REPRESENTATIONS OF
PARABOLIC AND BOREL SUBGROUPS

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ABSTRACT. We demonstrate a relationships between the representation theory of Borel subgroups and parabolic subgroups of general linear groups. In particular, we show that the representations of Borel subgroups could be computed from representations of certain maximal parabolic subgroups.

1. INTRODUCTION

Little is known about the representation theory of the Borel subgroups of general linear groups. The linear representations of these subgroups play an important role in the representation theory of the general linear group itself, so it is to be expected that further knowledge of the representation theory of Borels would be useful. In this paper, we investigate the representation theory of maximal parabolic groups. At first sight this might appear to be an easier problem, but we show that these groups present at least as many difficulties as the Borel subgroup. In particular, we show that computing the irreducible representations for a maximal parabolic subgroup is essentially equivalent to computing them for a collection of parabolic groups with more blocks but smaller degree—this collection will include Borel subgroups. The techniques in this paper are inspired by matrix problems [2].

2. BACKGROUND AND NOTATION

Throughout this paper k is a locally compact field (this includes the finite fields and the real and complex numbers). Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)$ be a composition, that is, a finite sequence of nonnegative integers. We write $\lambda \models m$ if $\sum_{i=1}^s \lambda_i = m$, and $n(\lambda) = \sum_{i=1}^s (i-1)\lambda_i$.

The parabolic subgroup with block sizes given by the parts of λ is

$$
P^\lambda = \begin{pmatrix}
\mathrm{GL}_{\lambda_1}(k) & \mathrm{M}_{\lambda_1,\lambda_2}(k) & \cdots & \mathrm{M}_{\lambda_1,\lambda_s}(k) \\
0 & \mathrm{GL}_{\lambda_2}(k) & \ddots & \vdots \\
\vdots & \ddots & \ddots & \mathrm{M}_{\lambda_{s-1},\lambda_s}(k) \\
0 & \cdots & 0 & \mathrm{GL}_{\lambda_s}(k)
\end{pmatrix}
$$

Note that the n-dimensional Borel subgroup is just $P^{(1^n)}$. We write $\mathrm{Irr}(G)$ for the set of irreducible unitary representations of a locally compact group G over the complex field. If A is an abelian group, we write $\hat{A}$ for the dual group consisting of all linear characters of A. We define $G = A \rtimes H$ to be a semidirect product of

A and H with A normal. If A is abelian, then the action of H on A induces an action of H on $\hat{A}$ by $h \cdot \phi(a) = \phi(h^{-1}ah)$. Finally, for H a subgroup of G, we write $\mathrm{Ind}_H^G(V)$ to denote the induction to G of the H-module V.

We assume for convenience that the irreducible representations of the general linear groups over k are known, although we never explicitly use them; for instance, see [3] when k is finite. Our main theorem on the representations of maximal parabolic groups can now be stated.

Theorem 1. *The set* $\mathrm{Irr}\left(P^{(m,n)}\right)$ *is in one-to-one correspondence with the disjoint union* $\bigcup_{(\lambda,p)} \mathrm{Irr}\left(P^\lambda \times \mathrm{GL}_p(k)\right)$ *where* (λ,p) *runs over pairs of a composition* λ *and a non-negative integer* p *such that* $\lambda \models m$ *and* $p = n - n(\lambda)$.

Furthermore, our proof gives this correspondence explicitly, so from the irreducible representations of $P^{(m,n)}$ you could construct the irreducible representations of each P^λ, and vice versa. We have used this result to compute explicit generic character tables of some small parabolic groups (for $m + n \leq 4$).

The proof of this theorem depends on the following standard result.

Theorem 2. *Let the group* G *be a semidirect product* $A \rtimes H$ *with* A *abelian. Let* X *be a set of orbit representatives for the action of* H *on* $\hat{A}$. *Then we have a one-to-one correspondence between the disjoint union* $\bigcup_{x \in X} \mathrm{Irr}(H_x)$ *and* $\mathrm{Irr}(G)$ *given by*

$$V \in \mathrm{Irr}(H_x) \mapsto \mathrm{Ind}_{A \rtimes H_x}^{A \rtimes H}(x \boxtimes V),$$

where $\boxtimes$ *denotes external direct product.*

This was proved for locally compact groups in [4, Theorem 14.1]. A proof for finite groups using Clifford theory can be found in [1, Proposition 11.8].

3. Representations of quotients of parabolic subgroups

In this section we prove a result relating the representations of a certain quotient of a parabolic subgroup to representations of other such quotients with smaller degree but more blocks. Suppose that λ is a composition and n is an integer. We define

$$N = \begin{pmatrix} I_{\lambda_1} & 0 & \dots & 0 & \mathrm{M}_{\lambda_1,n}(k) \\ 0 & I_{\lambda_2} & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \mathrm{M}_{\lambda_{s-1},n}(k) \\ 0 & \dots & 0 & I_{\lambda_s} & 0 \\ 0 & 0 & \dots & 0 & I_n \end{pmatrix},$$

which is easily seen to be a normal subgroup of $P^{(\lambda_1,\dots,\lambda_s,n)}$. Denote the quotient by

$$Q^{\lambda,n} = \begin{bmatrix} \mathrm{GL}_{\lambda_1}(k) & \mathrm{M}_{\lambda_1,\lambda_2}(k) & \dots & \mathrm{M}_{\lambda_1,\lambda_s}(k) & \circ \\ 0 & \mathrm{GL}_{\lambda_2}(k) & \ddots & \vdots & \circ \\ \vdots & & \ddots & \ddots & \mathrm{M}_{\lambda_{s-1},\lambda_s}(k) & \circ \\ 0 & & \dots & 0 & \mathrm{GL}_{\lambda_s}(k) & \mathrm{M}_{\lambda_s,n}(k) \\ 0 & & 0 & \dots & 0 & \mathrm{GL}_n(k) \end{bmatrix}.$$

Similarly we use square brackets for the image of a matrix in this quotient. Note that $Q^{(m),n} \cong P^{(m,n)}$, $Q^{\lambda,0} \cong P^\lambda$, and if $\lambda_s = 0$ then $Q^{\lambda,n} \cong P^\lambda \oplus \mathrm{GL}_n(k)$.

Proposition 3. *Suppose that λ is a composition with s parts and n is a natural number. Then there is a one-to-one correspondence between $\mathrm{Irr}(Q^{\lambda,n})$ and the disjoint union $\bigcup_{l=0}^{M} \mathrm{Irr}(Q^{\tilde{\lambda}^l, n-l})$, where $M = \min(\lambda_s, n)$ and*

$$\tilde{\lambda}^l := (\lambda_1, \ldots, \lambda_{s-1}, \lambda_s - l, l).$$

Proof. The group $G = Q^{\lambda,n}$ is a semidirect product of

$$A = \begin{bmatrix} I_{\lambda_1} & 0 & \ldots & 0 & \circ \\ 0 & I_{\lambda_2} & \ddots & \vdots & \circ \\ \vdots & \ddots & \ddots & 0 & \circ \\ 0 & \ldots & 0 & I_{\lambda_s} & \mathrm{M}_{\lambda_s,n}(k) \\ 0 & 0 & \ldots & 0 & I_n \end{bmatrix}$$

and the image H of $P^\lambda \oplus \mathrm{GL}_n(k)$ in G. Now A is easily identified with the additive group of $\lambda_s \times n$ matrices over k. Hence $\hat{A}$ can be identified with the same group—this identification is not natural, but is given with respect to the standard basis. An element of H is of the form

$$h = \begin{bmatrix} B_1 & \cdots & B_{1s} & \circ \\ \vdots & \ddots & \vdots & \circ \\ 0 & \cdots & B_s & 0 \\ 0 & \cdots & 0 & B_{s+1} \end{bmatrix},$$

with $B_1, \ldots, B_{s+1}$ invertible. The action of H on A is given by $h \cdot a = B_s a B_{s+1}^{-1}$ and so the action on $\hat{A}$ is $h \cdot v = B_s^t v B_{s+1}^{-t}$. This is essentially the natural two-sided action of $\mathrm{GL}_{\lambda_s}(k) \oplus \mathrm{GL}_{\lambda_{s+1}}(k)$ on the $\lambda_s \times \lambda_{s+1}$ matrices, so the orbits of this action have representatives of the form

$$x = \begin{pmatrix} 0 & 0 \\ I_l & 0 \end{pmatrix}$$

for $l = 0, 1, \ldots, \min(\lambda_s, n)$. The stabilizer H_x is just the set of matrices

$$\begin{bmatrix} B_1 & \cdots & B_{1s} & \circ \\ \vdots & \ddots & \vdots & \circ \\ 0 & \cdots & B_s & 0 \\ 0 & \cdots & 0 & B_{s+1} \end{bmatrix}$$

with $B_s = \left(\begin{smallmatrix} A & B \\ 0 & C \end{smallmatrix} \right)$ and $B_{s+1} = \left(\begin{smallmatrix} C^{-1} & D \\ 0 & E \end{smallmatrix} \right)$ where A, B, C, D, E are matrices of sizes $(\lambda_s - l) \times (\lambda_s - l)$, $(\lambda_s - l) \times l$, $l \times l$, $l \times (n - l)$, $(n - l) \times (n - l)$ respectively; and A, C, and E are invertible. It is now easy to show that this is isomorphic to $Q^{\tilde{\lambda}^l, n-l}$ where $\tilde{\lambda}^l = (\lambda_1, \ldots, \lambda_s - l, l)$.

This argument together with Theorem 2 gives us the desired result. $\qquad \square$

4. Application to Borel and parabolic subgroups

We start with $P^{(m,n)} = Q^{(m),n}$. Then $\mathrm{Irr}(P^{(m,n)})$ is in one-to-one correspondence with $\bigcup_{l_1} \mathrm{Irr}(Q^{(m-l_1,l_1),n-l_1})$, since $\widetilde{(m)}^{l_1} = (m-l_1, l_1)$. By repeated application of Proposition 3 we eventually get $\bigcup_{\lambda,p} \mathrm{Irr}(Q^{\lambda,p})$ where

$$\lambda = (m - l_1, l_1 - l_2, \ldots, l_{s-2} - l_{s-1}, l_{s-1}),$$
$$0 = l_s < l_{s-1} < \cdots < l_1 \leq m,$$
$$p = n - l_1 - \cdots - l_{s-1}.$$

Since $l_s = 0$, we have $Q^{\lambda,p} \cong P^\lambda \oplus \mathrm{GL}_p(k)$. Writing $\lambda = (\lambda_1, \ldots, \lambda_s)$ we get

$$\begin{aligned}
\sum_{i=0}^{s} \lambda_s &= (m - l_1) + (l_1 - l_2) + \cdots + (l_{s-2} - l_{s-1}) + l_{s-1} \\
&= m, \quad \text{and} \\
p &= n - (l_{s-1} + l_{s-2} + \cdots + l_1) \\
&= n - (\lambda_s + (\lambda_{s-1} + \lambda_s) + \cdots + (\lambda_2 + \cdots + \lambda_s)) \\
&= n - \sum_{i=1}^{s} (i-1)\lambda_i.
\end{aligned}$$

Hence we have proved Theorem 1.

By taking $\lambda = (1^n)$ we get, as an immediate corollary, that the irreducible representations of the n-dimensional Borel subgroup can be computed from the irreducible representations of $P^{(m,n)}$ where $m = \frac{1}{2}n(n-1)$.

Applying Theorem 1 with $m = 1$, we get $p = 0, 1, \ldots, n$ and $\lambda = (0^{n-p-1}, 1)$. So the irreducible representations of $P^{(1,n)}$ correspond to elements of

$$\bigcup_{p=0}^{n} \mathrm{Irr}(P^{(0^{n-p-1},1)} \times \mathrm{GL}_p(k)) = \bigcup_{p=0}^{n} \mathrm{Irr}(k^\times \times \mathrm{GL}_p(k)).$$

On the other hand, when $n = 1$, we get $p = 0$ and $\lambda = (m-1, 1)$, or $p = 1$ and $\lambda = (m)$. So the irreducible representations of $P^{(1,n)}$ correspond to elements of $\mathrm{Irr}(P^{(m-1,1)}) \cup \mathrm{Irr}(\mathrm{GL}_m(k) \times k^\times)$, which corresponds to $\bigcup_{p=0}^{m} \mathrm{Irr}(\mathrm{GL}_p(k) \times k^\times)$ by induction. These results were proved in [5]; they are the only cases in which repeated application of Theorem 1 reduces to a set involving no nontrivial parabolic subgroups.

Acknowledgment

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