r$. Then $\boldsymbol{\lambda} \sim_J \boldsymbol{\nu}$ by Proposition 3.6 (and Lemma 3.2). By the first line of the proof $\boldsymbol{\nu} \sim_J \boldsymbol{\mu}$, so $\boldsymbol{\lambda} \sim_J \boldsymbol{\mu}$ as required. □

In order to consider two multipartitions which are residue equivalent but have different multicores, we make the following definitions.

3.10. **Definition.** a) Suppose that $\boldsymbol{\lambda}$ is a multicore. If e is finite, define $s^{ab}_{ij}(\boldsymbol{\lambda})$ to be the multicore whose abacus display is obtained by moving a bead from runner i to runner j on the abacus for $\lambda^{(a)}$ and moving a bead from runner j to runner i on the abacus for $\lambda^{(b)}$. If $e = \infty$ and the abacus display for $\lambda^{(a)}$ contains a bead in position i but not in position j, while the abacus display for $\lambda^{(b)}$ contains a bead in position j but not in position i, define $s^{ab}_{ij}(\boldsymbol{\lambda})$ to be the multicore whose abacus display is obtained by moving a bead from position i to position j on the abacus for $\lambda^{(a)}$ and from j to position i on the abacus for $\lambda^{(b)}$.

b) Suppose that e is finite and let $\boldsymbol{\lambda}$ be a multipartition. Define $t^a_{iw}(\boldsymbol{\lambda})$ to be the multipartition whose abacus display is obtained by moving the lowest bead on runner i of the abacus for $\lambda^{(a)}$ down w rows.

3.11. **Lemma.** *Suppose that $\boldsymbol{\lambda} \sim_C \boldsymbol{\mu}$ and that $\overline{\boldsymbol{\mu}} = s^{ab}_{ij}(\overline{\boldsymbol{\lambda}})$. Then $\boldsymbol{\lambda} \sim_J \boldsymbol{\mu}$.*

Proof. Let $\boldsymbol{\nu} = t^a_{i\,\mathrm{w_e}(\boldsymbol{\lambda})}(\overline{\boldsymbol{\lambda}})$ and $\boldsymbol{\rho} = t^a_{j\,\mathrm{w_e}(\boldsymbol{\mu})}(\overline{\boldsymbol{\mu}})$. Then $\boldsymbol{\lambda} \sim_J \boldsymbol{\nu}$ and $\boldsymbol{\rho} \sim_J \boldsymbol{\mu}$ by Lemma 3.9. Furthermore, the multipartitions $\boldsymbol{\nu}$ and $\boldsymbol{\rho}$ satisfy the conditions of Proposition 3.6 (a), so $\boldsymbol{\lambda} \sim_J \boldsymbol{\nu} \sim_J \boldsymbol{\rho} \sim_J \boldsymbol{\mu}$ as required. □

We now need several results and definitions of Fayers from the papers [**14**, **15**]. In these papers Fayers assumes the classification of the blocks of the Ariki–Koike algebras. He remarks before [**15**, Theorem 1.5] that the paper [**15**] only ever uses the fact that if two Specht modules belong to the same block then they are residue equivalent. We have already proved this in Corollary 3.5. The paper [**14**] requires more careful consideration. In this paper, Fayers describes certain sets of multipartitions, each of which is of the form $\{\boldsymbol{\mu} \mid \boldsymbol{\mu} \sim_C \boldsymbol{\lambda}\}$ for some multipartition $\boldsymbol{\lambda}$. His construction does not rely on the assumption that $\boldsymbol{\lambda} \sim_C \boldsymbol{\mu}$ implies that the corresponding Specht modules lie in the same block. We may therefore use his descriptions.

3.12. **Definition** (Fayers [**15**, §2.1]). Suppose that $\boldsymbol{\lambda}$ is a multipartition. Then the e**–weight** of $\boldsymbol{\lambda}$ is the integer

$$\mathrm{wt}(\boldsymbol{\lambda}) = \sum_{j=1}^{r} C_{c_j}(\boldsymbol{\lambda}) - \frac{1}{2} \sum_{f \in \mathbb{Z}/e\mathbb{Z}} (C_f(\boldsymbol{\lambda}) - C_{f+1}(\boldsymbol{\lambda}))^2.$$

Fayers [**15**] shows that $\mathrm{wt}(\boldsymbol{\lambda}) \geq 0$ for all multipartitions $\boldsymbol{\lambda}$, and that if $r = 1$, it coincides with the usual definition $\mathrm{w}(\lambda)$ of weight. Further, if $\boldsymbol{\lambda} \sim_C \boldsymbol{\mu}$ then $\mathrm{wt}(\boldsymbol{\lambda}) = \mathrm{wt}(\boldsymbol{\mu})$, so the function $\mathrm{wt}(\cdot)$ is constant on the residue classes of $\Lambda^+_{r,n}$. The results of [**15**, Prop. 3.8] show how to use the abacus display of $\boldsymbol{\lambda}$ to calculate $\mathrm{wt}(\boldsymbol{\lambda})$. Combining this method with Lemma 3.16 below gives a way of computing $\mathrm{W_e}(\boldsymbol{\lambda})$ using the abacus display of $\boldsymbol{\lambda}$. We leave the details to the reader.

Recall that a node $(i,j,a) \in [\boldsymbol{\lambda}]$ is **removable** if $[\boldsymbol{\lambda}]\backslash\{(i,j,a)\}$ is the diagram of some multipartition $\boldsymbol{\nu} \in \Lambda^+_{r,n-1}$. Similarly, a node $(i,j,a) \notin [\boldsymbol{\lambda}]$ is a **addable** if $[\boldsymbol{\lambda}] \cup \{(i,j,a)\}$ is the diagram of some multipartition $\boldsymbol{\nu} \in \Lambda^+_{r,n+1}$. The node $x = (i,j,a)$ is an f–node if $\mathrm{res}(x) = f$.

Let $\boldsymbol{\lambda}$ be a multipartition. For $f \in \mathbb{Z}/e\mathbb{Z}$ and $a \in \{1,\dots,r\}$, define

$$\delta^a_f(\boldsymbol{\lambda}) = \#\{ \text{ removable } f\text{–nodes of } [\lambda^{(a)}] \} - \#\{ \text{ addable } f\text{–nodes of } [\lambda^{(a)}] \}$$

and set

$$\delta_f(\boldsymbol{\lambda}) = \sum_{j=1}^{r} \delta^k_i(\boldsymbol{\lambda}).$$

The sequence $(\delta_f(\boldsymbol{\lambda}) \mid f \in \mathbb{Z}/e\mathbb{Z})$ is the **hub** of $\boldsymbol{\lambda}$. The hub of $\boldsymbol{\lambda}$ can be read off the abacus display of $\boldsymbol{\lambda}$ using Lemma 3.2.

Observe that Corollary 3.3 implies that if e is finite then the hub is unchanged by wrapping he–hooks onto $[\boldsymbol{\lambda}]$, for $h \geq 1$. Furthermore, $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ have the same hub if $\boldsymbol{\mu} = s^{ab}_{ij}(\boldsymbol{\lambda})$, for some a, b, i, j.

3.13. **Proposition** (Fayers [**15**, Proposition 3.2 and Lemma 3.3]). *Suppose that $\boldsymbol{\lambda}$ is a multipartition of n and $\boldsymbol{\mu}$ is a multipartition of m. Then*

a) *If $e < \infty$ and $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ have the same hub then $m \equiv n \mod e$ and*

$$\mathrm{wt}(\boldsymbol{\lambda}) - \mathrm{wt}(\boldsymbol{\mu}) = \frac{r(n-m)}{e};$$

b) *If $n = m$ then $\boldsymbol{\lambda} \sim_C \boldsymbol{\mu}$ if and only if they have the same hub.*

Consequently, if $\boldsymbol{\mu}$ is obtained from $\boldsymbol{\lambda}$ by wrapping on an e–hook, then $\mathrm{wt}(\boldsymbol{\mu}) = \mathrm{wt}(\boldsymbol{\lambda}) + r$.

The next result will let us determine when $\mathrm{W_e}(\boldsymbol{\lambda}) = \mathrm{w_e}(\boldsymbol{\lambda})$.

3.14. **Proposition** (Fayers [**14**, Theorem 3.1]). *Suppose that $\boldsymbol{\lambda} \in \Lambda^+_{r,n}$ is a multipartition. Then the following are equivalent.*

a) *$\boldsymbol{\mu}$ is a multicore whenever $\boldsymbol{\mu} \sim_C \boldsymbol{\lambda}$.*

b) *$\mathrm{wt}(\boldsymbol{\mu}) \geq \mathrm{wt}(\boldsymbol{\lambda})$ whenever $\boldsymbol{\mu}$ and $\boldsymbol{\lambda}$ have the same hub.*

3.15. **Definition.** A multipartition $\boldsymbol{\lambda}$ is a **reduced multicore** if it satisfies the conditions of Proposition 3.14.

Not every multicore is reduced. If $\boldsymbol{\lambda}$ is a reduced multicore then the block which contains $\Delta(\boldsymbol{\lambda})$ is, in general, not simple. In contrast, when $r = 1$ every core is a reduced multicore and the block containing a core is always simple. If $\boldsymbol{\lambda}$ is an reduced multicore then Fayers [**14**] calls the set of multipartitions $\{ \boldsymbol{\mu} \mid \boldsymbol{\mu} \sim_C \boldsymbol{\lambda} \}$ a 'core block'.

3.16. **Lemma.** *Suppose that $\boldsymbol{\lambda} \in \Lambda^+_{n,r}$. Then $\overline{\boldsymbol{\lambda}}$ is a reduced multicore if and only if $\mathrm{w_e}(\boldsymbol{\lambda}) = \mathrm{W_e}(\boldsymbol{\lambda})$.*

Proof. Suppose $\mathrm{w_e}(\boldsymbol{\lambda}) \neq \mathrm{W_e}(\boldsymbol{\lambda})$. By definition, there exists a multipartition $\boldsymbol{\mu}$ such that $\boldsymbol{\mu} \sim_C \boldsymbol{\lambda}$ and $\mathrm{w_e}(\boldsymbol{\mu}) > \mathrm{w_e}(\boldsymbol{\lambda})$. Now $\overline{\boldsymbol{\mu}}$ and $\overline{\boldsymbol{\lambda}}$ have the same hub, and by Proposition 3.13, $\mathrm{wt}(\overline{\boldsymbol{\mu}}) < \mathrm{wt}(\overline{\boldsymbol{\lambda}})$, contradicting Condition (b) of Proposition 3.14. Therefore, $\overline{\boldsymbol{\lambda}}$ is not a reduced multicore.

Now suppose that $\overline{\boldsymbol{\lambda}}$ is not a reduced multicore. Then there exists a multipartition $\boldsymbol{\mu}$, which is not a multicore, such that $\boldsymbol{\mu} \sim_C \overline{\boldsymbol{\lambda}}$. Let $\boldsymbol{\nu} = t^1_{0\,\mathrm{w_e}(\boldsymbol{\lambda})}(\boldsymbol{\mu})$. Then $\boldsymbol{\nu} \sim_C \boldsymbol{\lambda}$ and $\mathrm{w_e}(\boldsymbol{\nu}) > \mathrm{w_e}(\boldsymbol{\lambda})$. Hence, $\mathrm{W_e}(\boldsymbol{\lambda}) > \mathrm{w_e}(\boldsymbol{\lambda})$. □

3.17. **Lemma** (Fayers [**14**, Proof of Proposition 3.7 (1)]). *Suppose that $\boldsymbol{\lambda}$ is a multicore which is not reduced. Then there exists a sequence of multicores $\boldsymbol{\lambda}_0 = \boldsymbol{\lambda}, \boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_k = \boldsymbol{\mu}$ such that $\mathrm{wt}(\boldsymbol{\mu}) < \mathrm{wt}(\boldsymbol{\lambda})$, and $\boldsymbol{\lambda}_{m+1} = s_{i_m j_m}^{a_m b_m}(\boldsymbol{\lambda}_m)$ and $\mathrm{wt}(\boldsymbol{\lambda}_m) \le \mathrm{wt}(\boldsymbol{\lambda})$, for $0 \le m < k$.*

3.18. **Lemma** (Fayers [**14**, Proof of Proposition 3.7 (2)]). *Suppose that $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are reduced multicores and that $\boldsymbol{\lambda} \sim_C \boldsymbol{\mu}$. Then there exists a sequence of multicores $\boldsymbol{\lambda}_0 = \boldsymbol{\lambda}, \boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_k = \boldsymbol{\mu}$ such that $\boldsymbol{\lambda}_{m+1} = s_{i_m j_m}^{a_m b_m}(\boldsymbol{\lambda}_m)$ and $\boldsymbol{\lambda}_{m+1} \sim_C \boldsymbol{\lambda}_m$, for $0 \le m < k$.*

We can now complete the proof of Theorem 2.11 when $q \ne 1$ and the parameters $Q_1, \dots, Q_r$ are non–zero. Consequently, this completes the proofs of Theorem A and Theorem B from the introduction.

3.19. **Theorem.** *Suppose that $q \ne 1$ and that the parameters $Q_1, \dots, Q_r$ are non–zero. Let $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ be multipartitions in $\Lambda_{n,r}^+$. Then $\boldsymbol{\lambda} \sim_C \boldsymbol{\mu}$ if and only if $\boldsymbol{\lambda} \sim_J \boldsymbol{\mu}$.*

Proof. By Corollary 3.5 if $\boldsymbol{\lambda} \sim_J \boldsymbol{\mu}$ then $\boldsymbol{\lambda} \sim_C \boldsymbol{\mu}$. Therefore, to prove the theorem it is sufficient to prove the following two statements.

a) Suppose that $\mathrm{w_e}(\boldsymbol{\lambda}) < \mathrm{W_e}(\boldsymbol{\lambda})$. Then $\boldsymbol{\mu} \sim_J \boldsymbol{\lambda}$ and $\mathrm{w_e}(\boldsymbol{\mu}) > \mathrm{w_e}(\boldsymbol{\lambda})$, for some $\boldsymbol{\mu} \in \Lambda_{r,n}^+$.

b) Suppose that $\boldsymbol{\lambda} \sim_C \boldsymbol{\mu}$ and that $\mathrm{w_e}(\boldsymbol{\lambda}) = \mathrm{W_e}(\boldsymbol{\lambda}) = \mathrm{w_e}(\boldsymbol{\mu})$. Then $\boldsymbol{\mu} \sim_J \boldsymbol{\lambda}$.

Suppose, as in (a), that $\mathrm{w_e}(\boldsymbol{\lambda}) < \mathrm{W_e}(\boldsymbol{\lambda})$. Then e is finite and by Lemma 3.16, $\overline{\boldsymbol{\lambda}}$ is not a reduced multicore. By Lemma 3.17, there exists a sequence of multicores $\boldsymbol{\lambda}_0 = \overline{\boldsymbol{\lambda}}, \boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_k = \boldsymbol{\mu}$ such that $\mathrm{wt}(\boldsymbol{\mu}) < \mathrm{wt}(\overline{\boldsymbol{\lambda}})$, $\boldsymbol{\lambda}_{m+1} = s_{i_m j_m}^{a_m b_m}(\boldsymbol{\lambda}_m)$ and $\mathrm{wt}(\boldsymbol{\lambda}_m) \le \mathrm{wt}(\overline{\boldsymbol{\lambda}})$, for $0 \le m < k$. Fix m with $0 \le m < k$. Since $\boldsymbol{\lambda}_m$ and $\overline{\boldsymbol{\lambda}}$ have the same hub, Proposition 3.13 says that $|\boldsymbol{\lambda}_m| \le |\overline{\boldsymbol{\lambda}}|$ and $|\overline{\boldsymbol{\lambda}}| \equiv |\boldsymbol{\lambda}_m| \pmod e$, and that $|\boldsymbol{\mu}| < |\overline{\boldsymbol{\lambda}}|$. Define $w_m = \mathrm{w_e}(\boldsymbol{\lambda}) + \frac{1}{e}(|\overline{\boldsymbol{\lambda}}| - |\boldsymbol{\lambda}_m|)$ and set $\boldsymbol{\nu}_m = t_{0 w_m}^1(\boldsymbol{\lambda}_m)$. Then $\boldsymbol{\nu}_m \sim_J \boldsymbol{\nu}_{m+1}$ by Lemma 3.11, so that $\boldsymbol{\lambda} \sim_J \boldsymbol{\mu}$. Moreover, $\mathrm{w_e}(\boldsymbol{\mu}) = \mathrm{w_e}(\boldsymbol{\lambda}) + \frac{1}{e}(|\overline{\boldsymbol{\lambda}}| - |\boldsymbol{\mu}|) > \mathrm{w_e}(\boldsymbol{\lambda})$ as required.

Now consider (b), that is, suppose that $\boldsymbol{\lambda} \sim_C \boldsymbol{\mu}$ and $\mathrm{w_e}(\boldsymbol{\lambda}) = \mathrm{W_e}(\boldsymbol{\lambda}) = \mathrm{w_e}(\boldsymbol{\mu})$. By Lemma 3.16, $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are reduced multicores. Then, by Lemma 3.18, there exist multicores $\boldsymbol{\lambda}_0 = \overline{\boldsymbol{\lambda}}, \boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_k = \overline{\boldsymbol{\mu}}$ such that $\boldsymbol{\lambda}_{m+1} = s_{i_m j_m}^{a_m b_m}(\boldsymbol{\lambda}_m)$ and $\boldsymbol{\lambda}_{m+1} \sim_C \boldsymbol{\lambda}_m$. For $0 \le m < k$, define $\boldsymbol{\nu}_m = t_{0\,\mathrm{w_e}(\boldsymbol{\lambda})}^1(\boldsymbol{\lambda}_m)$. Then by Lemma 3.11, $\boldsymbol{\nu}_m \sim_J \boldsymbol{\nu}_{m+1}$ and by Lemma 3.9, $\boldsymbol{\lambda} \sim_J \boldsymbol{\nu}_0 \sim_J \boldsymbol{\nu}_1 \sim_J \cdots \sim_J \boldsymbol{\nu}_k \sim_J \boldsymbol{\mu}$ as required. □

4. THE BLOCKS FOR ALGEBRAS WITH EXCEPTIONAL PARAMETERS

In this section we classify the blocks of the Ariki–Koike algebras for the remaining cases from (2.13). That is, we assume that the parameters satisfy one of the following four cases:

Case 2. $r = 1$, $q = 1$ and $Q_1 = 1$.
Case 3. $r > 1$, $q = 1$ and $Q_1 = \cdots = Q_r = 1$.
Case 4. $r > 1$, $q = 1$ and $Q_1 = \cdots = Q_r = 0$.
Case 5. $r > 1$, $q \ne 1$ and $Q_1 = \cdots = Q_r = 0$.

As in the previous section the basic strategy is to use the Jantzen sum formula to analyze the combinatorics of the Jantzen coefficients.

We distinguish between cases 2 and 3 because the blocks differ dramatically in these two cases. In fact, the blocks in Case 2 behave like the blocks when $q \ne 1$ and the parameters $Q_1, \dots, Q_r$ are non-zero. Quite surprisingly, the algebras $\mathscr{H}_{r,n}$ and $\mathscr{S}_{r,n}$ have only one block in Cases 3–5.

In all cases the blocks of the algebras $\mathscr{H}_{r,n}$ and $\mathscr{S}_{r,n}$ are determined by Jantzen equivalence by Proposition 2.9. This section gives an explicit description of when two multipartitions are Jantzen equivalent in cases 2–5 above.

4.1. The blocks when $r = 1$ and $q = 1$. Assume that we are in Case 2 above and let $\mathscr{H}_n = \mathscr{H}_{1,n}$ and $\mathscr{S}_n = \mathscr{S}_{1,n}$. In this case the Specht modules and Weyl modules are indexed by partitions, rather than multipartitions, so we write λ in place of $\boldsymbol{\lambda}$, and so on. The nodes in the diagrams of partitions are all of the form $(i,j,1)$, for $i,j\ge 1$, so we drop the trailing 1 from this notation and consider a node to be an ordered pair (i,j), so that $[\lambda]=\{\,(i,j)\mid 1\le j\le\lambda_i\,\}$.

As $q=1$ we have that $e=p$. Following section 3 define the **residue** of a node $x=(i,j)$ to be

$$\operatorname{res}(x)=(j-i)\pmod p.$$

Once again, $\{\,\operatorname{res}(x)\mid x\in[\boldsymbol\lambda]$ for some $\boldsymbol\lambda\in\Lambda^+_{r,n}\,\}\subseteq\mathbb Z/p\mathbb Z$. For a partition λ and $f\in\mathbb Z/p\mathbb Z$ put $C_f(\boldsymbol\lambda)=\#\{\,x\in[\boldsymbol\lambda]\mid\operatorname{res}(x)=f\,\}$ and define $\lambda\sim_C\mu$ if $C_f(\lambda)=C_f(\mu)$, for all $f\in\mathbb Z/p\mathbb Z$. Then it is well–known (and easy to prove using Corollary 3.3 (a)) that $\boldsymbol\lambda\sim_C\boldsymbol\mu$ if and only if λ and μ have the same p–core.

We can now prove Theorem 2.11 when $q=1$ and $r=1$. To prove this result we need to show that the Jantzen and residue equivalence relations on the set of partitions coincide. We follow the argument of the previous section.

The analogue of Lemma 3.4 in Case 2 is as follows.

4.1. **Lemma.** *Suppose that λ and μ are multipartitions of n and that $[\lambda]\backslash r^\lambda_x=[\mu]\backslash r^\mu_y$, for some nodes $x=(i,j)\in[\lambda]$ and $y=(k,l)\in[\mu]$. Then*

$$\nu_\pi\big(\operatorname{res}_{\mathcal O}(f^\lambda_x)-\operatorname{res}_{\mathcal O}(f^\mu_y)\big)=\nu_p\big(j-\lambda'_j-l+\mu'_l\big).$$

Proof. Let $i'=\lambda'_i$ and $k'=\mu'_k$ so that $f^\lambda_x=(i',j)$ and $f^\mu_y=(k',l)$. Then

$$\operatorname{res}_{\mathcal O}(f^\lambda_x)-\operatorname{res}_{\mathcal O}(f^\mu_y)=t^{na+j-i'}-t^{na+l-k'}=t^{na+l-k'}(t^{j-i'-l+k'}-1).$$

Mimicking the proof of Lemma 3.4, let $h=j-i'-l+k'$. Then

$$\nu_\pi\big(\operatorname{res}_{\mathcal O}(f^\lambda_x)-\operatorname{res}_{\mathcal O}(f^\mu_y)\big)=\nu_\pi(t^{j-i'-l+k'}-1)=1+\nu_\pi([h]_t).$$

Repeating the second half of the proof of Lemma 3.4 completes the proof. □

The only difference between Lemma 3.4 and Lemma 4.1 is that now $\nu_\pi\big(\operatorname{res}_{\mathcal O}(f^\lambda_x)-\operatorname{res}_{\mathcal O}(f^\mu_y)\big)$ is non–zero whenever $[\lambda]\backslash r^\lambda_x=[\mu]\backslash r^\mu_y$; that is, we no longer require that $\operatorname{res}(f^\lambda_x)=\operatorname{res}(f^\mu_y)$.

4.2. **Proposition.** *Let λ and μ are partitions of n. Then $J_{\lambda\mu}$ is non–zero only if p is finite and there exist nodes $x=(i,j),(i,m)\in[\lambda]$ such that $m<j$, $p\mid h^\lambda_{(i,m)}$ and μ is obtained by wrapping a rim hook of length h^λ_x onto $\lambda\backslash r^\lambda_x$ with its highest node in column m. In this case*

$$J_{\lambda\mu}=\begin{cases}(-1)^{\ell\ell(r^\lambda_x)+\ell\ell(r^\mu_y)}\nu_p(h^\lambda_{(i,m)}), & \text{if } p\nmid h^\lambda_{(i,j)},\\ (-1)^{\ell\ell(r^\lambda_x)+\ell\ell(r^\mu_y)}\Big(\nu_p(h^\lambda_{(i,m)})-\nu_p(h^\lambda_{(i,j)})\Big), & \text{if } p\mid h^\lambda_{(i,j)},\end{cases}$$

where the node $y\in[\mu]$ is determined by $[\mu]\backslash r^\mu_y=[\lambda]\backslash r^\lambda_x$.

Proof. Suppose that $J_{\lambda\mu}\ne0$. Then $\lambda\rhd\mu$ by Definition 2.5 and there exist nodes $x=(i,j)\in[\lambda]$ and $y=(k,l,b)\in[\mu]$ such that $[\lambda]\backslash r^\lambda_x=[\mu]\backslash r^\mu_y$.

Case 1. $\operatorname{res}(f^\lambda_x)\ne\operatorname{res}(f^\mu_y)$: Unwrapping the rim hook r^λ_x from λ moves a bead on the abacus for λ from runner $\operatorname{res}(f^\lambda_x)$ to runner r_1, say, and wrapping the rim hook r^μ_y back onto $\lambda\backslash r^\lambda_x$ moves a bead from runner r_2 to runner $\operatorname{res}(f^\mu_y)$. Since $\operatorname{res}(f^\lambda_x)\ne\operatorname{res}(f^\mu_y)$ we can also construct the partition μ from λ by moving a bead from runner $\operatorname{res}(f^\lambda_x)$ to runner r_2 and then moving a bead from runner r_1 to runner $\operatorname{res}(f^\mu_y)$. Comparing the abacus displays of λ and μ, there are no other ways of obtaining μ from λ by moving a single rim hook. As in the proof of Proposition 3.6, the sums of the leg lengths for the two different ways of changing λ into μ by moving a rim hook have different parities, so their contributions to $J_{\lambda\mu}$ cancel out. Hence, $J_{\lambda\mu}=0$ when $\operatorname{res}(f^\lambda_x)\ne\operatorname{res}(f^\mu_y)$.

Case 2. $\operatorname{res}(f^\lambda_x)=\operatorname{res}(f^\mu_y)$: The proof of Proposition 3.6 in the case when $a=b$ can now be repeated without change to complete the proof of the Proposition. □

4.3. **Corollary.** *Suppose that λ and μ are partitions of n. Then $\lambda \sim_J \mu$ if and only if $\lambda \sim_C \mu$.*

Proof. By Proposition 4.2, $\lambda \sim_C \mu$ whenever $\lambda \sim_J \mu$. The reverse implication follows by the argument of Proposition 3.7 since this proof only uses part (b) of Proposition 3.6, which is the same as the statement of Proposition 4.2. □

Remark. Corollary 4.3 completes the classification of the blocks of the q–Schur algebras and the Hecke algebras of type A; that is when $r = 1$. Unfortunately, the classification of the blocks of the q–Schur algebras given in [**20**, Theorem 4.24] (and reproduced in [**23**, Theorem 5.47]), contains a gap because these two proofs only consider the case of reducible Weyl modules. Fortunately, the classification of the blocks of the Hecke algebras of type A given in [**20**, Theorem 4.29] is correct – indeed, when $r = 1$ our proof is a streamlined version of this argument.

4.2. The blocks when $r > 1$ and $q = 1$ or $Q_1 = \dots = Q_r = 0$. We now consider the blocks in the remaining cases, that is, when $r > 1$ and either $q = 1$ or $Q_1 = \dots = Q_r = 0$. In this case all simple modules belong to the same block. We use the same strategy to prove Theorem 2.11 in these cases as in the previous sections.

Note that, in Cases 3–5, $\operatorname{res}(x) = (Q_a)$ for any node $x = (i, j, a)$. Therefore, in these cases, $\Lambda^+_{r,n}$ forms a single residue class. Hence, in order to prove Theorem 2.11, we need to show that any two multipartitions in $\Lambda^+_{r,n}$ are Jantzen equivalent. Consequently, in Cases 3–5 Theorem 2.11 asserts that the algebras $\mathscr{H}_{r,n}$ and $\mathscr{S}_{r,n}$ have only one block and, in particular, that they are both indecomposable algebras.

We adopt the same strategy for the proof. To state the analogue of Lemma 3.4 set

$$\epsilon = \begin{cases} 1, & \text{if } Q_1 = \dots = Q_r = 0 \text{ (cases 4 and 5)},\\ 0, & \text{otherwise.}\end{cases}$$

4.4. **Lemma.** *Suppose that $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are multipartitions of n and that $[\boldsymbol{\lambda}]\backslash r_x^{\boldsymbol{\lambda}} = [\boldsymbol{\mu}]\backslash r_y^{\boldsymbol{\mu}}$, for some nodes $x = (i, j, a) \in [\boldsymbol{\lambda}]$ and $y = (k, l, b) \in [\boldsymbol{\mu}]$. Then*

$$\nu_\pi\big(\operatorname{res}_{\mathcal{O}}(f_x^{\boldsymbol{\lambda}}) - \operatorname{res}_{\mathcal{O}}(f_y^{\boldsymbol{\mu}})\big) = \nu_p\big(n(a-b) + j - \boldsymbol{\lambda}_i^{(a)\prime} - l + \boldsymbol{\mu}_k^{(b)\prime}\big) + \epsilon$$

The proof of Lemma 4.4 is similar to proofs of Lemma 3.4 and Lemma 4.1, so we leave the details to the reader. Note in particular, that $\nu_\pi\big(\operatorname{res}_{\mathcal{O}}(f_x^{\boldsymbol{\lambda}}) - \operatorname{res}_{\mathcal{O}}(f_y^{\boldsymbol{\mu}})\big)$ is always non–zero when $a \neq b$. This crucial difference leads to $J_{\boldsymbol{\lambda\mu}}$ being non–zero whenever there exist nodes $x = (i, j, a) \in [\boldsymbol{\lambda}]$ and $y = (k, l, b) \in [\boldsymbol{\mu}]$ with $a < b$ and $[\boldsymbol{\lambda}]\backslash r_x^{\boldsymbol{\lambda}} = [\boldsymbol{\mu}]\backslash r_y^{\boldsymbol{\mu}}$. More explicitly, we have the following analogue of Propositions 3.6 and 4.2. Again, we leave details to the reader.

4.5. **Proposition.** *Let $\boldsymbol{\lambda} = (\boldsymbol{\lambda}^{(1)}, \dots, \boldsymbol{\lambda}^{(r)})$ and $\boldsymbol{\mu} = (\boldsymbol{\mu}^{(1)}, \dots, \boldsymbol{\mu}^{(r)})$ be multipartitions in $\Lambda^+_{r,n}$.*

a) *Suppose that there exist integers $a \neq b$ such that $\boldsymbol{\lambda}^{(c)} = \boldsymbol{\mu}^{(c)}$, for $c \neq a, b$. Then $J_{\boldsymbol{\lambda\mu}} \neq 0$ only if $a < b$ and there exist nodes $x = (i, j, a) \in [\boldsymbol{\lambda}]$ and $y = (k, l, b) \in [\boldsymbol{\mu}]$ such that $[\boldsymbol{\lambda}]\backslash r_x^{\boldsymbol{\lambda}} = [\boldsymbol{\mu}]\backslash r_y^{\boldsymbol{\mu}}$. In this case*

$$J_{\boldsymbol{\lambda\mu}} = (-1)^{\ell\ell(r_x^{\boldsymbol{\lambda}}) + \ell\ell(r_y^{\boldsymbol{\mu}})}\Big(\nu_p\big(n(a-b) + j - \boldsymbol{\lambda}_i^{(a)\prime} - l + \boldsymbol{\mu}_k^{(b)\prime}\big) + \epsilon\Big).$$

b) *Suppose that e is finite and for some integer a we have $\boldsymbol{\lambda}^{(c)} = \boldsymbol{\mu}^{(c)}$, for $c \neq a$. Then $J_{\boldsymbol{\lambda\mu}} \neq 0$ only if there exist nodes $x = (i, j, a), (i, m, a) \in [\boldsymbol{\lambda}]$ such that $m < j$, $e \mid h_{i,m,a)}^{\lambda}$ and $\boldsymbol{\mu}$ is obtained by wrapping a rim hook of length $h_x^{\boldsymbol{\lambda}}$ onto $\boldsymbol{\lambda}\backslash r_x^{\boldsymbol{\lambda}}$ with its highest node in column m. In this case*

$$J_{\boldsymbol{\lambda\mu}} = \begin{cases} (-1)^{\ell\ell(r_x^{\boldsymbol{\lambda}}) + \ell\ell(r_y^{\boldsymbol{\mu}})}\Big(\nu_p(h_{(i,m,a)}^{\boldsymbol{\lambda}}) + \epsilon\Big), & \text{if } e \nmid h_{(i,j,a)}^{\boldsymbol{\lambda}},\\ (-1)^{\ell\ell(r_x^{\boldsymbol{\lambda}}) + \ell\ell(r_y^{\boldsymbol{\mu}})}\Big(\nu_p(h_{(i,m,a)}^{\boldsymbol{\lambda}}) - \nu_p(h_{(i,j,a)}^{\boldsymbol{\lambda}})\Big), & \text{if } e \mid h_{(i,j,a)}^{\boldsymbol{\lambda}},\end{cases}$$

where $y \in [\boldsymbol{\mu}]$ is determined by $[\boldsymbol{\mu}]\backslash r_y^{\boldsymbol{\mu}} = [\boldsymbol{\lambda}]\backslash r_x^{\boldsymbol{\lambda}}$.

c) *In all other cases,* $J_{\boldsymbol{\lambda\mu}} = 0$.

We can now complete the proof of Theorem 2.11.

Proof of Theorem 2.11 for Cases 3–5. Let $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(r)})$ be a multipartition of n and fix an integer $a \neq b$ with $\lambda^{(a)} \neq (0)$ and $1 \leq a, b \leq r$. Let $\boldsymbol{\mu}$ be any multipartition that can be obtained by unwrapping a rim hook from $[\lambda^{(a)}]$ and wrapping it back on to component b of λ. Then $\boldsymbol{\lambda} \sim_J \boldsymbol{\mu}$ by Proposition 4.5(a). In particular, note that $\boldsymbol{\lambda} \sim_J \boldsymbol{\mu}$ if $\boldsymbol{\mu}$ is obtained from $\boldsymbol{\lambda}$ by moving a removable node from $\lambda^{(a)}$ to $\lambda^{(b)}$. Consequently, by moving the nodes in $[\boldsymbol{\lambda}]$ to the right, one by one, we see that $\boldsymbol{\lambda}$ is Janzten equivalent to a multipartition $\boldsymbol{\mu}$, where $\boldsymbol{\mu} = ((0), \dots, (0), \mu^{(r)})$. Similarly, moving nodes in $\boldsymbol{\mu}$ to the left, one by one, now shows that $\boldsymbol{\lambda} \sim_J \boldsymbol{\mu} \sim_J ((n), (0), \dots, (0))$. Hence, every multipartition in $\Lambda^+_{r,n}$ is Jantzen equivalent to $((n), (0), \dots, (0))$. This shows that there is only one block in Cases 3, 4 and 5, so the Theorem follows. $\square$

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