COMULTIPLICATION RULES FOR THE DOUBLE SCHUR FUNCTIONS AND CAUCHY IDENTITIES

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Abstract. The double Schur functions form a distinguished basis of the ring $\Lambda(x\,\|\,a)$ which is a multiparameter generalization of the ring of symmetric functions $\Lambda(x)$. The canonical comultiplication on $\Lambda(x)$ is extended to $\Lambda(x\,\|\,a)$ in a natural way so that the double power sums symmetric functions are primitive elements. We calculate the dual Littlewood–Richardson coefficients in two different ways thus providing comultiplication rules for the double Schur functions. We also prove multiparameter analogues of the Cauchy identity. A new family of Schur type functions plays the role of a dual object in the identities. We describe some properties of these dual Schur functions including a combinatorial presentation and an expansion formula in terms of the ordinary Schur functions. The dual Littlewood–Richardson coefficients provide a multiplication rule for the dual Schur functions.

Contents

1. Introduction

The ring $\Lambda = \Lambda(x)$ of symmetric functions in the set of variables $x = (x_1, x_2, \dots)$ admits a multiparameter generalization $\Lambda(x \,\|\, a)$, where a is a sequence of variables $a = (a_i)$, $i \in \mathbb{Z}$. Let $\mathbb{Q}[a]$ denote the ring of polynomials in the variables a_i with rational coefficients. The ring $\Lambda(x \,\|\, a)$ is generated over $\mathbb{Q}[a]$ by the *double power sums symmetric functions*

$$p_k(x \,\|\, a) = \sum_{i=1}^{\infty} (x_i^k - a_i^k). \tag{1.1}$$

Moreover, it possesses a distinguished basis over $\mathbb{Q}[a]$ formed by the *double Schur functions* $s_\lambda(x \,\|\, a)$ parameterized by partitions λ. The double Schur functions $s_\lambda(x \,\|\, a)$ are closely related to the 'factorial' or 'double' Schur polynomials $s_\lambda(x|a)$ which were introduced by Goulden and Greene [6] and Macdonald [14] as a generalization of the factorial Schur polynomials of Biedenharn and Louck [1, 2]. Moreover, the polynomials $s_\lambda(x|a)$ are also obtained a special case of the double Schubert polynomials of Lascoux and Schützenberger; see [3], [13]. A formal definition of the ring $\Lambda(x \,\|\, a)$ and its basis elements $s_\lambda(x \,\|\, a)$ can be found in a paper of Okounkov [21, Remark 2.11] and reproduced below in Section 2. The ring Λ is obtained from $\Lambda(x \,\|\, a)$ in the specialization $a_i = 0$ for all $i \in \mathbb{Z}$ while the elements $s_\lambda(x \,\|\, a)$ turn into the classical Schur functions $s_\lambda(x) \in \Lambda$; see Macdonald [15] for a detailed account of the properties of Λ.

Another specialization $a_i = -i + 1$ for all $i \in \mathbb{Z}$ yields the ring of *shifted symmetric functions* Λ^*, introduced and studied by Okounkov and Olshanski [22]. Many combinatorial results of [22] can be reproduced for the ring $\Lambda(x \,\|\, a)$ in a rather straightforward way. The respective specializations of the double Schur functions in Λ^*, known as the *shifted Schur functions* were studied in [20], [22] in relation with the higher Capelli identities and quantum immanants for the Lie algebra $\mathfrak{gl}_n$.

In a different kind of specialization, the double Schur functions become the equivariant Schubert classes on Grassmannians; see e.g. Knutson and Tao [9], Fulton [4] and Mihalcea [16]. The structure coefficients $c_{\lambda\mu}^{\nu}(a)$ of $\Lambda(x \,\|\, a)$ in the basis of $s_\lambda(x \,\|\, a)$, defined by the expansion

$$s_\lambda(x \,\|\, a)\, s_\mu(x \,\|\, a) = \sum_{\nu} c_{\lambda\mu}^{\nu}(a)\, s_\nu(x \,\|\, a), \tag{1.2}$$

were called the *Littlewood–Richardson polynomials* in [18]. Under the respective specializations they describe the multiplicative structure of the equivariant cohomology ring on the Grassmannian and the center of the enveloping algebra $\mathrm{U}(\mathfrak{gl}_n)$. The polynomials $c_{\lambda\mu}^{\nu}(a)$ possess the Graham positivity property: they are polynomials in the differences $a_i - a_j$, $i < j$, with positive integer coefficients; see [7]. Explicit positive formulas for the polynomials $c_{\lambda\mu}^{\nu}(a)$ were found in [9], [10] and [18]; an earlier formula found in [19] lacks the positivity property. The Graham positivity brings natural combinatorics of polynomials into the structure theory of $\Lambda(x\,\|\,a)$. Namely, the entries of some transition matrices between bases of $\Lambda(x\,\|\,a)$ such as analogues of the Kostka numbers, turn out to be Graham positive.

The *comultiplication* on the ring $\Lambda(x\,\|\,a)$ is the $\mathbb{Q}[a]$-linear ring homomorphism

$$\Delta : \Lambda(x\,\|\,a) \to \Lambda(x\,\|\,a) \otimes_{\mathbb{Q}[a]} \Lambda(x\,\|\,a)$$

defined on the generators by

$$\Delta\big(p_k(x\,\|\,a)\big) = p_k(x\,\|\,a) \otimes 1 + 1 \otimes p_k(x\,\|\,a).$$

In the specialization $a_i = 0$ this homomorphism turns into the comultiplication on the ring of symmetric functions Λ; see [15, Chapter 1]. Define the *dual Littlewood–Richardson polynomials* $\widehat{c}_{\lambda\mu}^{\,\nu}(a)$ as the coefficients in the expansion

$$\Delta\big(s_\nu(x\,\|\,a)\big) = \sum_{\lambda,\,\mu} \widehat{c}_{\lambda\mu}^{\,\nu}(a)\, s_\lambda(x\,\|\,a) \otimes s_\mu(x\,\|\,a).$$

The central problem we address in this paper is calculation of the polynomials $\widehat{c}_{\lambda\mu}^{\,\nu}(a)$ in an explicit form. Note that if $|\nu| = |\lambda| + |\mu|$ then $c_{\lambda\mu}^{\nu}(a) = \widehat{c}_{\lambda\mu}^{\,\nu}(a) = c_{\lambda\mu}^{\nu}$ is the Littlewood–Richardson coefficient. Moreover,

$$c_{\lambda\mu}^{\nu}(a) = 0 \quad \text{unless} \quad |\nu| \leqslant |\lambda| + |\mu|, \quad \text{and} \quad \widehat{c}_{\lambda\mu}^{\,\nu}(a) = 0 \quad \text{unless} \quad |\nu| \geqslant |\lambda| + |\mu|.$$

We will show that the polynomials $\widehat{c}_{\lambda\mu}^{\,\nu}(a)$ can be interpreted as the multiplication coefficients for certain analogues of the Schur functions,

$$\widehat{s}_\lambda(x\,\|\,a)\, \widehat{s}_\mu(x\,\|\,a) = \sum_{\nu} \widehat{c}_{\lambda\mu}^{\,\nu}(a)\, \widehat{s}_\nu(x\,\|\,a),$$

where the $\widehat{s}_\lambda(x\,\|\,a)$ are symmetric functions in x which we call the *dual Schur functions*. They can be given by the combinatorial formula

$$\widehat{s}_\lambda(x\,\|\,a) = \sum_{T} \prod_{\alpha\in\lambda} X_{T(\alpha)}(a_{-c(\alpha)+1}, a_{-c(\alpha)}), \tag{1.3}$$

summed over the *reverse* λ-tableaux T, where

$$X_i(g,h) = \frac{x_i\,(1 - g\,x_{i-1}) \dots (1 - g\,x_1)}{(1 - h\,x_i) \dots (1 - h\,x_1)},$$

and $c(\alpha) = j - i$ denotes the content of the box $\alpha = (i,j)$; see Section 3 below.

We calculate in an explicit form the coefficients of the expansion of $\widehat{s}_\lambda(x\,\|\,a)$ as a series of the Schur functions $s_\mu(x)$ and vice versa. This makes it possible to express $\widehat{c}_{\lambda\mu}^{\,\nu}(a)$ explicitly as polynomials in the a_i with the use of the Littlewood–Richardson coefficients $c_{\lambda\mu}^{\nu}$.

The combinatorial formula (1.3) can be used to define the skew dual Schur functions, and we show that the following decomposition holds

$$\widehat{s}_{\nu/\mu}(x\,\|\,a) = \sum_{\lambda} c_{\lambda\mu}^{\nu}(a)\, \widehat{s}_\lambda(x\,\|\,a),$$

where the $c_{\lambda\mu}^{\nu}(a)$ are the Littlewood–Richardson polynomials.

The functions $\widehat{s}_{\lambda}(x\,\|\,a)$ turn out to be dual to the double Schur functions via the following analogue of the classical Cauchy identity:

$$\prod_{i,\,j\geqslant 1}\frac{1-a_i\,y_j}{1-x_i\,y_j}=\sum_{\lambda\in\mathcal{P}}s_{\lambda}(x\,\|\,a)\,\widehat{s}_{\lambda}(y\,\|\,a),\tag{1.4}$$

where $\mathcal{P}$ denotes the set of all partitions and $y=(y_1,y_2,\dots)$ is a set of variables.

The dual Schur functions $\widehat{s}_{\lambda}(x\,\|\,a)$ are elements of the extended ring $\widehat{\Lambda}(x\,\|\,a)$ of formal series of elements of $\Lambda(x)$ whose coefficients are polynomials in the a_i. If $x=(x_1,x_2,\dots,x_n)$ is a finite set of variables (i.e., $x_i=0$ for $i\geqslant n+1$), then $\widehat{s}_{\lambda}(x\,\|\,a)$ can be defined as the ratio of alternants by analogy with the classical Schur polynomials. With this definition of the dual Schur functions, the identity (1.4) can be deduced from the 'dual Cauchy formula' obtained in [14, (6.17)] and which is a particular case of the Cauchy identity for the double Schubert polynomials [12]. An independent proof of a version of (1.4) for the shifted Schur functions (i.e., in the specialization $a_i=-i+1$) was given by Olshanski [23]. In the specialization $a_i=0$ each $\widehat{s}_{\lambda}(x\,\|\,a)$ becomes the Schur function $s_{\lambda}(x)$, and (1.4) turns into the classical Cauchy identity.

We will also need a super version of the ring of symmetric functions. The elements

$$p_k(x/y)=\sum_{i=1}^{\infty}\big(x_i^k+(-1)^{k-1}y_i^k\big)\tag{1.5}$$

with $k=1,2,\dots$ are generators of the ring of *supersymmetric functions* which we will regard as a $\mathbb{Q}[a]$-module and denote by $\Lambda(x/y\,\|\,a)$. A distinguished basis of $\Lambda(x/y\,\|\,a)$ was introduced by Olshanski, Regev and Vershik [24]. In a certain specialization the basis elements become the *Frobenius–Schur functions* Fs_{λ} associated with the relative dimension function on partitions; see [24]. In order to indicate dependence on the variables, we will denote the basis elements by $s_{\lambda}(x/y\,\|\,a)$ and call them the (*multiparameter*) *supersymmetric Schur functions*. They are closely related to the *factorial supersymmetric Schur polynomials* introduced in [17]; see Section 2 for precise formulas. Note that the evaluation map $y_i\mapsto -a_i$ for all $i\geqslant 1$ defines an isomorphism

$$\Lambda(x/y\,\|\,a)\to\Lambda(x\,\|\,a).\tag{1.6}$$

The images of the generators (1.5) under this isomorphism are the double power sums symmetric functions (1.1). We will show that under the isomorphism (1.6) we have

$$s_{\lambda}(x/y\,\|\,a)\mapsto s_{\lambda}(x\,\|\,a).\tag{1.7}$$

Due to [24], the supersymmetric Schur functions possess a remarkable combinatorial presentation in terms of diagonal-strict or 'shuffle' tableaux. The isomorphism (1.6) implies the corresponding combinatorial presentation for $s_\lambda(x\,\|\,a)$ and allows us to introduce the *skew double Schur functions* $s_{\nu/\mu}(x\,\|\,a)$. The dual Littlewood–Richardson polynomials $\widehat{c}_{\lambda\mu}^{\,\nu}(a)$ can then be found from the expansion

$$s_{\nu/\mu}(x\,\|\,a) = \sum_{\lambda} \widehat{c}_{\lambda\mu}^{\,\nu}(a)\, s_\lambda(x\,\|\,a), \tag{1.8}$$

which leads to an alternative rule for the calculation of $\widehat{c}_{\lambda\mu}^{\,\nu}(a)$. This rule relies on the combinatorial objects called 'barred tableaux' which were introduced in [19] for the calculation of the polynomials $c_{\lambda\mu}^{\nu}(a)$; see also [10], [11] and [18].

The coefficients in the expansion of $s_\mu(x)$ in terms of the $\widehat{s}_\lambda(x\,\|\,a)$ turn out to coincide with those in the decomposition of $s_\lambda(x/y\,\|\,a)$ in terms of the ordinary supersymmetric Schur functions $s_\lambda(x/y)$ thus providing another expression for these coefficients; cf. [24].

The identity (1.4) allows us to introduce a pairing between the rings $\Lambda(x\,\|\,a)$ and $\widehat{\Lambda}(x\,\|\,a)$ so that the respective families $\{s_\lambda(x\,\|\,a)\}$ and $\{\widehat{s}_\lambda(x\,\|\,a)\}$ are dual to each other. This leads to a natural definition of the monomial and forgotten symmetric functions in $\Lambda(x\,\|\,a)$ and $\widehat{\Lambda}(x\,\|\,a)$ by analogy with [15] and provides a relationship between the transition matrices relating different bases of these rings.

It is well known that the ring of symmetric functions Λ admits an involutive automorphism $\omega : \Lambda \to \Lambda$ which interchanges the elementary and complete symmetric functions; see [15]. We show that there is an isomorphism $\omega_a : \Lambda(x\,\|\,a) \to \Lambda(x\,\|\,a')$, and ω_a has the property $\omega_{a'} \circ \omega_a = \mathrm{id}$, where a' denotes the sequence of parameters with $(a')_i = -a_{-i+1}$. Moreover, the images of the natural bases elements of $\Lambda(x\,\|\,a)$ with respect to ω_a can be explicitly described; see also [22] where such an involution was constructed for the specialization $a_i = -i + 1$, and [24] for its super version. Furthermore, using a symmetry property of the supersymmetric Schur functions, we derive the symmetry properties of the Littlewood–Richardson polynomials and their dual counterparts

$$c_{\lambda\mu}^{\nu}(a) = c_{\lambda'\mu'}^{\nu'}(a') \qquad \text{and} \qquad \widehat{c}_{\lambda\mu}^{\,\nu}(a) = \widehat{c}_{\lambda'\mu'}^{\,\nu'}(a'),$$

where ρ' denotes the conjugate partition to any partition ρ. In the context of equivariant cohomology, the first relation is a consequence of the Grassmann duality; see e.g. [4, Lecture 8] and [9].

An essential role in the proof of (1.4) is played by interpolation formulas for symmetric functions. The interpolation approach goes back to the work of Okounkov [20, 21], where the key *vanishing theorem* for the double Schur functions $s_\lambda(x\,\|\,a)$ was proved; see also [22]. In a more general context, the Newton interpolation for polynomials in several variables relies on the theory of Schubert polynomials of Lascoux and Schützenberger; see [13]. The interpolation approach leads to a recurrence relation for the coefficients $c_{P,\mu}^{\nu}(a)$ in the expansion

$$P(x)\, s_\mu(x\,\|\,a) = \sum_\nu c_{P,\mu}^{\nu}(a)\, s_\nu(x\,\|\,a), \qquad P(x) \in \Lambda(x\,\|\,a), \tag{1.9}$$

as well as to an explicit formula for the $c_{P,\mu}^{\nu}(a)$ in terms of the values of $P(x)$; see [19]. Therefore, the (dual) Littlewood–Richardson polynomials and the entries of the transition matrices between various bases of $\Lambda(x\,\|\,a)$ can be given as rational functions in the variables a_i. Under appropriate specializations, these formulas imply some combinatorial identities involving Kostka numbers, irreducible characters of the symmetric group and dimensions of skew diagrams; cf. [22].

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2. Double and supersymmetric Schur functions

2.1. **Definitions and preliminaries.** Recall the definition of the ring $\Lambda(x\,\|\,a)$ from [21, Remark 2.11]; see also [18]. For each nonnegative integer n denote by Λ_n the ring of symmetric polynomials in $x_1, \dots, x_n$ with coefficients in $\mathbb{Q}[a]$ and let Λ_n^k denote the $\mathbb{Q}[a]$-submodule of Λ_n which consists of the polynomials $P_n(x_1, \dots, x_n)$ such that the total degree of P_n in the variables x_i does not exceed k. Consider the evaluation maps

$$\varphi_n : \Lambda_n^k \to \Lambda_{n-1}^k, \qquad P_n(x_1, \dots, x_n) \mapsto P_n(x_1, \dots, x_{n-1}, a_n) \tag{2.1}$$

and the corresponding inverse limit

$$\Lambda^k = \varprojlim \Lambda_n^k, \qquad n \to \infty.$$

The elements of Λ^k are sequences $P = (P_0, P_1, P_2, \dots)$ with $P_n \in \Lambda_n^k$ such that

$$\varphi_n(P_n) = P_{n-1} \qquad \text{for} \quad n = 1, 2, \dots .$$

Then the union

$$\Lambda(x\,\|\,a) = \bigcup_{k \geqslant 0} \Lambda^k$$

is a ring with the product

$$PQ = (P_0 Q_0, P_1 Q_1, P_2 Q_2, \dots), \qquad Q = (Q_0, Q_1, Q_2, \dots).$$

The elements of $\Lambda(x\,\|\,a)$ may be regarded as formal series in the variables x_i with coefficients in $\mathbb{Q}[a]$. For instance, the sequence of polynomials

$$\sum_{i=1}^n (x_i^k - a_i^k), \qquad n \geqslant 0,$$

determines the *double power sums symmetric function* (1.1).

Note that if k is fixed, then the evaluation maps (2.1) are isomorphisms for all sufficiently large values of n. This allows one to establish many properties of $\Lambda(x\,\|\,a)$ by working with finite sets of variables $x = (x_1, \dots, x_n)$.

Now we recall the definition and some key properties of the double Schur functions. We basically follow [14, 6th Variation] and [21], although our notation is slightly different. A partition λ is a weakly decreasing sequence $\lambda = (\lambda_1, \dots, \lambda_l)$ of integers λ_i such that $\lambda_1 \geqslant \dots \geqslant \lambda_l \geqslant 0$. Sometimes this sequence is considered to be completed by a finite or infinite sequence of zeros. We will identify λ with its diagram represented graphically as the array of left justified rows of unit boxes with λ_1 boxes in the top row, λ_2 boxes in the second row, etc. The total number of boxes in λ will be denoted by $|\lambda|$ and the number of nonzero rows will be called the length of λ and denoted $\ell(\lambda)$. The transposed diagram $\lambda' = (\lambda'_1, \dots, \lambda'_p)$ is obtained from λ by applying the symmetry with respect to the main diagonal, so that λ'_j is the number of boxes in the j-th column of λ. If μ is a diagram contained in λ, then the skew diagram λ/μ is the set-theoretical difference of diagrams λ and μ.

Suppose now that $x = (x_1, \dots, x_n)$ is a finite set of variables. For any n-tuple of nonnegative integers $\alpha = (\alpha_1, \dots, \alpha_n)$ set

$$A_\alpha(x\,\|\,a) = \det \big[(x_i\,\|\,a)^{\alpha_j}\big]_{i,j=1}^n,$$

where $(x_i\,\|\,a)^0 = 1$ and

$$(x_i\,\|\,a)^r = (x_i - a_n)(x_i - a_{n-1}) \dots (x_i - a_{n-r+1}), \qquad r \geqslant 1.$$

For any partition $\lambda = (\lambda_1, \dots, \lambda_n)$ of length not exceeding n set

$$s_\lambda(x\,\|\,a) = \frac{A_{\lambda+\delta}(x\,\|\,a)}{A_\delta(x\,\|\,a)},$$

where $\delta = (n-1, \dots, 1, 0)$. Note that $A_\delta(x\,\|\,a)$ coincides with the Vandermonde determinant,

$$A_\delta(x\,\|\,a) = \prod_{1 \leqslant i < j \leqslant n} (x_i - x_j)$$

and so $s_\lambda(x\,\|\,a)$ belongs to the ring Λ_n. Moreover,

$$s_\lambda(x\,\|\,a) = s_\lambda(x) + \text{ lower degree terms in } x,$$

where $s_\lambda(x)$ is the Schur polynomial; see e.g. [15, Chapter 1]. We also set $s_\lambda(x\,\|\,a) = 0$ if $\ell(\lambda) > n$. Then under the evaluation map (2.1) we have

$$\varphi_n : s_\lambda(x\,\|\,a) \mapsto s_\lambda(x'\,\|\,a), \qquad x' = (x_1, \dots, x_{n-1}),$$

so that the sequence $\big(s_\lambda(x\,\|\,a) \mid n \geqslant 0\big)$ defines an element of the ring $\Lambda(x\,\|\,a)$. We will keep the notation $s_\lambda(x\,\|\,a)$ for this element of $\Lambda(x\,\|\,a)$, where x is now understood as the infinite sequence of variables, and call it the *double Schur function*.

By a *reverse λ-tableau* T we will mean a tableau obtained by filling in the boxes of λ with the positive integers in such a way that the entries weakly decrease along the rows and strictly decrease down the columns. If $\alpha = (i, j)$ is a box of λ in row i and column j, we let $T(\alpha) = T(i, j)$ denote the entry of T in the box α and let $c(\alpha) = j - i$ denote the content of this box. The double Schur functions admit the following tableau presentation

$$s_\lambda(x \,\|\, a) = \sum_T \prod_{\alpha\in\lambda} (x_{T(\alpha)} - a_{T(\alpha)-c(\alpha)}), \tag{2.2}$$

summed over all reverse λ-tableaux T.

When the entries of T are restricted to the set $\{1, \dots, n\}$, formula (2.2) provides the respective tableau presentation of the polynomials $s_\lambda(x \,\|\, a)$ with $x = (x_1, \dots, x_n)$. Moreover, in this case the formula can be extended to skew diagrams and we define the corresponding polynomials by

$$\widetilde{s}_\theta(x \,\|\, a) = \sum_T \prod_{\alpha\in\theta} (x_{T(\alpha)} - a_{T(\alpha)-c(\alpha)}), \tag{2.3}$$

summed over all reverse θ-tableaux T with entries in $\{1, \dots, n\}$, where θ is a skew diagram. We suppose that $\widetilde{s}_\theta(x \,\|\, a) = 0$ unless all columns of θ contain at most n boxes.

Remark 2.1. (i) Although the polynomials (2.3) belong to the ring Λ_n, they are generally not consistent with respect to the evaluation maps (2.1). We used different notation in (2.2) and (2.3) in order to distinguish between the polynomials $\widetilde{s}_\theta(x \,\|\, a)$ and the skew double Schur functions $s_\theta(x \,\|\, a)$ to be introduced in Definition 2.7 below.

(ii) In order to relate our notation to [14], note that for the polynomials $\widetilde{s}_\theta(x \,\|\, a)$ with $x = (x_1, \dots, x_n)$ we have

$$\widetilde{s}_\theta(x \,\|\, a) = s_\theta(x \,|\, u),$$

where the sequences $a = (a_i)$ and $u = (u_i)$ are related by

$$u_i = a_{n-i+1}, \qquad i \in \mathbb{Z}. \tag{2.4}$$

The polynomials $s_\theta(x \,|\, u)$ are often called the *factorial Schur polynomials* (*functions*) in the literature. They can be given by the combinatorial formula

$$s_\theta(x \,|\, u) = \sum_T \prod_{\alpha\in\theta} (x_{T(\alpha)} - u_{T(\alpha)+c(\alpha)}), \tag{2.5}$$

summed over all *semistandard* θ-tableaux T with entries in $\{1, \dots, n\}$; the entries of T weakly increase along the rows and strictly increase down the columns.

(iii) If we replace a_i with c_{-i} and index the variables x with nonnegative integers, the double Schur functions $s_\lambda(x \,\|\, a)$ will become the corresponding symmetric functions of [21]; cf. formula (3.7) in that paper. Moreover, under the specialization $a_i = -i+1$

for all $i \in \mathbb{Z}$ the double Schur functions become the *shifted Schur functions* of [22] in the variables $y_i = x_i + i - 1$. $\square$

2.2. **Analogues of classical bases.** The *double elementary* and *complete symmetric functions* are defined respectively by

$$e_k(x\,\|\,a) = s_{(1^k)}(x\,\|\,a), \qquad h_k(x\,\|\,a) = s_{(k)}(x\,\|\,a)$$

and hence, they can be given by the formulas

$$e_k(x\,\|\,a) = \sum_{i_1 > \dots > i_k} (x_{i_1} - a_{i_1}) \dots (x_{i_k} - a_{i_k+k-1}),$$

$$h_k(x\,\|\,a) = \sum_{i_1 \geqslant \dots \geqslant i_k} (x_{i_1} - a_{i_1}) \dots (x_{i_k} - a_{i_k-k+1}).$$

Their generating functions can be written by analogy with the classical case as in [15] and they take the form

$$1 + \sum_{k=1}^{\infty} \frac{e_k(x\,\|\,a)\, t^k}{(1 + a_1 t) \dots (1 + a_k t)} = \prod_{i=1}^{\infty} \frac{1 + x_i t}{1 + a_i t}, \tag{2.6}$$

$$1 + \sum_{k=1}^{\infty} \frac{h_k(x\,\|\,a)\, t^k}{(1 - a_0 t) \dots (1 - a_{-k+1} t)} = \prod_{i=1}^{\infty} \frac{1 - a_i t}{1 - x_i t}; \tag{2.7}$$

see e.g. [14], [22].

Given a partition $\lambda = (\lambda_1, \dots, \lambda_l)$, set

$$p_\lambda(x\,\|\,a) = p_{\lambda_1}(x\,\|\,a) \dots p_{\lambda_l}(x\,\|\,a),$$
$$e_\lambda(x\,\|\,a) = e_{\lambda_1}(x\,\|\,a) \dots e_{\lambda_l}(x\,\|\,a),$$
$$h_\lambda(x\,\|\,a) = h_{\lambda_1}(x\,\|\,a) \dots h_{\lambda_l}(x\,\|\,a).$$

The following proposition is easy to deduce from the properties of the classical symmetric functions; see [15].

Proposition 2.2. *Each of the families $p_\lambda(x\,\|\,a)$, $e_\lambda(x\,\|\,a)$, $h_\lambda(x\,\|\,a)$ and $s_\lambda(x\,\|\,a)$, parameterized by all partitions λ, forms a basis of $\Lambda(x\,\|\,a)$ over $\mathbb{Q}[a]$.* $\square$

In particular, each of the families $p_k(x\,\|\,a)$, $e_k(x\,\|\,a)$ and $h_k(x\,\|\,a)$ with $k \geqslant 1$ is a set of algebraically independent generators of $\Lambda(x\,\|\,a)$ over $\mathbb{Q}[a]$. Under the specialization $a_i = 0$, the bases of Proposition 2.2 turn into the classical bases $p_\lambda(x)$, $e_\lambda(x)$, $h_\lambda(x)$ and $s_\lambda(x)$ of Λ. The ring of symmetric functions Λ possesses two more bases $m_\lambda(x)$ and $f_\lambda(x)$; see [15, Chapter 1]. The *monomial symmetric functions* $m_\lambda(x)$ are defined by

$$m_\lambda(x) = \sum_\sigma x_{\sigma(1)}^{\lambda_1} x_{\sigma(2)}^{\lambda_2} \dots x_{\sigma(l)}^{\lambda_l},$$

summed over permutations σ of the x_i which give distinct monomials. The basis elements $f_\lambda(x)$ are called the *forgotten symmetric functions*, they are defined as the images of the $m_\lambda(x)$ under the involution $\omega:\Lambda\to\Lambda$ which takes $e_\lambda(x)$ to $h_\lambda(x)$; see [15]. The corresponding basis elements $m_\lambda(x\,\|\,a)$ and $f_\lambda(x\,\|\,a)$ in $\Lambda(x\,\|\,a)$ will be defined in Section 5.

2.3. **Duality isomorphism.** Introduce the sequence of variables a' which is related to the sequence a by the rule

$$(a')_i = -a_{-i+1}, \qquad i\in\mathbb{Z}.$$

The operation $a\mapsto a'$ is clearly involutive so that $(a')'=a$. Note that any element of the polynomial ring $\mathbb{Q}[a']$ can be identified with the element of $\mathbb{Q}[a]$ obtained by replacing each $(a')_i$ by $-a_{-i+1}$. Define the ring homomorphism

$$\omega_a:\Lambda(x\,\|\,a)\to\Lambda(x\,\|\,a')$$

as the $\mathbb{Q}[a]$-linear map such that

$$\omega_a: e_k(x\,\|\,a)\mapsto h_k(x\,\|\,a'), \qquad k=1,2,\dots. \tag{2.8}$$

An arbitrary element of $\Lambda(x\,\|\,a)$ can be written as a unique linear combination of the basis elements $e_\lambda(x\,\|\,a)$ with coefficients in $\mathbb{Q}[a]$. The image of such a linear combination under ω_a is then found by

$$\omega_a:\sum_\lambda c_\lambda(a)\,e_\lambda(x\,\|\,a)\mapsto\sum_\lambda c_\lambda(a)\,h_\lambda(x\,\|\,a'), \qquad c_\lambda(a)\in\mathbb{Q}[a],$$

and $c_\lambda(a)$ is regarded as an element of $\mathbb{Q}[a']$. Clearly, ω_a is a ring isomorphism, since the $h_k(x\,\|\,a')$ are algebraically independent generators of $\Lambda(x\,\|\,a')$ over $\mathbb{Q}[a']$. In the case of finite set of variables $x=(x_1,\dots,x_n)$ the respective isomorphism ω_a is defined by the same rule (2.8) with the values $k=1,\dots,n$.

Proposition 2.3. *We have* $\omega_{a'}\circ\omega_a=\mathrm{id}_{\Lambda(x\,\|\,a)}$ *and*

$$\omega_a: s_\lambda(x\,\|\,a)\mapsto s_{\lambda'}(x\,\|\,a'), \tag{2.9}$$

$$h_\lambda(x\,\|\,a)\mapsto e_\lambda(x\,\|\,a'). \tag{2.10}$$

Proof. Relations (2.6) and (2.7) imply that

$$\Bigg(\sum_{k=0}^{\infty}\frac{(-1)^k\,e_k(x\,\|\,a)\,t^k}{(1-a_1t)\dots(1-a_kt)}\Bigg)\Bigg(\sum_{r=0}^{\infty}\frac{h_r(x\,\|\,a)\,t^r}{(1-a_0t)\dots(1-a_{-r+1}t)}\Bigg)=1.$$

Applying the isomorphism ω_a, we get

$$\Bigg(\sum_{k=0}^{\infty}\frac{(-1)^k\,h_k(x\,\|\,a')\,t^k}{(1+(a')_0t)\dots(1+(a')_{-k+1}t)}\Bigg)\Bigg(\sum_{r=0}^{\infty}\frac{\omega_a\big(h_r(x\,\|\,a)\big)\,t^r}{(1+(a')_1t)\dots(1+(a')_rt)}\Bigg)=1.$$

Replacing here t by $-t$ and comparing with the previous identity, we can conclude that $\omega_a\big(h_r(x\,\|\,a)\big) = e_r(x\,\|\,a')$. This proves (2.10) and the first part of the proposition, because $\omega_{a'}\big(h_r(x\,\|\,a')\big) = e_r(x\,\|\,a)$.

In order to prove (2.9), we will work with a finite set of variables $x = (x_1, \dots, x_n)$. By (2.6) we have

$$\sum_{k=0}^{n} \frac{e_k(x\,\|\,\tau^{-1}a)\, t^k}{(1+a_0t)\dots(1+a_{k-1}t)} = \frac{1+a_nt}{1+a_0t} \sum_{k=0}^{n} \frac{e_k(x\,\|\,a)\, t^k}{(1+a_1t)\dots(1+a_kt)},$$

where τ denotes the shift operator on sequences,

$$(\tau^k a)_i = a_{k+i} \qquad \text{for} \quad k \in \mathbb{Z}.$$

Hence, the image of $e_k(x\,\|\,\tau^{-1}a)$ under ω_a can be found from the relation

$$\sum_{k=0}^{n} \frac{\omega_a\big(e_k(x\,\|\,\tau^{-1}a)\big)\, t^k}{(1+a_0t)\dots(1+a_{k-1}t)} = \frac{1+a_nt}{1+a_0t} \sum_{k=0}^{n} \frac{h_k(x\,\|\,a')\, t^k}{(1+a_1t)\dots(1+a_kt)}.$$

On the other hand, applying (2.7) to the sequence a' instead of a, we get

$$\sum_{k=0}^{\infty} \frac{h_k(x\,\|\,\tau a')\, t^k}{(1+a_0t)\dots(1+a_{k-1}t)} = \frac{1+a_nt}{1+a_0t} \sum_{k=0}^{\infty} \frac{h_k(x\,\|\,a')\, t^k}{(1+a_1t)\dots(1+a_kt)}.$$

Comparing the coefficients of $t, t^2, \dots, t^n$ in the two series we conclude that

$$\omega_a\big(e_k(x\,\|\,\tau^{-1}a)\big) = h_k(x\,\|\,\tau a'), \qquad k = 1, \dots, n.$$

Therefore, (2.9) follows from the Jacobi–Trudi and Nägelsbach–Kostka formulas; see [14, (6.7)]. Namely, if λ is a partition of length not exceeding n, then

$$s_\lambda(x\,\|\,a) = \det\big[h_{\lambda_i-i+j}(x\,\|\,\tau^{j-1}a)\big] \tag{2.11}$$

and

$$s_\lambda(x\,\|\,a) = \det\big[e_{\lambda'_i-i+j}(x\,\|\,\tau^{-j+1}a)\big], \tag{2.12}$$

where the determinants are taken over the respective sets of indices $i, j = 1, \dots, \ell(\lambda)$ and $i, j = 1, \dots, \ell(\lambda')$. □

2.4. **Skew double Schur functions.** Consider now the ring of supersymmetric functions $\Lambda(x/y\,\|\,a)$ defined in the Introduction. Taking two finite sets of variables $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$, define the *supersymmetric Schur polynomial* $s_{\nu/\mu}(x/y\,\|\,a)$ associated with a skew diagram ν/μ by the formula

$$s_{\nu/\mu}(x/y\,\|\,a) = \sum_{\mu\subseteq\rho\subseteq\lambda} \widetilde{s}_{\nu/\rho}(x\,\|\,a)\, s_{\rho'/\mu'}(y\,|\,{-a}), \tag{2.13}$$

where the polynomials $\widetilde{s}_{\nu/\rho}(x\,\|\,a)$ and $s_{\rho'/\mu'}(y\,|\,{-a})$ are defined by the respective combinatorial formulas (2.3) and (2.5). The polynomials (2.13) coincide with the factorial supersymmetric Schur polynomials $s_{\nu/\mu}(x/y\,|\,u)$ of [17] associated with the sequence

u related to a by (2.4). It was observed in [24] that the sequence of polynomials $\big(s_{\nu/\mu}(x/y\,\|\,a)\,|\,n\geqslant 1\big)$ is consistent with respect to the evaluations $x_n=y_n=0$ and hence, it defines the *supersymmetric Schur function* $s_{\nu/\mu}(x/y\,\|\,a)$, where x and y are infinite sequences of variables (in fact, Proposition 3.4 in [24] needs to be extended to skew diagrams which is immediate). Moreover, in [24] these functions were given by new combinatorial formulas. In order to write them down, consider the ordered alphabet

$$\mathbb{A}=\{1'<1<2'<2<\dots\}.$$

Given a skew diagram θ, an $\mathbb{A}$-*tableau* T of shape θ is obtained by filling in the boxes of θ with the elements of $\mathbb{A}$ in such a way that the entries of T weakly increase along each row and down each column, and for each $i=1,2,\dots$ there is at most one symbol i' in each row and at most one symbol i in each column of T. The following formula gives the supersymmetric Schur function $s_\theta(x/y\,\|\,a)$ associated with θ:

$$s_\theta(x/y\,\|\,a)=\sum_T\prod_{\substack{\alpha\in\theta\\ T(\alpha)\text{ unprimed}}}\big(x_{T(\alpha)}-a_{-c(\alpha)+1}\big)\prod_{\substack{\alpha\in\theta\\ T(\alpha)\text{ primed}}}\big(y_{T(\alpha)}+a_{-c(\alpha)+1}\big),\tag{2.14}$$

summed over all $\mathbb{A}$-tableaux T of shape θ, where the subscripts of the variables y_i are identified with the primed indices. An alternative formula is obtained by using a different ordering of the alphabet:

$$\mathbb{A}'=\{1<1'<2<2'<\dots\}.$$

The $\mathbb{A}'$-tableaux T of shape θ are defined in exactly the same way as the $\mathbb{A}$-tableaux, only taking into account the new ordering. Then

$$s_\theta(x/y\,\|\,a)=\sum_T\prod_{\substack{\alpha\in\theta\\ T(\alpha)\text{ unprimed}}}\big(x_{T(\alpha)}-a_{-c(\alpha)}\big)\prod_{\substack{\alpha\in\theta\\ T(\alpha)\text{ primed}}}\big(y_{T(\alpha)}+a_{-c(\alpha)}\big),\tag{2.15}$$

summed over all $\mathbb{A}'$-tableaux T of shape θ.

The supersymmetric Schur functions have the following symmetry property

$$s_\theta(x/y\,\|\,a)=s_{\theta'}(y/x\,\|\,a').\tag{2.16}$$

Moreover, if $x_i=y_i=0$ for all $i\geqslant n+1$, then only tableaux T with entries in $\{1,1',\dots,n,n'\}$ make nonzero contributions in either (2.14) or (2.15).

Remark 2.4. The supersymmetric Schur function $s_\theta(x/y\,\|\,a)$ given in (2.14) coincides with $\Sigma_{\theta;-a'}(x;y)$ as defined in [24, Proposition 4.4]. In order to derive (2.15), first use (2.16), then apply the transposition of the tableaux with respect to the main diagonal and swap i and i' for each i. Note that [24] also contains an equivalent combinatorial formula for $\Sigma_{\theta;a}(x;y)$ in terms of skew hooks. $\square$

Proposition 2.5. *The image of the supersymmetric Schur function $s_\nu(x/y\,\|\,a)$ associated with a (nonskew) diagram ν under the isomorphism* (1.6) *coincides with the double Schur function $s_\nu(x\,\|\,a)$; that is,*

$$s_\nu(x/y\,\|\,a)\big|_{y=-a} = s_\nu(x\,\|\,a),$$

where $y=-a$ denotes the evaluation $y_i=-a_i$ for $i\geqslant 1$.

Proof. We may assume that the sets of variables x and y are finite, $x=(x_1,\dots,x_n)$ and $y=(y_1,\dots,y_n)$. The claim now follows from relation (2.13) with $\mu=\varnothing$, if we observe that $s_{\rho'}(y\,|-a)\big|_{y=-a}=0$ unless $\rho=\varnothing$. □

The symmetry property (2.16) implies the following dual version of Proposition 2.5.

Corollary 2.6. *Under the isomorphism $\Lambda(x/y\,\|\,a)\to\Lambda(y\,\|\,a')$ defined by the evaluation $x_i=-(a')_i$ for all $i\geqslant 1$ we have*

$$s_\theta(x/y\,\|\,a)\big|_{x=-a'} = s_{\theta'}(y\,\|\,a').$$

Proposition 2.5 leads to the following definition.

Definition 2.7. For any skew diagram θ define the *skew double Schur function* $s_\theta(x\,\|\,a)\in\Lambda(x\,\|\,a)$ as the image of $s_\theta(x/y\,\|\,a)\in\Lambda(x/y\,\|\,a)$ under the isomorphism (1.6); that is,

$$s_\theta(x\,\|\,a) = s_\theta(x/y\,\|\,a)\big|_{y=-a}.$$

Equivalently,

$$s_\theta(x\,\|\,a) = \sum_T \prod_{\substack{\alpha\in\theta\\ T(\alpha)\text{ unprimed}}} \big(x_{T(\alpha)} - a_{-c(\alpha)+1}\big) \prod_{\substack{\alpha\in\theta\\ T(\alpha)\text{ primed}}} \big(a_{-c(\alpha)+1} - a_{T(\alpha)}\big), \tag{2.17}$$

summed over all $\mathbb{A}$-tableaux T of shape θ; and

$$s_\theta(x\,\|\,a) = \sum_T \prod_{\substack{\alpha\in\theta\\ T(\alpha)\text{ unprimed}}} \big(x_{T(\alpha)} - a_{-c(\alpha)}\big) \prod_{\substack{\alpha\in\theta\\ T(\alpha)\text{ primed}}} \big(a_{-c(\alpha)} - a_{T(\alpha)}\big), \tag{2.18}$$

summed over all $\mathbb{A}'$-tableaux T of shape θ. Furthermore, by (2.13) the skew double Schur function $s_{\nu/\mu}(x\,\|\,a)$ can also be defined as the sequence of polynomials

$$s_{\nu/\mu}(x\,\|\,a) = \sum_{\mu\subseteq\rho\subseteq\lambda} \widetilde{s}_{\nu/\rho}(x\,\|\,a)\, s_{\rho'/\mu'}(-a^{(n)}\,|-a), \qquad n=1,2,\dots, \tag{2.19}$$

where $x=(x_1,\dots,x_n)$ and $a^{(n)}=(a_1,\dots,a_n)$. □

For any partition μ introduce the sequence a_μ and the series $|a_\mu|$ by

$$a_\mu = (a_{1-\mu_1}, a_{2-\mu_2},\dots) \qquad\text{and}\qquad |a_\mu| = a_{1-\mu_1}+a_{2-\mu_2}+\dots.$$

We will write $\rho \to \sigma$ if the diagram σ is obtained from the diagram ρ by adding one box. Given any element $P(x) \in \Lambda(x\,\|\,a)$, the value $P(a_\mu)$ is a well-defined element of $\mathbb{Q}[a]$. The vanishing theorem of Okounkov [20, 21] states that

$$s_\lambda(a_\rho\,\|\,a) = 0 \qquad \text{unless} \quad \lambda \subseteq \rho,$$

and

$$s_\lambda(a_\lambda\,\|\,a) = \prod_{(i,j)\in\lambda} \big(a_{i-\lambda_i} - a_{\lambda'_j - j+1}\big). \tag{2.20}$$

This theorem can be used to derive the interpolation formulas given in the next proposition. In a slightly different situation this derivation was performed in [19, Propositions 3.3 & 3.4] relying on the approach of [22], and an obvious modification of those arguments works in the present context; see also [4], [9]. The expressions like $|a_\nu| - |a_\mu|$ used below are understood as the polynomials $\sum_{i\geqslant 1}(a_{i-\nu_i} - a_{i-\mu_i})$.

Proposition 2.8. *Given an element $P(x) \in \Lambda(x\,\|\,a)$, define the polynomials $c_{P,\,\mu}^{\nu}(a)$ by the expansion*

$$P(x)\, s_\mu(x\,\|\,a) = \sum_\nu c_{P,\,\mu}^{\nu}(a)\, s_\nu(x\,\|\,a). \tag{2.21}$$

Then $c_{P,\,\mu}^{\nu}(a) = 0$ unless $\mu \subseteq \nu$, and $c_{P,\,\mu}^{\mu}(a) = P(a_\mu)$. Moreover, if $\mu \subsetneq \nu$, then

$$c_{P,\,\mu}^{\nu}(a) = \frac{1}{|a_\nu| - |a_\mu|}\left(\sum_{\mu^+,\,\mu\to\mu^+} c_{P,\,\mu^+}^{\nu}(a) - \sum_{\nu^-,\,\nu^-\to\nu} c_{P,\,\mu}^{\nu^-}(a)\right).$$

The same coefficient can also be found by the formula

$$c_{P,\,\mu}^{\nu}(a) = \sum_R \sum_{k=0}^{l} \frac{P(a_{\rho^{(k)}})}{(|a_{\rho^{(k)}}| - |a_{\rho^{(0)}}|) \dots \wedge \dots (|a_{\rho^{(k)}}| - |a_{\rho^{(l)}}|)}, \tag{2.22}$$

summed over all sequences of partitions R of the form

$$\mu = \rho^{(0)} \to \rho^{(1)} \to \dots \to \rho^{(l-1)} \to \rho^{(l)} = \nu,$$

where the symbol $\wedge$ indicates that the zero factor should be skipped. □

3. Cauchy identities and dual Schur functions

3.1. **Definition of dual Schur functions and Cauchy identities.** We let $\widehat{\Lambda}(x\,\|\,a)$ denote the ring of formal series of the symmetric functions in the set of indeterminates $x = (x_1, x_2, \dots)$ with coefficients in $\mathbb{Q}[a]$. More precisely,

$$\widehat{\Lambda}(x\,\|\,a) = \Big\{ \sum_{\lambda\in\mathcal{P}} c_\lambda(a)\, s_\lambda(x) \mid c_\lambda(a) \in \mathbb{Q}[a] \Big\}. \tag{3.1}$$

The Schur functions $s_\lambda(x)$ can certainly be replaced here by any other classical basis of Λ parameterized by the set of partitions $\mathcal{P}$. We will use the symbol $\widehat{\Lambda}_n = \widehat{\Lambda}_n(x\,\|\,a)$

to indicate the ring defined as in (3.1) for the case of the finite set of variables $x = (x_1, \dots, x_n)$. An element of $\widehat{\Lambda}(x\,\|\,a)$ can be viewed as a sequence of elements of $\widehat{\Lambda}_n$ with $n = 0, 1, \dots$, consistent with respect to the evaluation maps

$$\psi_n : \widehat{\Lambda}_n \to \widehat{\Lambda}_{n-1}, \qquad Q(x_1, \dots, x_n) \mapsto Q(x_1, \dots, x_{n-1}, 0).$$

For any n-tuple of nonnegative integers $\beta = (\beta_1, \dots, \beta_n)$ set

$$A_\beta(x, a) = \det \big[(x_i, a)^{\beta_j}\,(1 - a_{n-\beta_j-1}x_i)(1 - a_{n-\beta_j-2}\,x_i) \dots (1 - a_{1-\beta_j}x_i)\big]_{i,j=1}^{n},$$

where $(x_i, a)^0 = 1$ and

$$(x_i, a)^r = \frac{x_i^r}{(1 - a_0\,x_i)(1 - a_{-1}\,x_i) \dots (1 - a_{1-r}\,x_i)}, \qquad r \geqslant 1. \tag{3.2}$$

Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be a partition of length not exceeding n. Denote by d the number of boxes on the diagonal of λ. That is, d is uniquely determined by the condition that $\lambda_{d+1} \leqslant d \leqslant \lambda_d$. The (i, j) entry A_{ij} of the determinant $A_{\lambda+\delta}(x, a)$ can be written more explicitly as

$$A_{ij} = \begin{cases} \dfrac{x_i^{\lambda_j+n-j}}{(1 - a_0\,x_i)(1 - a_{-1}\,x_i) \dots (1 - a_{j-\lambda_j}\,x_i)} & \text{for} \quad j = 1, \dots, d, \\[1em] x_i^{\lambda_j+n-j}\,(1 - a_1\,x_i)(1 - a_2\,x_i) \dots (1 - a_{j-\lambda_j-1}\,x_i) & \text{for} \quad j = d+1, \dots, n. \end{cases}$$

Observe that the determinant $A_\delta(x, a)$ corresponding to the empty partition equals the Vandermonde determinant,

$$A_\delta(x, a) = \prod_{1 \leqslant i < j \leqslant n} (x_i - x_j).$$

Hence, the formula

$$\widehat{s}_\lambda(x\,\|\,a) = \frac{A_{\lambda+\delta}(x, a)}{A_\delta(x, a)} \tag{3.3}$$

defines an element of the ring $\widehat{\Lambda}_n$. Furthermore, setting $\widehat{s}_\lambda(x\,\|\,a) = 0$ if the length of λ exceeds the number of the x variables, we obtain that the evaluation of the element $\widehat{s}_\lambda(x\,\|\,a) \in \widehat{\Lambda}_n$ at $x_n = 0$ yields the corresponding element of $\widehat{\Lambda}_{n-1}$ associated with λ. Thus, the sequence $\widehat{s}_\lambda(x\,\|\,a) \in \widehat{\Lambda}_n$ for $n = 0, 1, \dots$ defines an element $\widehat{s}_\lambda(x\,\|\,a)$ of $\widehat{\Lambda}(x\,\|\,a)$ which we call the *dual Schur function*. The lowest degree component of $\widehat{s}_\lambda(x\,\|\,a)$ in x coincides with the Schur function $s_\lambda(x)$. Moreover, if a is specialized to the sequence of zeros, then $\widehat{s}_\lambda(x\,\|\,a)$ specializes to $s_\lambda(x)$.

Now we prove an analogue of the Cauchy identity involving the double and dual Schur functions. Consider one more set of variables $y = (y_1, y_2, \dots)$.

Theorem 3.1. *The following identity holds*

$$\prod_{i,\,j\geqslant 1}\frac{1-a_i\,y_j}{1-x_i\,y_j}=\sum_{\lambda\in\mathcal{P}}s_\lambda(x\,\|\,a)\,\widehat{s}_\lambda(y\,\|\,a). \tag{3.4}$$

Proof. We use a modification of the argument applied in [15, Chapter 1] for the proof of the classical Cauchy identity. It will be sufficient to prove the identity in the case of finite sets of variables $x=(x_1,\dots,x_n)$ and $y=(y_1,\dots,y_n)$. We have

$$A_\delta(x\,\|\,a)\,A_\delta(y,a)\sum_{\lambda\in\mathcal{P}}s_\lambda(x\,\|\,a)\,\widehat{s}_\lambda(y\,\|\,a)=\sum_{\gamma}A_\gamma(x\,\|\,a)\,A_\gamma(y,a), \tag{3.5}$$

summed over n-tuples $\gamma=(\gamma_1,\dots,\gamma_n)$ with $\gamma_1>\dots>\gamma_n\geqslant 0$. Since

$$A_\gamma(y,a)=\sum_{\sigma\in\mathfrak{S}_n}\operatorname{sgn}\sigma\,\prod_{i=1}^n(y_i,a)^{\gamma_{\sigma(i)}}\,(1-a_{n-\gamma_{\sigma(i)}-1}y_i)\dots(1-a_{1-\gamma_{\sigma(i)}}y_i)$$

and $A_\gamma(x\,\|\,a)$ is skew-symmetric under permutations of the components of γ, we can write (3.5) in the form

$$\sum_{\beta}A_\beta(x\,\|\,a)\prod_{i=1}^n(y_i,a)^{\beta_i}\,(1-a_{n-\beta_i-1}y_i)\dots(1-a_{1-\beta_i}y_i), \tag{3.6}$$

summed over n-tuples $\beta=(\beta_1,\dots,\beta_n)$ on nonnegative integers. Due to the Jacobi–Trudi formula (2.11), we have

$$A_\beta(x\,\|\,a)=A_\delta(x\,\|\,a)\sum_{\sigma\in\mathfrak{S}_n}\operatorname{sgn}\sigma\cdot h_{\beta_{\sigma(1)}-n+1}(x\,\|\,a)\dots h_{\beta_{\sigma(n)}}(x\,\|\,\tau^{n-1}a).$$

Hence, (3.6) becomes

$$\begin{aligned}A_\delta(x\,\|\,a)&\sum_{\alpha}h_{\alpha_1}(x\,\|\,a)\dots h_{\alpha_n}(x\,\|\,\tau^{n-1}a)\\&\times\sum_{\sigma\in\mathfrak{S}_n}\operatorname{sgn}\sigma\cdot\prod_{i=1}^n(y_{\sigma(i)},a)^{\alpha_i+n-i}\,(1-a_{i-\alpha_i-1}y_{\sigma(i)})\dots(1-a_{i-\alpha_i-n+1}y_{\sigma(i)}),\end{aligned} \tag{3.7}$$

summed over n-tuples $\alpha=(\alpha_1,\dots,\alpha_n)$ on nonnegative integers. However, using (2.7), for each $i=1,\dots,n$ we obtain

$$\begin{aligned}\sum_{k=0}^{\infty}&h_k(x\,\|\,\tau^{i-1}a)\,(z,a)^{k+n-i}\,(1-a_{i-\alpha_i-1}z)\dots(1-a_{i-\alpha_i-n+1}z)\\&=z^{n-i}\,(1-a_1\,z)\dots(1-a_{i-1}\,z)\sum_{k=0}^{\infty}h_k(x\,\|\,\tau^{i-1}a)\,(z,\tau^{i-1}a)^k\\&=z^{n-i}\,(1-a_1\,z)\dots(1-a_{i-1}\,z)\prod_{r=1}^n\frac{1-a_{i+r-1}\,z}{1-x_r\,z},\end{aligned}$$

where we put $z = y_{\sigma(i)}$. Therefore, (3.7) simplifies to

$$\begin{aligned} A_\delta(x\,\|\,a) \prod_{i,j=1}^n \frac{1-a_i y_j}{1-x_i y_j} \sum_{\sigma\in\mathfrak{S}_n} \operatorname{sgn}\sigma \cdot \prod_{i=1}^n y_{\sigma(i)}^{n-i} (1-a_{n+1}y_{\sigma(i)})\dots(1-a_{n+i-1}y_{\sigma(i)}) \\ = A_\delta(x\,\|\,a)\, A_\delta(y,a) \prod_{i,j=1}^n \frac{1-a_i y_j}{1-x_i y_j}, \end{aligned}$$

thus completing the proof. □

Let $z = (z_1, z_2, \dots)$ be another set of variables.

Corollary 3.2. *The following identity holds*

$$\prod_{i,\,j\geqslant 1} \frac{1+y_i\, z_j}{1-x_i\, z_j} = \sum_{\lambda\in\mathcal{P}} s_\lambda(x/y\,\|\,a)\, \widehat{s}_\lambda(z\,\|\,a),$$

Proof. The elements $\widehat{s}_\lambda(z\,\|\,a) \in \widehat{\Lambda}(z\,\|\,a)$ are uniquely determined by the relation. Hence, the claim follows by the application of Proposition 2.5 and Theorem 3.1. □

Some other identities of this kind are immediate from the symmetry property (2.16) and Corollary 3.2.

Corollary 3.3. *We have the identities*

$$\prod_{i,\,j\geqslant 1} \frac{1+x_i\, z_j}{1-y_i\, z_j} = \sum_{\lambda\in\mathcal{P}} s_\lambda(x/y\,\|\,a)\, \widehat{s}_{\lambda'}(z\,\|\,a')$$

and

$$\prod_{i,\,j\geqslant 1} \frac{1+x_i\, y_j}{1+a_i\, y_j} = \sum_{\lambda\in\mathcal{P}} s_\lambda(x\,\|\,a)\, \widehat{s}_{\lambda'}(y\,\|\,a').$$ □

3.2. **Combinatorial presentation.** Given a skew diagram θ, introduce the corresponding *skew dual Schur function* $\widehat{s}_\theta(x\,\|\,a)$ by the formula

$$\widehat{s}_\theta(x\,\|\,a) = \sum_T \prod_{\alpha\in\theta} X_{T(\alpha)}(a_{-c(\alpha)+1}, a_{-c(\alpha)}), \tag{3.8}$$

summed over the reverse θ-tableaux T, where

$$X_i(g,h) = \frac{x_i\,(1-g\,x_{i-1})\dots(1-g\,x_1)}{(1-h\,x_i)\dots(1-h\,x_1)}.$$

Theorem 3.4. *For any partition μ the following identity holds*

$$\prod_{i,\,j\geqslant 1} \frac{1-a_{i-\mu_i}\, y_j}{1-x_i\, y_j}\, s_\mu(x\,\|\,a) = \sum_\nu s_\nu(x\,\|\,a)\, \widehat{s}_{\nu/\mu}(y\,\|\,a), \tag{3.9}$$

summed over partitions ν containing μ. In particular, if $\theta = \lambda$ is a normal diagram, then the dual Schur function $\widehat{s}_\lambda(x\,\|\,a)$ admits the tableau presentation (3.8).

Proof. It will be sufficient to consider the case where the set of variables y is finite, $y=(y_1,\dots,y_n)$. We will argue by induction on n and suppose that $n\geqslant 1$. By the induction hypothesis, the identity (3.9) holds for the set of variables $y'=(y_2,\dots,y_n)$. Hence, we need to verify that

$$\prod_{i\geqslant 1}\frac{1-a_{i-\mu_i}\,y_1}{1-x_i\,y_1}\sum_{\lambda}s_{\lambda}(x\,\|\,a)\,\widehat{s}_{\lambda/\mu}(y'\,\|\,a)=\sum_{\nu}s_{\nu}(x\,\|\,a)\,\widehat{s}_{\nu/\mu}(y\,\|\,a).$$

However, due to (2.7),

$$\prod_{i\geqslant 1}\frac{1-a_{i-\mu_i}\,y_1}{1-x_i\,y_1}=\sum_{k=0}^{\infty}\frac{h_k(x\,\|\,a^{\mu})\,y_1^k}{(1-a_0\,y_1)\dots(1-a_{-k+1}\,y_1)},$$

where a^{μ} denotes the sequence of parameters such that $(a^{\mu})_i=a_{i-\mu_i}$ for $i\geqslant 1$ and $(a^{\mu})_i=a_i$ for $i\leqslant 0$. Now define polynomials $c_{\lambda,(k)}^{\nu}(a,a^{\mu})\in\mathbb{Q}[a]$ by the expansion

$$s_{\lambda}(x\,\|\,a)\,h_k(x\,\|\,a^{\mu})=\sum_{\nu}c_{\lambda,(k)}^{\nu}(a,a^{\mu})\,s_{\nu}(x\,\|\,a).$$

Hence, the claim will follow if we show that

$$\widehat{s}_{\nu/\mu}(y\,\|\,a)=\sum_{\lambda,\,k}c_{\lambda,(k)}^{\nu}(a,a^{\mu})\,\widehat{s}_{\lambda/\mu}(y'\,\|\,a)\,\frac{y_1^k}{(1-a_0\,y_1)\dots(1-a_{-k+1}\,y_1)}.\tag{3.10}$$

The definition (3.8) of the skew dual Schur functions implies that

$$\widehat{s}_{\nu/\mu}(y\,\|\,a)=\sum_{\lambda}\widehat{s}_{\lambda/\mu}(y'\,\|\,a)\prod_{\alpha\in\lambda/\mu}\frac{1-a_{-c(\alpha)+1}\,y_1}{1-a_{-c(\alpha)}\,y_1}\prod_{\beta\in\nu/\lambda}\frac{y_1}{1-a_{-c(\alpha)}\,y_1},$$

summed over diagrams λ such that $\mu\subseteq\lambda\subseteq\nu$ and ν/λ is a horizontal strip (i.e., every column of this diagram contains at most one box). Therefore, (3.10) will follow from the relation

$$\sum_{k}c_{\lambda,(k)}^{\nu}(a,a^{\mu})\,\frac{y_1^k}{(1-a_0\,y_1)\dots(1-a_{-k+1}\,y_1)}$$
$$=\prod_{\alpha\in\lambda/\mu}\frac{1-a_{-c(\alpha)+1}\,y_1}{1-a_{-c(\alpha)}\,y_1}\prod_{\beta\in\nu/\lambda}\frac{y_1}{1-a_{-c(\alpha)}\,y_1}$$

which takes more convenient form after the substitution $t=y_1^{-1}$:

$$\sum_{k}\frac{c_{\lambda,(k)}^{\nu}(a,a^{\mu})}{(t-a_0)\dots(t-a_{-k+1})}=\prod_{\alpha\in\lambda/\mu}\big(t-a_{-c(\alpha)+1}\big)\prod_{\beta\in\nu/\mu}\big(t-a_{-c(\alpha)}\big)^{-1}.\tag{3.11}$$

We will verify the latter by induction on $|\nu|-|\lambda|$. Suppose first that $\nu=\lambda$. Then $c_{\lambda,(k)}^{\lambda}(a,a^{\mu})=h_k(a_{\lambda}\,\|\,a^{\mu})$ by Proposition 2.8, and relation (2.7) implies that

$$\sum_{k}\frac{h_k(a_{\lambda}\,\|\,a^{\mu})}{(t-a_0)\dots(t-a_{-k+1})}=\prod_{i\geqslant 1}\frac{t-a_{i-\mu_i}}{t-a_{i-\lambda_i}}.$$

This expression coincides with

$$\prod_{\alpha\in\lambda/\mu}\frac{t-a_{-c(\alpha)+1}}{t-a_{-c(\alpha)}},$$

thus verifying (3.11) in the case under consideration. Suppose now that $|\nu|-|\lambda|\geqslant 1$. By Proposition 2.8, we have

$$c^{\nu}_{\lambda,(k)}(a,a^{\mu})=\frac{1}{|a_\nu|-|a_\lambda|}\Bigg(\sum_{\lambda^+,\,\lambda\to\lambda^+}c^{\nu}_{\lambda^+,(k)}(a,a^{\mu})-\sum_{\nu^-,\,\nu^-\to\nu}c^{\nu^-}_{\lambda,(k)}(a,a^{\mu})\Bigg).$$

Hence, applying the induction hypothesis, we can write the left hand side of (3.11) in the form

$$\begin{aligned}\frac{1}{|a_\nu|-|a_\lambda|}\Big(\sum_{\lambda^+}\prod_{\alpha\in\lambda^+/\mu}\big(t-a_{-c(\alpha)+1}\big)\prod_{\beta\in\nu/\mu}\big(t-a_{-c(\alpha)}\big)^{-1}\\ -\sum_{\nu^-}\prod_{\alpha\in\lambda/\mu}\big(t-a_{-c(\alpha)+1}\big)\prod_{\beta\in\nu^-/\mu}\big(t-a_{-c(\alpha)}\big)^{-1}\Big).\end{aligned}$$

Since ν/λ is a horizontal strip, we have

$$\sum_{\alpha=\lambda^+/\lambda}\big(t-a_{-c(\alpha)+1}\big)-\sum_{\alpha=\nu/\nu^-}\big(t-a_{-c(\alpha)}\big)=|a_\nu|-|a_\lambda|,$$

so that the previous expression simplifies to

$$\prod_{\alpha\in\lambda/\mu}\big(t-a_{-c(\alpha)+1}\big)\prod_{\beta\in\nu/\mu}\big(t-a_{-c(\alpha)}\big)^{-1}$$

completing the proof of (3.11).

The second part of the proposition follows from Theorem 3.1 and the fact that the elements $\widehat{s}_\lambda(y\,\|\,a)\in\widehat{\Lambda}(y\,\|\,a)$ are uniquely determined by the relation (3.4). □

Remark 3.5. Under the specialization $a_i=0$ the identity of Theorem 3.4 turns into a particular case of the identity in [15, Examples, Ch. I]. □

Since the skew dual Schur functions are uniquely determined by the expansion (3.9), the following corollary is immediate from Theorem 3.4.

Corollary 3.6. *The skew dual Schur functions defined in* (3.8) *belong to the ring* $\widehat{\Lambda}(x\,\|\,a)$. *In particular, they are symmetric in the variables* x. □

Recall the Littlewood–Richardson polynomials defined by (1.2).

Proposition 3.7. *For any skew diagram* ν/μ *we have the expansion*

$$\widehat{s}_{\nu/\mu}(y\,\|\,a)=\sum_{\lambda}c^{\nu}_{\lambda\mu}(a)\,\widehat{s}_\lambda(y\,\|\,a).$$

Proof. We use an argument similar to [15, Chapter 1]. Consider the set of variables (y, y'), where $y = (y_1, y_2, \dots)$ and $y' = (y'_1, y'_2, \dots)$ and assume they are ordered in the way that each y_i precedes each y'_j. By the tableau presentation (3.8) of the dual Schur functions, we get

$$\widehat{s}_\nu(y, y' \,\|\, a) = \sum_{\mu \subseteq \nu} \widehat{s}_{\nu/\mu}(y \,\|\, a)\, \widehat{s}_\mu(y' \,\|\, a). \tag{3.12}$$

On the other hand, by Theorem 3.1,

$$\sum_\nu s_\nu(x \,\|\, a)\, \widehat{s}_\nu(y, y' \,\|\, a) = \prod_{i,j \geqslant 1} \frac{1 - a_i y_j}{1 - x_i y_j} \prod_{i,k \geqslant 1} \frac{1 - a_i y'_k}{1 - x_i y'_k}$$
$$= \sum_{\lambda,\, \mu} s_\lambda(x \,\|\, a)\, \widehat{s}_\lambda(y \,\|\, a)\, s_\mu(x \,\|\, a)\, \widehat{s}_\mu(y' \,\|\, a) = \sum_{\lambda,\, \mu,\, \nu} c_{\lambda\mu}^{\nu}(a)\, s_\nu(x \,\|\, a)\, \widehat{s}_\lambda(y \,\|\, a)\, \widehat{s}_\mu(y' \,\|\, a)$$

which proves that

$$\widehat{s}_\nu(y, y' \,\|\, a) = \sum_{\lambda,\, \mu} c_{\lambda\mu}^{\nu}(a)\, \widehat{s}_\lambda(y \,\|\, a)\, \widehat{s}_\mu(y' \,\|\, a). \tag{3.13}$$

The desired relation now follows by comparing (3.12) and (3.13). □

3.3. **Jacobi–Trudi-type formulas.** Introduce the *dual elementary* and *complete symmetric functions* by

$$\widehat{e}_k(x \,\|\, a) = \widehat{s}_{(1^k)}(x \,\|\, a), \qquad \widehat{h}_k(x \,\|\, a) = \widehat{s}_{(k)}(x \,\|\, a).$$

By Theorem 3.4,

$$\widehat{e}_k(x \,\|\, a) = \sum_{i_1 > \dots > i_k} X_{i_1}(a_1, a_0) X_{i_2}(a_2, a_1) \dots X_{i_k}(a_k, a_{k-1}),$$
$$\widehat{h}_k(x \,\|\, a) = \sum_{i_1 \geqslant \dots \geqslant i_k} X_{i_1}(a_1, a_0) X_{i_2}(a_0, a_{-1}) \dots X_{i_k}(a_{-k+2}, a_{-k+1}).$$

Proposition 3.8. *We have the following generating series formulas*

$$1 + \sum_{k=1}^{\infty} \widehat{e}_k(x \,\|\, a)\, (t + a_0)(t + a_1) \dots (t + a_{k-1}) = \prod_{i=1}^{\infty} \frac{1 + t x_i}{1 - a_0 x_i},$$
$$1 + \sum_{k=1}^{\infty} \widehat{h}_k(x \,\|\, a)\, (t - a_1)(t - a_0) \dots (t - a_{-k+2}) = \prod_{i=1}^{\infty} \frac{1 - a_1 x_i}{1 - t x_i}.$$

Proof. The first relation follows from the second identity in Corollary 3.3 by taking $x = (t)$ and then replacing a by a' and y_i by x_i for all i. Similarly, the second relation follows from Theorem 3.1 by taking $x = (t)$ and replacing y_i by x_i. □

We can now prove an analogue of the Jacobi–Trudi formula for the dual Schur functions.

Proposition 3.9. *If λ and μ are partitions of length not exceeding n, then*

$$\widehat{s}_{\lambda/\mu}(x\,\|\,a) = \det\big[\widehat{h}_{\lambda_i-\mu_j-i+j}(x\,\|\,\tau^{-\mu_j+j-1}a)\big]_{i,j=1}^{n}. \tag{3.14}$$

Proof. Apply Theorem 3.4 for the finite set of variables $x = (x_1, \dots, x_n)$ and multiply both sides of (3.9) by $A_\delta(x\,\|\,a)$. This gives

$$\prod_{j\geqslant 1}\prod_{i=1}^{n}\frac{1-a_{i-\mu_i}\,y_j}{1-x_i\,y_j}\,A_{\mu+\delta}(x\,\|\,a) = \sum_{\lambda} A_{\lambda+\delta}(x\,\|\,a)\,\widehat{s}_{\lambda/\mu}(y\,\|\,a). \tag{3.15}$$

For any $\sigma \in \mathfrak{S}_n$ we have

$$\prod_{j\geqslant 1}\prod_{i=1}^{n}\frac{1-a_{i-\mu_i}\,y_j}{1-x_i\,y_j} = \prod_{j\geqslant 1}\prod_{i=1}^{n}\frac{1-a_{i-\mu_i}\,y_j}{1-x_{\sigma(i)}\,y_j}.$$

By the second formula of Proposition 3.8,

$$\prod_{j\geqslant 1}\frac{1-a_{i-\mu_i}\,y_j}{1-x_{\sigma(i)}\,y_j} = \sum_{k=0}^{\infty}\widehat{h}_k(y\,\|\,\tau^{-\mu_i+i-1}a)\,(x_{\sigma(i)}-a_{i-\mu_i})\dots(x_{\sigma(i)}-a_{i-\mu_i-k+1}).$$

Since

$$A_{\mu+\delta}(x\,\|\,a) = \sum_{\sigma\in\mathfrak{S}_n}\operatorname{sgn}\sigma\cdot(x_{\sigma(1)}\,\|\,a)^{\mu_1+n-1}\dots(x_{\sigma(n)}\,\|\,a)^{\mu_n},$$

the left hand side of (3.15) can be written in the form

$$\sum_{\sigma\in\mathfrak{S}_n}\operatorname{sgn}\sigma\prod_{i=1}^{n}\sum_{k_i=0}^{\infty}(x_{\sigma(i)}-a_n)\dots(x_{\sigma(i)}-a_{i-\mu_i-k_i+1})\,h_{k_i}(y,\tau^{-\mu_i+i-1}a).$$

Hence, comparing the coefficients of $(x_1\,\|\,a)^{\lambda_1+n-1}\dots(x_n\,\|\,a)^{\lambda_n}$ on both sides of (3.15), we get

$$\widehat{s}_{\lambda/\mu}(y\,\|\,a) = \sum_{\rho\in\mathfrak{S}_n}\operatorname{sgn}\rho\prod_{i=1}^{n}\widehat{h}_{\lambda_i-\mu_{\rho(i)}-i+\rho(i)}(y\,\|\,\tau^{-\mu_{\rho(i)}+\rho(i)-1}a),$$

as required. □

Proposition 3.9 implies that the dual Schur functions may be regarded as a specialization of the generalized Schur functions described in [14, 9th Variation]. Namely, in the notation of that paper, specialize the variables h_{rs} by

$$h_{rs} = \widehat{h}_r(x\,\|\,\tau^{-s}a), \qquad r\geqslant 1, \quad s\in\mathbb{Z}. \tag{3.16}$$

Then the Schur functions $s_{\lambda/\mu}$ of [14] become $\widehat{s}_{\lambda/\mu}(x\,\|\,a)$. Hence the following corollaries are immediate from (9.6′) and (9.7) in [14] and Proposition 3.9. The first of them is an analogue of the Nägelsbach–Kostka formula.

Corollary 3.10. *If λ and μ are partitions such that the lengths of λ' and μ' do not exceed m, then*

$$\widehat{s}_{\lambda/\mu}(x\,\|\,a) = \det\big[\widehat{e}_{\lambda'_i-\mu'_j-i+j}(x\,\|\,\tau^{\mu'_j-j+1}a)\big]_{i,j=1}^{m}. \tag{3.17}$$

Suppose that λ is a diagram with d boxes on the main diagonal. Write λ in the Frobenius notation

$$\lambda = (\alpha_1, \dots, \alpha_d \,|\, \beta_1, \dots, \beta_d) = (\alpha \,|\, \beta),$$

where $\alpha_i = \lambda_i - i$ and $\beta_i = \lambda'_i - i$. The following is an analogue of the Giambelli formula.

Corollary 3.11. *We have the identity*

$$\widehat{s}_{(\alpha\,|\,\beta)}(x\,\|\,a) = \det\big[\widehat{s}_{(\alpha_i\,|\,\beta_j)}(x\,\|\,a)\big]_{i,j=1}^{d}. \tag{3.18}$$

3.4. **Expansions in terms of Schur functions.** We will now deduce expansions of the dual Schur functions in terms of the Schur functions $s_\lambda(x)$ whose coefficients are elements of $\mathbb{Q}[a]$ written explicitly as certain determinants. In Theorem 3.17 below we will give alternative tableau presentations for these coefficients. Suppose that μ is a diagram containing d boxes on the main diagonal.

Proposition 3.12. *The dual Schur function $\widehat{s}_\mu(x\,\|\,a)$ can be written as the series*

$$\widehat{s}_\mu(x\,\|\,a) = \sum_{\lambda} (-1)^{n(\lambda/\mu)} \det\big[h_{\lambda_i-\mu_j-i+j}(a_0, a_{-1}, \dots, a_{j-\mu_j})\big]_{i,j=1}^{d}$$
$$\times \det\big[e_{\lambda_i-\mu_j-i+j}(a_1, a_2, \dots, a_{j-\mu_j-1})\big]_{i,j=d+1}^{n} s_\lambda(x),$$

summed over diagrams λ which contain μ and such that λ has d boxes on the main diagonal, where $n(\lambda/\mu)$ denotes the total number of boxes in the diagram λ/μ in rows $d+1, d+2, \dots, n = \ell(\lambda)$.

Proof. It will be sufficient to prove the formula for the case of finite set of variables $x = (x_1, \dots, x_n)$. We use the definition (3.3) of the dual Schur functions. The entries A_{ij} of the determinant $A_{\mu+\delta}(x, a)$ can be written as

$$A_{ij} = \begin{cases} \displaystyle\sum_{p_j \geqslant 0} h_{p_j}(a_0, a_{-1}, \dots, a_{j-\mu_j})\, x_i^{\mu_j+p_j+n-j} & \text{for} \quad j = 1, \dots, d, \\[2ex] \displaystyle\sum_{p_j \geqslant 0} (-1)^{p_j}\, e_{p_j}(a_1, a_2, \dots, a_{j-\mu_j-1})\, x_i^{\mu_j+p_j+n-j} & \text{for} \quad j = d+1, \dots, n. \end{cases}$$

Hence, (3.3) gives

$$\widehat{s}_\mu(x\,\|\,a) = \sum_{p_1,\dots,p_n} \prod_{j=1}^{d} h_{p_j}(a_0, a_{-1}, \dots, a_{j-\mu_j}) \prod_{j=d+1}^{n} (-1)^{p_j}\, e_{p_j}(a_1, a_2, \dots, a_{j-\mu_j-1})$$
$$\times \det[x_i^{\mu_j+p_j+n-j}] / \det[x_i^{n-j}].$$

The ratio of the determinants in this formula is nonzero only if

$$\mu_{\sigma(j)} + p_{\sigma(j)} + n - \sigma(j) = \lambda_j + n - j, \qquad j = 1, \dots, n,$$

for some diagram λ containing μ and some permutation σ of the set $\{1,\dots,n\}$. Moreover, since $e_{p_j}(a_1,a_2,\dots,a_{j-\mu_j-1})=0$ for $p_j>j-\mu_j-1$, the number of diagonal boxes in λ equals d. The ratio can then be written as

$$\det[x_i^{\mu_j+p_j+n-j}]/\det[x_i^{n-j}]=\operatorname{sgn}\sigma\cdot s_\lambda(x),$$

which gives the desired formula for the coefficients. □

Corollary 3.13. *Using the Frobenius notation $(\alpha\,|\,\beta)$ for the hook diagram $(\alpha+1,1^\beta)$, we have*

$$\widehat{s}_{(\alpha\,|\,\beta)}(x\,\|\,a)=\sum_{p,\,q\geqslant 0}(-1)^q\,h_p(a_0,a_{-1},\dots,a_{-\alpha})\,h_q(a_1,a_2,\dots,a_{\beta+1})\,s_{(\alpha+p\,|\,\beta+q)}(x).$$

Proof. By Proposition 3.12, the coefficient of $s_{(\alpha+p\,|\,\beta+q)}(x)$ in the expansion of the dual Schur function $\widehat{s}_{(\alpha\,|\,\beta)}(x\,\|\,a)$ equals

$$(-1)^q\,h_p(a_0,a_{-1},\dots,a_{-\alpha})\,\det\big[e_{j-i+1}(a_1,a_2,\dots,a_{\beta+j})\big]_{i,j=1}^q. \tag{3.19}$$

Using the relations for the elementary symmetric polynomials

$$e_k(a_1,a_2,\dots,a_{\beta+j})=e_k(a_1,a_2,\dots,a_{\beta+j-1})+e_{k-1}(a_1,a_2,\dots,a_{\beta+j-1})\,a_{\beta+j},$$

it is not difficult to bring the determinant which occurs in (3.19) to the form

$$\det\big[e_{j-i+1}(a_1,a_2,\dots,a_{\beta+j})\big]_{i,j=1}^q=\det\big[e_{j-i+1}(a_1,a_2,\dots,a_{\beta+1})\big]_{i,j=1}^q.$$

However, this coincides with $h_q(a_1,a_2,\dots,a_{\beta+1})$ due to the Nägelsbach–Kostka formula (i.e., (3.17) with the zero sequence a). □

Example 3.14. The dual Schur function corresponding to the single box diagram is given by

$$\widehat{s}_{(1)}(x\,\|\,a)=\sum_{p,\,q\geqslant 0}(-1)^q\,a_0^p\,a_1^q\,s_{(p\,|\,q)}(x).$$

□

Recall that the involution $\omega:\Lambda\to\Lambda$ on the ring of symmetric functions in x takes $s_\lambda(x)$ to $s_{\lambda'}(x)$; see [15, Chapter 1] or Section 2 above. Let us extend ω to the $\mathbb{Q}[a]$-linear involution

$$\widehat{\omega}:\widehat{\Lambda}(x\,\|\,a)\to\widehat{\Lambda}(x\,\|\,a),\qquad \sum_{\lambda\in\mathcal{P}}c_\lambda(a)\,s_\lambda(x)\mapsto\sum_{\lambda\in\mathcal{P}}c_\lambda(a)\,s_{\lambda'}(x), \tag{3.20}$$

where $c_\lambda(a)\in\mathbb{Q}[a]$. We will find the images of the dual Schur functions under $\widehat{\omega}$. As before, by a' we denote the sequence of variables such that $(a')_i=-a_{-i+1}$ for all $i\in\mathbb{Z}$.

Corollary 3.15. *For any skew diagram λ/μ we have*

$$\widehat{\omega}:\widehat{s}_{\lambda/\mu}(x\,\|\,a)\mapsto\widehat{s}_{\lambda'/\mu'}(x\,\|\,a'). \tag{3.21}$$

Proof. By Corollary 3.13, for any $m \in \mathbb{Z}$

$$\widehat{\omega} : \widehat{s}_{(\alpha\,|\,\beta)}(x\,\|\,\tau^m a) \mapsto \widehat{s}_{(\beta\,|\,\alpha)}(x\,\|\,\tau^{-m} a').$$

In particular,

$$\widehat{\omega} : \widehat{h}_k(x\,\|\,\tau^m a) \mapsto \widehat{e}_k(x\,\|\,\tau^{-m} a'), \qquad k \geqslant 0.$$

The statement now follows from (3.14) and (3.17). □

Note that (3.21) with $\mu = \varnothing$ also follows from Corollary 3.13 and the Giambelli formula (3.18).

Remark 3.16. The involution $\widehat{\omega}$ does not coincide with the involution introduced in [15, (9.6)]. The latter is defined on the ring generated by the elements h_{rs} and takes the generalized Schur function $s_{\lambda/\mu}$ to $s_{\lambda'/\mu'}$. Therefore, under the specialization (3.16), the image of $\widehat{s}_{\lambda/\mu}(x\,\|\,a)$ would be $\widehat{s}_{\lambda'/\mu'}(x\,\|\,a)$ which is different from (3.21). □

We can now derive an alternative expansion of the dual Schur functions in terms of the Schur functions $s_\lambda(x)$; cf. Proposition 3.12. Suppose that λ is a diagram which contains μ and such that μ and λ have the same number of boxes d on the diagonal. By a *hook λ/μ-tableau T* we will mean a tableau obtained by filling in the boxes of λ/μ with integers in the following way. The entries in the first d rows weakly increase along the rows and strictly increase down the columns, and all entries in row i belong to the set $\{i-\mu_i,\dots,-1,0\}$ for $i=1,\dots,d$; the entries in the first d columns weakly decrease down the columns and strictly decrease along the rows, and all entries in column j belong to the set $\{1,2,\dots,\mu'_j-j+1\}$ for $j=1,\dots,d$. Then we define the corresponding flagged Schur function $\varphi_{\lambda/\mu}(a)$ by the formula

$$\varphi_{\lambda/\mu}(a) = \sum_T \prod_{\alpha\in\lambda/\mu} a_{T(\alpha)},$$

summed over the hook λ/μ-tableaux T.

Theorem 3.17. *We have the expansion of the dual Schur function $\widehat{s}_\mu(x\,\|\,a)$*

$$\widehat{s}_\mu(x\,\|\,a) = \sum_\lambda (-1)^{n(\lambda/\mu)}\, \varphi_{\lambda/\mu}(a)\, s_\lambda(x),$$

summed over diagrams λ which contain μ and such that λ has d boxes on the main diagonal, where $n(\lambda/\mu)$ denotes the total number of boxes in the diagram λ/μ in rows $d+1, d+2, \dots$.

Proof. Consider the expansions of $\widehat{s}_\mu(x\,\|\,a)$ and $\widehat{s}_{\mu'}(x\,\|\,a')$ provided by Proposition 3.12. By Corollary 3.15, $\widehat{s}_\mu(x\,\|\,a)$ is the image of $\widehat{s}_{\mu'}(x\,\|\,a')$ under the involution $\widehat{\omega}$. Since $\widehat{\omega} : s_\lambda(x) \mapsto s_{\lambda'}(x)$, taking $\lambda_i = \mu_i$ for $i = 1,\dots,d$ and comparing the coefficients of

$s_\lambda(x)$ in the expansions of $\widehat{s}_\mu(x\,\|\,a)$ and $\widehat{\omega}\big(\widehat{s}_{\mu'}(x\,\|\,a')\big)$, we can conclude that

$$(-1)^{n(\lambda/\mu)} \det\big[e_{\lambda_i-\mu_j-i+j}(a_1,a_2,\dots,a_{j-\mu_j-1})\big]_{i,j\geqslant d+1} \\ = \det\big[h_{\lambda'_i-\mu'_j-i+j}(a'_0,a'_{-1},\dots,a'_{j-\mu'_j})\big]_{i,j=1}^{d}$$

so that

$$\det\big[e_{\lambda_i-\mu_j-i+j}(a_1,a_2,\dots,a_{j-\mu_j-1})\big]_{i,j\geqslant d+1} = \det\big[h_{\lambda'_i-\mu'_j-i+j}(a_1,a_2,\dots,a_{\mu'_j-j+1})\big]_{i,j=1}^{d}.$$

On the other hand, if λ is a diagram containing μ and such that λ has d boxes on the main diagonal, both determinants

$$\det\big[h_{\lambda_i-\mu_j-i+j}(a_0,a_{-1},\dots,a_{j-\mu_j})\big]_{i,j=1}^{d}, \quad \det\big[h_{\lambda'_i-\mu'_j-i+j}(a_1,a_2,\dots,a_{\mu'_j-j+1})\big]_{i,j=1}^{d}$$

coincide with the respective row-flagged Schur functions and they admit the required tableau presentations; see [14, (8.2)], [25]. □

It is clear from the definition of the flagged Schur function $\varphi_{\lambda/\mu}(a)$ that it can be written as the product of two polynomials. More precisely, suppose that the diagram λ contains μ and both λ and μ have d boxes on their main diagonals. Let $(\lambda/\mu)_+$ denote the part of the skew diagram λ/μ contained in the top d rows. With this notation, the hook flagged Schur function $\varphi_{\lambda/\mu}(a)$ can be written as

$$\varphi_{\lambda/\mu}(a) = (-1)^{n(\lambda/\mu)}\, \varphi_{(\lambda/\mu)_+}(a)\, \varphi_{(\lambda'/\mu')_+}(a'). \tag{3.22}$$

In addition to the tableau presentation of the polynomial $\varphi_{(\lambda/\mu)_+}(a)$ given above, we can get an alternative presentation based on the column-flagged Schur functions; see [14, (8.2′)], [25]. Due to (3.22), this also gives alternative formulas for the coefficients in the expansion of $\widehat{s}_\mu(x\,\|\,a)$.

Corollary 3.18. *We have the tableau presentation*

$$\varphi_{(\lambda/\mu)_+}(a) = \sum_T \prod_{\alpha\in(\lambda/\mu)_+} a_{T(\alpha)},$$

summed over the $(\lambda/\mu)_+$-tableaux T whose entries in column j belong to the set $\{0,-1,\dots,-j+\mu'_j+2\}$ for $j\geqslant d+1$, and the entries weakly increase along the rows and strictly increase down the columns. □

Our next goal is to derive the inverse formulas expressing the Schur functions $s_\mu(x)$ as a series of the dual Schur functions $\widehat{s}_\lambda(x\,\|\,a)$.

Proposition 3.19. *We have the expansion*

$$s_\mu(x) = \sum_\lambda (-1)^{m(\lambda/\mu)} \det\big[e_{\lambda_i-\mu_j-i+j}(a_0,a_{-1},\dots,a_{i-\lambda_i+1})\big]_{i,j=1}^{d} \\ \times \det\big[h_{\lambda_i-\mu_j-i+j}(a_1,a_2,\dots,a_{i-\lambda_i})\big]_{i,j\geqslant d+1}\, \widehat{s}_\lambda(x\,\|\,a),$$

summed over diagrams λ which contain μ and such that λ has d boxes on the main diagonal, where $m(\lambda/\mu)$ denotes the total number of boxes in the diagram λ/μ in rows $1, \dots, d$.

Proof. We will work with a finite set of variables $x = (x_1, \dots, x_n)$. The one variable specialization of (2.7) gives

$$1 + \sum_{k=1}^{\infty} \frac{(x - a_1)(x - a_0) \dots (x - a_{-k+2})\, t^k}{(1 - a_0 t) \dots (1 - a_{-k+1} t)} = \frac{1 - a_1 t}{1 - x t}.$$

This implies

$$\sum_{k=1}^{\infty} \frac{(x - a_0) \dots (x - a_{-k+2})\, t^k}{(1 - a_0 t) \dots (1 - a_{-k+1} t)} = \frac{t}{1 - x t}. \tag{3.23}$$

Writing

$$(x - a_0) \dots (x - a_{-k+2}) = \sum_{i=1}^{k} (-1)^{k-i} e_{k-i}(a_0, a_{-1}, \dots, a_{-k+2})\, x^{i-1}$$

and comparing the coefficients of x^{r-1} on both sides of (3.23) we come to the relation

$$t^r = \sum_{k=r}^{\infty} \frac{(-1)^{k-r}\, e_{k-r}(a_0, a_{-1}, \dots, a_{-k+2})\, t^k}{(1 - a_0 t) \dots (1 - a_{-k+1} t)}, \qquad r \geqslant 1. \tag{3.24}$$

Similarly, writing

$$\frac{1}{(1 - a_0 t) \dots (1 - a_{-k+1} t)} = \sum_{j=0}^{\infty} h_j(a_0, a_{-1}, \dots, a_{-k+1})\, t^j$$

and comparing the coefficients of t^{r+1} on both sides of (3.23) we come to

$$x^r = \sum_{k=0}^{r} h_{r-k}(a_0, a_{-1}, \dots, a_{-k})\, (x - a_0)(x - a_{-1}) \dots (x - a_{-k+1}), \qquad r \geqslant 0. \tag{3.25}$$

Assuming that the length of μ does not exceed n, represent $s_\mu(x)$ as the ratio of determinants

$$s_\mu(x) = \frac{A_{\mu+\delta}(x)}{A_\delta(x)},$$

where

$$A_\alpha(x) = \det \big[x_i^{\alpha_j} \big]_{i,j=1}^{n}, \qquad \alpha = (\alpha_1, \dots, \alpha_n).$$

By (3.24), for any $j = 1, \dots, d$ we have

$$x_i^{\mu_j - j + 1} = \sum_{p=\mu_j - j + 1}^{\infty} \frac{(-1)^{p - \mu_j + j - 1}\, e_{p - \mu_j + j - 1}(a_0, a_{-1}, \dots, a_{-p+2})\, x_i^p}{(1 - a_0 x_i) \dots (1 - a_{-p+1} x_i)}.$$

Similarly, for $j=d+1,\dots,n$ we find from (3.25) applied for $x=x_i^{-1}$ and $r=j-\mu_j-1$ that

$$x_i^{\mu_j-j+1}=\sum_{p=0}^{j-\mu_j-1} h_{j-\mu_j-p-1}(a_1,a_2,\dots,a_{p+1})\,x_i^{-p}(1-a_1x_i)(1-a_2x_i)\dots(1-a_px_i).$$

Multiplying both sides of these relations by x_i^{n-1} we get the respective expansions of $x_i^{\mu_j+n-j}$ which allow us to write

$$A_{\mu+\delta}(x)=\sum_{\beta_1,\dots,\beta_n}\prod_{j=1}^{d}(-1)^{\beta_j-\mu_j+j-1}\,e_{\beta_j-\mu_j+j-1}(a_0,a_{-1},\dots,a_{-\beta_j+2})\\ \times\prod_{j=d+1}^{n} h_{j-\mu_j-\beta_j-1}(a_1,a_2,\dots,a_{\beta_j+1})\,A_\beta(x,a).$$

Nonzero summands here correspond to the n-tuples β of the form

$$\beta_j=\lambda_{\sigma(j)}-\sigma(j)+1,\qquad j=1,\dots,d,$$

and

$$\beta_j=-\lambda_{\tau(j)}+\tau(j)-1,\qquad j=d+1,\dots,n,$$

where σ is a permutation of $\{1,\dots,d\}$ and τ is a permutation of $\{d+1,\dots,n\}$, and λ is a diagram containing μ such that λ has d boxes on the main diagonal. Dividing both sides of the above relation by the Vandermonde determinant, we get the desired expansion formula. $\square$

Now we obtain a tableau presentation of the coefficients in the expansion of $s_\mu(x)$; cf. Theorem 3.17. By a *dual hook λ/μ-tableau T* we will mean a tableau obtained by filling in the boxes of λ/μ with integers in the following way. The entries in the first d rows strictly decrease along the rows and weakly decrease down the columns, and all entries in row i belong to the set $\{0,-1,\dots,i-\lambda_i+1\}$ for $i=1,\dots,d$; the entries in the first d columns strictly increase down the columns and weakly increase along the rows, and all entries in column j belong to the set $\{1,2,\dots,\lambda'_j-j\}$ for $j=1,\dots,d$. Then we define the corresponding dual flagged Schur function $\psi_{\lambda/\mu}(a)$ by the formula

$$\psi_{\lambda/\mu}(a)=\sum_T\prod_{\alpha\in\lambda/\mu}a_{T(\alpha)},$$

summed over the dual hook λ/μ-tableaux T.

Theorem 3.20. *We have the expansion of the Schur function $s_\mu(x)$*

$$s_\mu(x)=\sum_\lambda(-1)^{m(\lambda/\mu)}\,\psi_{\lambda/\mu}(a)\,\widehat{s}_\lambda(x\,\|\,a),$$

summed over diagrams λ which contain μ and such that λ has d boxes on the main diagonal, where $m(\lambda/\mu)$ denotes the total number of boxes in the diagram λ/μ in rows $1,\dots,d$.

Proof. This is deduced from Proposition 3.19 and the formulas for the flagged Schur functions in [14, 8th Variation] exactly as in the proof of Theorem 3.17. □

Corollary 3.21. *For the expansion of the hook Schur function we have*

$$s_{(\alpha\,|\,\beta)}(x)=\sum_{p,\,q\geqslant 0}(-1)^p\,e_p(a_0,a_{-1},\dots,a_{-\alpha-p+1})\,e_q(a_1,a_2,\dots,a_{\beta+q})\,\widehat{s}_{(\alpha+p\,|\,\beta+q)}(x\,\|\,a).$$

Example 3.22. We have

$$s_{(1)}(x)=\sum_{p,\,q\geqslant 0}(-1)^p\,a_0\,a_{-1}\dots a_{-p+1}\,a_1\,a_2\dots a_q\,\widehat{s}_{(p\,|\,q)}(x\,\|\,a).$$

As with the flagged Schur functions $\varphi_{\lambda/\mu}(a)$, we have the following factorization formula

$$\psi_{\lambda/\mu}(a)=(-1)^{m(\lambda/\mu)}\,\psi_{(\lambda/\mu)_-}(a)\,\psi_{(\lambda'/\mu')_-}(a'), \tag{3.26}$$

where $(\lambda/\mu)_-$ denotes the part of the skew diagram λ/μ whose boxes lie in the rows $d+1,d+2,\dots$. An alternative tableau presentation for the polynomials $\psi_{(\lambda/\mu)_-}(a)$ is implied by the formulas [14, (8.2)], [25]. By (3.26), this also gives alternative formulas for the coefficients in the expansion of $s_\mu(x)$.

Corollary 3.23. *We have the tableau presentation*

$$\psi_{(\lambda/\mu)_-}(a)=\sum_T\prod_{\alpha\in(\lambda/\mu)_-}a_{T(\alpha)},$$

where the sum is taken over the $(\lambda/\mu)_-$-tableaux T whose entries in row i belong to the set $\{1,2,\dots,i-\lambda_i\}$ for $i=d+1,d+2,\dots,$ and the entries weakly increase along the rows and strictly increase down the columns.

Completing this section we note that the canonical comultiplication on the ring Λ is naturally extended to the comultiplication

$$\Delta:\widehat{\Lambda}(x\,\|\,a)\to\widehat{\Lambda}(x\,\|\,a)\otimes_{\mathbb{Q}[a]}\widehat{\Lambda}(x\,\|\,a)$$

defined on the generators by

$$\Delta\big(p_k(x)\big)=p_k(x)\otimes 1+1\otimes p_k(x).$$

Hence, Proposition 3.7 can be interpreted in terms of Δ as the following decomposition of the image of the dual Schur function

$$\Delta\big(\widehat{s}_\nu(x\,\|\,a)\big)=\sum_\mu\widehat{s}_{\nu/\mu}(x\,\|\,a)\otimes\widehat{s}_\mu(x\,\|\,a)=\sum_{\lambda,\,\mu}c_{\lambda\mu}^{\,\nu}(a)\,\widehat{s}_\lambda(x\,\|\,a)\otimes\widehat{s}_\mu(x\,\|\,a).$$

4. Dual Littlewood–Richardson polynomials

It was pointed out in [24, Remark 3.3] that the ring of supersymmetric functions $\Lambda(x/y\,\|\,a)$ is equipped with the comultiplication Δ such that

$$\Delta\big(p_k(x/y)\big) = p_k(x/y)\otimes 1 + 1\otimes p_k(x/y);$$

cf. [15, Chapter 1]. The isomorphism (1.6) allows us to transfer the comultiplication to the ring of double symmetric functions $\Lambda(x\,\|\,a)$ so that Δ is a $\mathbb{Q}[a]$-linear ring homomorphism

$$\Delta : \Lambda(x\,\|\,a) \to \Lambda(x\,\|\,a) \otimes_{\mathbb{Q}[a]} \Lambda(x\,\|\,a)$$

such that

$$\Delta\big(p_k(x\,\|\,a)\big) = p_k(x\,\|\,a)\otimes 1 + 1\otimes p_k(x\,\|\,a).$$

Definition 4.1. The *dual Littlewood–Richardson polynomials* $\widehat{c}_{\lambda\mu}^{\,\nu}(a)$ are defined as the coefficients in the expansion

$$\Delta\big(s_\nu(x\,\|\,a)\big) = \sum_{\lambda,\,\mu} \widehat{c}_{\lambda\mu}^{\,\nu}(a)\, s_\lambda(x\,\|\,a)\otimes s_\mu(x\,\|\,a).$$

Equivalently, these polynomials can be found from the decomposition

$$s_{\nu/\mu}(x\,\|\,a) = \sum_{\lambda} \widehat{c}_{\lambda\mu}^{\,\nu}(a)\, s_\lambda(x\,\|\,a). \tag{4.1}$$

□

In order to verify the equivalence of the definitions, note that by [24, Remark 3.3],

$$\Delta\big(s_\nu(x/y\,\|\,a)\big) = \sum_{\mu} s_{\nu/\mu}(x/y\,\|\,a)\otimes s_\mu(x/y\,\|\,a).$$

The claim now follows by the application of Proposition 2.5.

It is clear from the definition that the polynomial $\widehat{c}_{\lambda\mu}^{\,\nu}(a)$ is nonzero only if the inequality $|\nu| \geqslant |\lambda| + |\mu|$ holds. In this case it is a homogeneous polynomial in the variables a_i of degree $|\nu| - |\lambda| - |\mu|$. Moreover, in the particular case $|\nu| = |\lambda| + |\mu|$ the constant $\widehat{c}_{\lambda\mu}^{\,\nu}(a)$ equals $c_{\lambda\mu}^{\,\nu}$, the Littlewood–Richardson coefficient.

Recall that the Littlewood–Richardson polynomials $c_{\lambda\mu}^{\,\nu}(a)$ are defined by the expansion (1.2); see [18].

Corollary 4.2. *We have the following symmetry properties*

$$c_{\lambda\mu}^{\,\nu}(a) = c_{\lambda'\mu'}^{\,\nu'}(a') \qquad \textit{and} \qquad \widehat{c}_{\lambda\mu}^{\,\nu}(a) = \widehat{c}_{\lambda'\mu'}^{\,\nu'}(a').$$

Proof. By Proposition 2.5 and Definition 2.7, we have

$$s_\lambda(x/y\,\|\,a)\, s_\mu(x/y\,\|\,a) = \sum_{\nu} c_{\lambda\mu}^{\,\nu}(a)\, s_\nu(x/y\,\|\,a)$$

and

$$s_{\nu/\mu}(x/y\,\|\,a) = \sum_{\lambda} \widehat{c}_{\lambda\mu}^{\,\nu}(a)\, s_\lambda(x/y\,\|\,a).$$

The desired relations now follow from the symmetry property (2.16). □

We can now prove that the dual Littlewood–Richardson polynomials $\widehat{c}_{\lambda\mu}^{\,\nu}(a)$ introduced in Definition 4.1 describe the multiplication rule for the dual Schur functions.

Theorem 4.3. *We have the expansion*

$$\widehat{s}_\lambda(x\,\|\,a)\,\widehat{s}_\mu(x\,\|\,a) = \sum_\nu \widehat{c}_{\lambda\mu}^{\,\nu}(a)\,\widehat{s}_\nu(x\,\|\,a).$$

Proof. We argue as in the proof of the classical analogue of this result; see [15, Chapter 1]. Applying Corollary 3.2 for the families of variables $x = x' \cup x''$ and $y = y' \cup y''$ we get

$$\begin{aligned}\sum_{\nu\in\mathcal{P}} s_\nu(x/y\,\|\,a)\,\widehat{s}_\nu(z\,\|\,a) &= \prod_{i,\,j\geqslant 1}\frac{1+y'_i\,z_j}{1-x'_i\,z_j}\prod_{i,\,j\geqslant 1}\frac{1+y''_i\,z_j}{1-x''_i\,z_j}\\ &= \sum_{\lambda,\,\mu\in\mathcal{P}} s_\lambda(x'/y'\,\|\,a)\,\widehat{s}_\lambda(z\,\|\,a)\,s_\mu(x''/y''\,\|\,a)\,\widehat{s}_\mu(z\,\|\,a).\end{aligned}$$

On the other hand, an alternative expansion of the sum on the left hand side is obtained by using the relation

$$s_\nu(x/y\,\|\,a) = \sum_{\lambda\subseteq\nu} s_\lambda(x'/y'\,\|\,a)\,s_{\nu/\lambda}(x''/y''\,\|\,a) = \sum_{\lambda,\,\mu} s_\lambda(x'/y'\,\|\,a)\,\widehat{c}_{\lambda\mu}^{\,\nu}(a)\,s_\mu(x''/y''\,\|\,a),$$

implied by the combinatorial formula (2.15). Therefore, the required relation follows by comparing the two expansions. □

An explicit formula for the polynomials $\widehat{c}_{\lambda\mu}^{\,\nu}(a)$ is provided by the following corollary, where the $c_{\alpha\beta}^{\gamma}$ denote the classical Littlewood–Richardson coefficients defined by the decomposition of the product of the Schur functions

$$s_\alpha(x)\,s_\beta(x) = \sum_\gamma c_{\alpha\beta}^{\gamma}\,s_\gamma(x).$$

Corollary 4.4. *We have*

$$\widehat{c}_{\lambda\mu}^{\,\nu}(a) = \sum_{\alpha,\beta,\gamma}(-1)^{n(\alpha/\lambda)+n(\beta/\mu)+m(\nu/\gamma)}\,c_{\alpha\beta}^{\gamma}\,\varphi_{\alpha/\lambda}(a)\,\varphi_{\beta/\mu}(a)\,\psi_{\nu/\gamma}(a),$$

summed over diagrams α, β, γ. In particular, $\widehat{c}_{\lambda\mu}^{\,\nu}(a) = 0$ unless $\lambda\subseteq\nu$ and $\mu\subseteq\nu$.

Proof. The formula follows from Theorems 3.17, 3.20 and 4.3. The second statement is implied by the same property of the Littlewood–Richardson coefficients. □

Example 4.5. If $k\leqslant l$ and $k+l\leqslant m$ then

$$\widehat{c}_{(k)(l)}^{\,(m)}(a) = \sum_{r+s=m-k-l}(-1)^s\,h_r(a_0,a_{-1},\dots,a_{-k+1})\,e_s(a_{-l},a_{-l-1},\dots,a_{-m+2}).$$

In particular,

$$\widehat{c}_{(1)(l)}^{\,(m)}(a) = (a_0-a_{-l})(a_0-a_{-l-1})\dots(a_0-a_{-m+2}).$$

Applying Corollary 4.2, we also get

$$\widehat{c}_{(1^k)(1^l)}^{\,(1^m)}(a) = \sum_{r+s=m-k-l} (-1)^r\, h_r(a_1, a_2, \dots, a_k)\, e_s(a_{l+1}, a_{l+2}, \dots, a_{m-1})$$

and

$$\widehat{c}_{(1)(1^l)}^{\,(1^m)}(a) = (a_{l+1} - a_1)(a_{l+2} - a_1) \dots (a_{m-1} - a_1).$$

These relations provide explicit formulas for the images of the double elementary and complete symmetric functions $h_m(x \,\|\, a)$ and $e_m(x \,\|\, a)$ with respect to the comultiplication Δ. □

Another formula for the dual Littlewood–Richardson polynomials $\widehat{c}_{\lambda\mu}^{\,\nu}(a)$ can be obtained with the use of the decomposition (4.1). We will consider the skew double Schur function as the sequence of polynomials $s_{\nu/\mu}(x \,\|\, a)$ defined in (2.19). For a given skew diagram ν/μ consider the finite set of variables $x = (x_1, \dots, x_n)$, where $\nu'_j - \mu'_j \leqslant n$ for all j; that is, the number of boxes in each column of ν/μ does not exceed n. Since the skew double Schur functions are consistent with the evaluation homomorphisms (2.1), the polynomials $\widehat{c}_{\lambda\mu}^{\,\nu}(a)$ are determined by the decomposition (4.1), where x is understood as the above finite set of variables.

In order to formulate the result, introduce *ν/μ-supertableaux* T which are obtained by filling in the boxes of ν/μ with the symbols $1, 1', \dots, n, n'$ in such a way that in each row (resp. column) each primed index is to the left (resp. above) of each unprimed index; unprimed indices weakly decrease along the rows and strictly decrease down the columns; primed indices strictly increase along the rows and weakly increase down the columns.

Introduce the ordering on the set of boxes of a skew diagram by reading them by columns from left to right and from bottom to top in each column. We call this the *column order*. We shall write $\alpha \prec \beta$ if α (strictly) precedes β with respect to the column order.

Suppose that λ is a diagram. Given a sequence of diagrams R of the form

$$\varnothing = \rho^{(0)} \to \rho^{(1)} \to \dots \to \rho^{(l-1)} \to \rho^{(l)} = \lambda, \tag{4.2}$$

we let r_i denote the row number of the box added to the diagram $\rho^{(i-1)}$. The sequence $r_1 r_2 \dots r_l$ is called the *Yamanouchi symbol* of R. Construct the set $\mathcal{T}(\nu/\mu, R)$ of *barred* ν/μ-supertableaux T such that T contains boxes $\alpha_1, \dots, \alpha_l$ with

$$\alpha_1 \prec \dots \prec \alpha_l \qquad \text{and} \qquad T(\alpha_i) = r_i, \quad 1 \leqslant i \leqslant l,$$

where all entries r_i are unprimed and the boxes are listed in the column order which is restricted to the subtableau of T formed by the unprimed indices.

We will distinguish the entries in $\alpha_1, \dots, \alpha_l$ by barring each of them. So, an element of $\mathcal{T}(\nu/\mu, R)$ is a pair consisting of a ν/μ-supertableau and a chosen sequence of barred entries compatible with R. We shall keep the notation T for such a pair.

For each box α with $\alpha_i \prec \alpha \prec \alpha_{i+1}$, $0 \leqslant i \leqslant l$, which is occupied by an unprimed index, set $\rho(\alpha) = \rho^{(i)}$.

Theorem 4.6. *The dual Littlewood–Richardson polynomials can be given by*

$$\widehat{c}_{\lambda\mu}^{\,\nu}(a) = \sum_{R}\sum_{T} \prod_{\substack{\alpha\in\nu/\mu\\ T(\alpha)\ \textit{unprimed, unbarred}}} \big(a_{T(\alpha)-\rho(\alpha)_{T(\alpha)}} - a_{T(\alpha)-c(\alpha)}\big) \times \prod_{\substack{\alpha\in\nu/\mu\\ T(\alpha)\ \textit{primed}}} \big(a_{T(\alpha)-c(\alpha)} - a_{T(\alpha)}\big),$$

summed over sequences R of the form (4.2) *and barred supertableaux $T \in \mathcal{T}(\nu/\mu, R)$.*

Proof. Due to (2.19), we have

$$\widehat{c}_{\lambda\mu}^{\,\nu}(a) = \sum_{\mu\subseteq\rho\subseteq\lambda} \widetilde{c}_{\lambda\rho}^{\,\nu}(a)\, s_{\rho'/\mu'}(-a^{(n)}\,|\,{-a}),$$

where the polynomials $\widetilde{c}_{\lambda\rho}^{\,\nu}(a)$ are defined by the decomposition

$$\widetilde{s}_{\nu/\rho}(x\,\|\,a) = \sum_{\lambda} \widetilde{c}_{\lambda\rho}^{\,\nu}(a)\, s_{\lambda}(x\,\|\,a).$$

The desired formula is now implied by [18, Lemma 2.4] which gives the combinatorial expression for the coefficients $\widetilde{c}_{\lambda\rho}^{\,\nu}(a)$ and thus takes care of the unprimed part of T; the expression for the primed part is implied by (2.5). $\square$

Remark 4.7. Both the formulas for $\widehat{c}_{\lambda\mu}^{\,\nu}(a)$ provided by Corollary 4.4 and Theorem 4.6 involve some terms which cancel pairwise. It would be interesting to find a combinatorial presentation of the polynomials $\widehat{c}_{\lambda\mu}^{\,\nu}(a)$ analogous to [9], [10] or [18] and to understand their positivity properties. A possible way to find such a presentation could rely on the vanishing theorem of the supersymmetric Schur functions obtained in [24, Theorems 5.1 & 5.2]; see also [17, Theorem 4.4] for a similar result. $\square$

Example 4.8. In order to calculate the polynomial $\widehat{c}_{(1)\,(2)}^{\,(2^2)}(a)$, take $\lambda = (1)$, $\mu = (2)$, $\nu = (2^2)$ and $n = 1$. The barred supertableaux compatible with the sequence $\varnothing \to (1)$ are

$$\begin{array}{|c|c|}\hline & \\ \hline \overline{1} & 1 \\ \hline \end{array} \qquad\qquad \begin{array}{|c|c|}\hline & \\ \hline 1 & \overline{1} \\ \hline \end{array} \qquad\qquad \begin{array}{|c|c|}\hline & \\ \hline 1' & \overline{1} \\ \hline \end{array}$$

so that

$$\widehat{c}_{(1)\,(2)}^{\,(2^2)}(a) = a_0 - a_1 + a_1 - a_2 + a_2 - a_2 = a_0 - a_1.$$

Alternatively, we can take $\lambda = (2)$, $\mu = (1)$, $\nu = (2^2)$ and $n = 2$. The barred supertableaux compatible with the sequence $\varnothing \to (1) \to (2)$ are

$$\begin{array}{|c|c|}\hline & 2 \\ \hline \bar{1} & \bar{1} \\ \hline \end{array} \qquad\qquad \begin{array}{|c|c|}\hline & 1' \\ \hline \bar{1} & \bar{1} \\ \hline \end{array} \qquad\qquad \begin{array}{|c|c|}\hline & 2' \\ \hline \bar{1} & \bar{1} \\ \hline \end{array}$$

so that

$$\widehat{c}^{\,(2^2)}_{(1)\,(2)}(a) = a_2 - a_1 + a_0 - a_1 + a_1 - a_2 = a_0 - a_1.$$

This agrees with the previous calculation and the formula implied by Corollary 4.4.

Example 4.9. Theorem 4.6 gives formulas for the polynomials $\widehat{c}^{\,(m)}_{(k)(l)}(a)$ and $\widehat{c}^{\,(1^m)}_{(1^k)(1^l)}(a)$ in a different form as compared to Example 4.5. If $k + l \leqslant m$ then

$$\begin{aligned} \widehat{c}^{\,(m)}_{(k)(l)}(a) = \sum (a_0 - a_{-l})(a_0 - a_{-l-1}) \cdots (a_0 - a_{-l-i_1+1}) \\ \times (a_{-1} - a_{-l-i_1-1}) \cdots (a_{-1} - a_{-l-i_2+1}) \\ \times \ \cdots (a_{-k+1} - a_{-l-i_{k-1}-1}) \cdots (a_{-k+1} - a_{-m+2}) \end{aligned}$$

summed over the sets of indices $0 \leqslant i_1 < \cdots < i_{k-1} \leqslant m - l - 2$. A similar expression for $\widehat{c}^{\,(1^m)}_{(1^k)(1^l)}(a)$ follows by the application of Corollary 4.2.

5. Transition matrices

5.1. **Pairing between the double and dual symmetric functions.** We now prove alternative expansion formulas for the infinite product which occurs in the Cauchy formula (3.4). These formulas turn into the well known identities when a is specialized to the sequence of zeros; see [15, Chapter 1].

Let $\lambda = (\lambda_1, \dots, \lambda_l)$ be a partition and suppose that the length of λ does not exceed l. Using the notation (3.2), introduce the *dual monomial symmetric function* $\widehat{m}_\lambda(x \,\|\, a) \in \widehat{\Lambda}(x \,\|\, a)$ by the formula

$$\widehat{m}_\lambda(x \,\|\, a) = \sum_{\sigma} (x_{\sigma(1)}, a)^{\lambda_1} \, (x_{\sigma(2)}, a)^{\lambda_2} \dots (x_{\sigma(l)}, a)^{\lambda_l},$$

summed over permutations σ of the x_i which give distinct monomials.

For a partition $\lambda = (1^{m_1} \, 2^{m_2} \dots)$ set $z_\lambda = \prod_{i \geqslant 1} i^{m_i} \, m_i!$.

Proposition 5.1. *We have the expansions*

$$\prod_{i,\, j \geqslant 1} \frac{1 - a_i \, y_j}{1 - x_i \, y_j} = \sum_{\lambda \in \mathcal{P}} h_\lambda(x \,\|\, a) \, \widehat{m}_\lambda(y \,\|\, a) \tag{5.1}$$

and

$$\prod_{i,\, j \geqslant 1} \frac{1 - a_i \, y_j}{1 - x_i \, y_j} = \sum_{\lambda \in \mathcal{P}} z_\lambda^{-1} \, p_\lambda(x \,\|\, a) \, p_\lambda(y). \tag{5.2}$$

Proof. Let us set

$$H(t) = \prod_{i=1}^{\infty} \frac{1 - a_i t}{1 - x_i t}.$$

Then using (2.7) and arguing as in [15, Chapter 1], we can write

$$\prod_{i,\,j\geqslant 1}\frac{1-a_i\,y_j}{1-x_i\,y_j}=\prod_{j\geqslant 1}H(y_j)=\prod_{j\geqslant 1}\sum_{k=0}^{\infty}h_k(x\,\|\,a)\,(y_j,a)^k=\sum_{\lambda\in\mathcal{P}}h_\lambda(x\,\|\,a)\,\widehat{m}_\lambda(y\,\|\,a),$$

which proves (5.1). For the proof of (5.2) note that

$$\begin{aligned}\ln H(t)&=\sum_{i\geqslant 1}\Big(\ln(1-a_it)-\ln(1-x_it)\Big)\\&=\sum_{i\geqslant 1}\sum_{k\geqslant 1}\Big(\frac{x_i^k\,t^k}{k}-\frac{a_i^k\,t^k}{k}\Big)=\sum_{k\geqslant 1}\frac{p_k(x\,\|\,a)t^k}{k}.\end{aligned}$$

Hence,

$$H(t)=\sum_{\lambda\in\mathcal{P}}z_\lambda^{-1}\,p_\lambda(x\,\|\,a)\,t^{|\lambda|}.$$

Now apply this relation to the sets of variables x and a respectively replaced with the sets $\{x_iy_j\}$ and $\{a_iy_j\}$. Then $p_k(x\,\|\,a)$ is replaced by $p_\lambda(x\,\|\,a)\,p_\lambda(y)$, and (5.2) follows by putting $t=1$. □

Now define the $\mathbb{Q}[a]$-bilinear pairing between the rings $\Lambda(x\,\|\,a)$ and $\widehat{\Lambda}(y\,\|\,a)$,

$$\langle\ ,\ \rangle:\big(\Lambda(x\,\|\,a),\widehat{\Lambda}(y\,\|\,a)\big)\to\mathbb{Q}[a], \tag{5.3}$$

by setting

$$\big\langle h_\lambda(x\,\|\,a),\widehat{m}_\mu(y\,\|\,a)\big\rangle=\delta_{\lambda\mu}. \tag{5.4}$$

Clearly, $\langle u,\widehat{v}\rangle$ is a well-defined polynomial in a for any elements $u\in\Lambda(x\,\|\,a)$ and $\widehat{v}\in\widehat{\Lambda}(y\,\|\,a)$ which is determined from (5.4) by linearity.

The following is an analogue of the duality properties of the classical bases of the ring of symmetric functions; see [15, Chapter 1].

Proposition 5.2. *Let $\{u_\lambda(x\,\|\,a)\}$ and $\{\widehat{v}_\lambda(y\,\|\,a)\}$ be families of elements of rings $\Lambda(x\,\|\,a)$ and $\widehat{\Lambda}(y\,\|\,a)$, respectively, which are parameterized by all partitions. Suppose that for any $n\geqslant 0$ the highest degree components in x (resp., the lowest degree components in y) of the elements $u_\lambda(x\,\|\,a)$ (resp., $\widehat{v}_\lambda(y\,\|\,a)$) with $|\lambda|=n$ form a basis of the space of homogeneous symmetric functions in x (resp., y) of degree n. Then the following conditions are equivalent:*

$$\big\langle u_\lambda(x\,\|\,a),\widehat{v}_\mu(y\,\|\,a)\big\rangle=\delta_{\lambda\mu},\qquad \textit{for all}\quad\lambda,\mu; \tag{5.5}$$

$$\sum_{\lambda\in\mathcal{P}}u_\lambda(x\,\|\,a)\,\widehat{v}_\lambda(y\,\|\,a)=\prod_{i,\,j\geqslant 1}\frac{1-a_i\,y_j}{1-x_i\,y_j}. \tag{5.6}$$

Proof. We only need to slightly modify the respective argument of [15, Chapter 1]. Write

$$u_\lambda(x\,\|\,a)=\sum_\rho A_{\lambda\rho}(a)\,h_\rho(x\,\|\,a),\qquad \widehat{v}_\mu(y\,\|\,a)=\sum_\sigma B_{\mu\sigma}(a)\,\widehat{m}_\sigma(y\,\|\,a),$$

where the first sum is taken over partitions ρ with $|\rho|\leqslant|\lambda|$, while the second is taken over partitions σ with $|\sigma|\geqslant|\mu|$. Then

$$\big\langle u_\lambda(x\,\|\,a),\widehat{v}_\mu(y\,\|\,a)\big\rangle=\sum_\rho A_{\lambda\rho}(a)\,B_{\mu\rho}(a).$$

Hence, condition (5.5) is equivalent to

$$\sum_\rho A_{\lambda\rho}(a)\,B_{\mu\rho}(a)=\delta_{\lambda\mu}. \tag{5.7}$$

On the other hand, due to (5.1), (5.6) can be written as

$$\sum_{\lambda\in\mathcal{P}}u_\lambda(x\,\|\,a)\,\widehat{v}_\lambda(y\,\|\,a)=\sum_{\rho\in\mathcal{P}}h_\rho(x\,\|\,a)\,\widehat{m}_\rho(y\,\|\,a),$$

which is equivalent to

$$\sum_\lambda A_{\lambda\rho}(a)\,B_{\lambda\sigma}(a)=\delta_{\rho\sigma}.$$

This condition is easily verified to be equivalent to (5.7). □

Applying Theorem 3.1 and Proposition 5.1 we get the following corollary.

Corollary 5.3. *Under the pairing* (5.3) *we have*

$$\big\langle s_\lambda(x\,\|\,a),\widehat{s}_\mu(y\,\|\,a)\big\rangle=\delta_{\lambda\mu}\qquad\textit{and}\qquad\big\langle p_\lambda(x\,\|\,a),p_\mu(y)\big\rangle=\delta_{\lambda\mu}\,z_\lambda. \qquad \square$$

Thus, the symmetric functions $\widehat{s}_\lambda(y\,\|\,a)$ are dual to the double Schur functions $s_\lambda(x\,\|\,a)$ in sense of the pairing (5.3).

Using the isomorphism (1.6) and the pairing (5.3), we get another $\mathbb{Q}[a]$-bilinear pairing

$$\langle\ ,\ \rangle:\big(\Lambda(x/y\,\|\,a),\widehat{\Lambda}(z\,\|\,a)\big)\to\mathbb{Q}[a] \tag{5.8}$$

such that

$$\big\langle s_\lambda(x/y\,\|\,a),\widehat{s}_\mu(z\,\|\,a)\big\rangle=\delta_{\lambda\mu}. \tag{5.9}$$

Note that Proposition 5.2 can be easily reformulated for the pairing (5.8). In particular, the condition (5.6) is now replaced by

$$\sum_{\lambda\in\mathcal{P}}u_\lambda(x/y\,\|\,a)\,\widehat{v}_\lambda(z\,\|\,a)=\prod_{i,\,j\geqslant1}\frac{1+y_i\,z_j}{1-x_i\,z_j}. \tag{5.10}$$

This implies that

$$\big\langle s_\lambda(x/y),s_\mu(z)\big\rangle=\delta_{\lambda\mu}, \tag{5.11}$$

where $s_\lambda(x/y)$ denoted the ordinary supersymmetric Schur function which is obtained from $s_\lambda(x/y\,\|\,a)$ by the specialization $a_i = 0$. Together with Theorems 3.17 and 3.20, the relations (5.9) and (5.11) imply the following expansions for the supersymmetric Schur functions.

Corollary 5.4. *We have the decompositions*

$$s_\lambda(x/y\,\|\,a) = \sum_\mu (-1)^{m(\lambda/\mu)}\, \psi_{\lambda/\mu}(a)\, s_\mu(x/y),$$

summed over diagrams μ contained in λ and such that λ and μ have the same number of boxes on the main diagonal; and

$$s_\mu(x/y) = \sum_\lambda (-1)^{n(\lambda/\mu)}\, \varphi_{\lambda/\mu}(a)\, s_\lambda(x/y\,\|\,a),$$

summed over diagrams λ which contain μ and such that λ and μ have the same number of boxes on the main diagonal. □

Note that expressions for $\psi_{\lambda/\mu}(a)$ and $\varphi_{\lambda/\mu}(a)$ in terms of determinants as in Propositions 3.12 and 3.19 were given in [24]. Corollary 5.4 gives new tableau formulas for these coefficients. Moreover, under the specialization $a_i = -i + 1/2$ the supersymmetric Schur functions $s_\lambda(x/y\,\|\,a)$ turn into the *Frobenius–Schur functions* Fs_μ; see [24]. Hence, the transition coefficients between the Fs_μ and the Schur functions can be found as follows; cf. [24, Theorem 2.6].

Corollary 5.5. *We have the decompositions*

$$Fs_\lambda = \sum_\mu (-1)^{m(\lambda/\mu)}\, \psi_{\lambda/\mu}\, s_\mu(x/y)$$

and

$$s_\mu(x/y) = \sum_\lambda (-1)^{n(\lambda/\mu)}\, \varphi_{\lambda/\mu}\, Fs_\lambda,$$

where $\psi_{\lambda/\mu}$ and $\varphi_{\lambda/\mu}$ are the respective values of the polynomials $\psi_{\lambda/\mu}(a)$ and $\varphi_{\lambda/\mu}(a)$ at $a_i = -i + 1/2$, $i \in \mathbb{Z}$. □

Using the notation of Corollary 5.4 and applying the isomorphism (1.6) we get the respective expansion formulas involving the double Schur functions.

Corollary 5.6. *We have the decompositions*

$$s_\lambda(x\,\|\,a) = \sum_\mu (-1)^{m(\lambda/\mu)}\, \psi_{\lambda/\mu}(a)\, s_\mu(x)$$

and

$$s_\mu(x) = \sum_\lambda (-1)^{n(\lambda/\mu)}\, \varphi_{\lambda/\mu}(a)\, s_\lambda(x\,\|\,a).$$

□

Some other expressions for the coefficients in the expansions relating the double and ordinary Schur functions or polynomials can be found in [10], [11], [14], [18] and [19].

Let us now recall the isomorphism $\omega_a : \Lambda(x\|a) \to \Lambda(x\|a')$ and the involution $\widehat{\omega} : \widehat{\Lambda}(x\|a) \to \widehat{\Lambda}(x\|a)$; see (2.8) and (3.20). Since every polynomial $c(a) \in \mathbb{Q}[a]$ can be regarded as an element of $\mathbb{Q}[a']$, the ring $\widehat{\Lambda}(x\|a')$ can be naturally identified with $\widehat{\Lambda}(x\|a)$ via the map $c(a) \mapsto c'(a')$, where $c'(a') = c(a)$ as polynomials in the a_i, $i \in \mathbb{Z}$.

Proposition 5.7. *For any elements $u \in \Lambda(x\|a)$ and $\widehat{v} \in \widehat{\Lambda}(y\|a)$ we have*

$$\big\langle \omega_a u,\, \widehat{\omega}\,\widehat{v}\big\rangle' = \big\langle u, \widehat{v}\big\rangle,$$

where $\langle\ ,\ \rangle'$ denotes the pairing (5.3) *between $\Lambda(x\|a')$ and $\widehat{\Lambda}(y\|a) \simeq \widehat{\Lambda}(y\|a')$.*

Proof. It suffices to take $u = s_\lambda(x\|a)$ and $\widehat{v} = \widehat{s}_\mu(y\|a)$. Using (2.9) and (3.20), we get

$$\big\langle \omega_a s_\lambda(x\|a),\, \widehat{\omega}\,\widehat{s}_\mu(y\|a)\big\rangle' = \big\langle s_{\lambda'}(x\|a'),\, \widehat{s}_{\mu'}(y\|a')\big\rangle'.$$

By Corollary 5.3 this equals $\delta_{\lambda\mu}$, and hence coincides with $\langle s_\lambda(x\|a),\, \widehat{s}_\mu(y\|a)\rangle$. $\square$

Introduce the *dual forgotten symmetric functions* $\widehat{f}_\lambda(y\|a) \in \widehat{\Lambda}(y\|a)$ as the images of the dual monomial symmetric functions under the involution $\widehat{\omega}$, that is,

$$\widehat{f}_\lambda(y\|a) = \widehat{\omega}\,\widehat{m}_\lambda(y\|a'), \qquad \lambda \in \mathcal{P}.$$

Furthermore, for any partition λ define the *double monomial symmetric functions* $m_\lambda(x\|a) \in \Lambda(x\|a)$ and the *double forgotten symmetric functions* $f_\lambda(x\|a) \in \Lambda(x\|a)$ by the relations

$$\prod_{i,\,j\geqslant 1} \frac{1 - a_i\, y_j}{1 - x_i\, y_j} = \sum_{\lambda\in\mathcal{P}} m_\lambda(x\|a)\,\widehat{h}_\lambda(y\|a) \tag{5.12}$$

and

$$\prod_{i,\,j\geqslant 1} \frac{1 - a_i\, y_j}{1 - x_i\, y_j} = \sum_{\lambda\in\mathcal{P}} f_\lambda(x\|a)\,\widehat{e}_\lambda(y\|a). \tag{5.13}$$

Hence, by Proposition 5.2, under the pairing (5.3) we have

$$\big\langle m_\lambda(x\|a), \widehat{h}_\mu(y\|a)\big\rangle = \delta_{\lambda\mu} \qquad \text{and} \qquad \big\langle f_\lambda(x\|a), \widehat{e}_\mu(y\|a)\big\rangle = \delta_{\lambda\mu}. \tag{5.14}$$

Moreover, Proposition 2.3 and Corollary 5.7 imply

$$\omega_a : m_\lambda(x\|a) \mapsto h_\lambda(x\|a'), \qquad f_\lambda(x\|a) \mapsto e_\lambda(x\|a'), \qquad p_\lambda(x\|a) \mapsto \varepsilon_\lambda\, p_\lambda(x\|a'),$$

where $\varepsilon_\lambda = (-1)^{|\lambda|-\ell(\lambda)}$. To check the latter relation we need to recall that under the involution ω of the ring of symmetric functions we have $\omega : p_\lambda(y) \mapsto \varepsilon_\lambda\, p_\lambda(y)$; see [15, Chapter 1].

We can now obtain analogues of the decomposition of Corollary 3.3 for other families of symmetric functions.

Corollary 5.8. *We have the decompositions*

$$\prod_{i,\,j\geqslant 1}\frac{1+x_i\,y_j}{1+a_i\,y_j}=\sum_{\lambda\in\mathcal{P}} e_\lambda(x\,\|\,a)\,\widehat{m}_\lambda(y\,\|\,a'),$$

$$\prod_{i,\,j\geqslant 1}\frac{1+x_i\,y_j}{1+a_i\,y_j}=\sum_{\lambda\in\mathcal{P}} \varepsilon_\lambda\,z_\lambda^{-1}\,p_\lambda(x\,\|\,a)\,p_\lambda(y),$$

$$\prod_{i,\,j\geqslant 1}\frac{1+x_i\,y_j}{1+a_i\,y_j}=\sum_{\lambda\in\mathcal{P}} m_\lambda(x\,\|\,a)\,\widehat{e}_\lambda(y\,\|\,a').$$

Proof. The relations follow by the application of ω_a to the expansions (5.1), (5.2) and by the application of $\widehat{\omega}$ to (5.12). □

Note that relations of this kind involving the forgotten symmetric functions can be obtained in a similar way.

5.2. **Kostka-type and character polynomials.** The entries of the transition matrices between the classical bases of the ring of symmetric functions can be expressed in terms of the Kostka numbers $K_{\lambda\mu}$ and the values χ^λ_μ of the irreducible characters of the symmetric groups; see [15, Chapter 1]. By analogy with the classical case, introduce the *Kostka-type polynomials* $K_{\lambda\mu}(a)$ and the *character polynomials* $\chi^\lambda_\mu(a)$ as well as their dual counterparts $\widehat{K}_{\lambda\mu}(a)$ and $\widehat{\chi}^\lambda_\mu(a)$ by the respective expansions

$$s_\lambda(x\,\|\,a)=\sum_\mu K_{\lambda\mu}(a)\,m_\mu(x\,\|\,a),\qquad \widehat{s}_\lambda(y\,\|\,a)=\sum_\mu \widehat{K}_{\lambda\mu}(a)\,\widehat{m}_\mu(y\,\|\,a),$$

and

$$p_\mu(x\,\|\,a)=\sum_\lambda \chi^\lambda_\mu(a)\,s_\lambda(x\,\|\,a),\qquad p_\mu(y)=\sum_\lambda \widehat{\chi}^\lambda_\mu(a)\,\widehat{s}_\lambda(y\,\|\,a).$$

If $|\lambda|=|\mu|$, then

$$K_{\lambda\mu}(a)=\widehat{K}_{\lambda\mu}(a)=K_{\lambda\mu}\qquad\text{and}\qquad \chi^\lambda_\mu(a)=\widehat{\chi}^\lambda_\mu(a)=\chi^\lambda_\mu. \tag{5.15}$$

Moreover, $K_{\lambda\mu}(a)$ and $\widehat{\chi}^\lambda_\mu(a)$ are zero unless $|\lambda|\geqslant|\mu|$, while $\widehat{K}_{\lambda\mu}(a)$ and $\chi^\lambda_\mu(a)$ are zero unless $|\lambda|\leqslant|\mu|$.

Using the duality properties of the double and dual symmetric functions, we can get all other transition matrices in the same way as this is done in [15, Chapter 1]. In particular, we have the relations

$$h_\mu(x\,\|\,a)=\sum_\lambda \widehat{K}_{\lambda\mu}(a)\,s_\lambda(x\,\|\,a),\qquad \widehat{h}_\mu(y\,\|\,a)=\sum_\lambda K_{\lambda\mu}(a)\,\widehat{s}_\lambda(y\,\|\,a).$$

The Littlewood–Richardson polynomials $c_{\lambda\mu}^{\nu}(a)$ defined in (1.2) are Graham positive as they can be written as polynomials in the differences $a_i - a_j$, $i < j$, with positive integer coefficients; see [7]. Explicit positive formulas for $c_{\lambda\mu}^{\nu}(a)$ were found in [9], [10] and [18]. Using the fact that $h_k(x\,\|\,a)$ coincides with $s_{(k)}(x\,\|\,a)$, we come to the following expression for the polynomials $\widehat{K}_{\lambda\mu}(a)$:

$$\widehat{K}_{\lambda\mu}(a) = \sum_{\rho^{(1)},\dots,\rho^{(l-2)}} c_{(\mu_1)\,\rho^{(1)}}^{\lambda}(a)\, c_{(\mu_2)\,\rho^{(2)}}^{\rho^{(1)}}(a)\, \dots c_{(\mu_{l-2})\,\rho^{(l-2)}}^{\rho^{(l-3)}}(a)\, c_{(\mu_{l-1})\,(\mu_l)}^{\rho^{(l-2)}}(a),$$

summed over partitions $\rho^{(i)}$, where $\mu = (\mu_1, \dots \mu_l)$. In particular, each dual Kostka-type polynomial $\widehat{K}_{\lambda\mu}(a)$ is Graham positive. For an explicit tableau presentation of the polynomials $\widehat{K}_{\lambda\mu}(a)$ see [5].

Example 5.9. We have

$$\widehat{K}_{(3\,2)\,(3\,2\,1)}(a) = \sum_{\rho} c_{(3)\,\rho}^{(3\,2)}(a)\, c_{(2)\,(1)}^{\rho}(a) = c_{(3)\,(2)}^{(3\,2)}(a)\, c_{(2)\,(1)}^{(2)}(a) + c_{(3)\,(2\,1)}^{(3\,2)}(a)\, c_{(2)\,(1)}^{(2\,1)}(a).$$

Now, $c_{(3)\,(2)}^{(3\,2)}(a) = c_{(3)\,(2)}^{(3\,2)} = 1$ and $c_{(2)\,(1)}^{(2\,1)}(a) = c_{(2)\,(1)}^{(2\,1)} = 1$, while applying [18, Theorem 2.1] we get $c_{(2)\,(1)}^{(2)}(a) = a_{-1} - a_1$ and $c_{(3)\,(2\,1)}^{(3\,2)}(a) = a_{-2} - a_2$. Hence,

$$\widehat{K}_{(3\,2)\,(3\,2\,1)}(a) = a_{-2} + a_{-1} - a_1 - a_2.$$

□

The polynomials $K_{\lambda\mu}(a)$ can be calculated by the following procedure. Given a partition $\mu = (\mu_1, \dots, \mu_l)$, write each dual complete symmetric function $\widehat{h}_{\mu_i}(y\,\|\,a)$ as a series of the hook Schur functions with coefficients in $\mathbb{Q}[a]$ using Corollary 3.13. Then multiply the Schur functions using the classical Littlewood–Richardson rule. Finally, use Theorem 3.20 to represent each Schur function as a series of the dual Schur functions.

Example 5.10. By Example 3.14, $\widehat{h}_1(y\,\|\,a)^2$ equals

$$\big(s_{(1)}(y) + a_0\, s_{(2)}(y) - a_1\, s_{(1^2)}(y) + a_0^2\, s_{(3)}(y) - a_0\, a_1\, s_{(2\,1)}(y) + a_1^2\, s_{(1^3)}(y) + \dots\big)^2.$$

Hence, multiplying the Schur functions, we find that

$$\begin{aligned}\widehat{h}_1(y\,\|\,a)^2 &= s_{(2)}(y) + s_{(1^2)}(y) + 2\,a_0\, s_{(3)}(y) + 2\,(a_0 - a_1)\, s_{(2\,1)}(y) - 2\,a_1\, s_{(1^3)}(y)\\ &\quad + 3\,a_0^2\, s_{(4)}(y) + (3\,a_0^2 - 4\,a_0\, a_1)\, s_{(3\,1)}(y) + (a_0^2 + a_1^2 - 2\,a_0\, a_1)\, s_{(2^2)}(y)\\ &\quad + (3\,a_1^2 - 4\,a_0\, a_1)\, s_{(2\,1^2)}(y) + 3\,a_1^2\, s_{(1^4)}(y) + \cdots .\end{aligned}$$

Expanding now each Schur function with the use of Theorem 3.20 or Corollary 3.21, we come to

$$\begin{aligned}\widehat{h}_1(y\,\|\,a)^2 &= \widehat{s}_{(2)}(y\,\|\,a) + \widehat{s}_{(1^2)}(y\,\|\,a) + (a_0 - a_{-1})\,\widehat{s}_{(3)}(y\,\|\,a) + (a_0 - a_1)\,\widehat{s}_{(2\,1)}(y\,\|\,a)\\ &+ (a_2 - a_1)\,\widehat{s}_{(1^3)}(y\,\|\,a) + (a_0 - a_{-2})\,(a_0 - a_{-1})\,\widehat{s}_{(4)}(y\,\|\,a)\\ &+ (a_0 - a_1)\,(a_0 - a_{-1})\,\widehat{s}_{(3\,1)}(y\,\|\,a) + (a_0 - a_1)^2\,\widehat{s}_{(2^2)}(y\,\|\,a)\\ &+ (a_1 - a_0)\,(a_1 - a_2)\,\widehat{s}_{(2\,1^2)}(y\,\|\,a) + (a_1 - a_2)\,(a_1 - a_3)\,\widehat{s}_{(1^4)}(y\,\|\,a) + \cdots,\end{aligned}$$

thus calculating the first few polynomials $K_{\lambda\,(1^2)}(a)$. □

Example 5.11. Using Example 5.10, we can calculate the first few double monomial symmetric functions:

$$\begin{aligned}m_{(1)}(x\,\|\,a) &= s_{(1)}(x\,\|\,a), \qquad m_{(1^2)}(x\,\|\,a) = s_{(1^2)}(x\,\|\,a)\\ m_{(2)}(x\,\|\,a) &= s_{(2)}(x\,\|\,a) - s_{(1^2)}(x\,\|\,a)\\ m_{(1^3)}(x\,\|\,a) &= s_{(1^3)}(x\,\|\,a) + (a_1 - a_2)\,s_{(1^2)}(x\,\|\,a)\\ m_{(2\,1)}(x\,\|\,a) &= s_{(2\,1)}(x\,\|\,a) - 2\,s_{(1^3)}(x\,\|\,a) + (2a_2 - a_1 - a_0)\,s_{(1^2)}(x\,\|\,a)\\ m_{(3)}(x\,\|\,a) &= s_{(3)}(x\,\|\,a) - s_{(2\,1)}(x\,\|\,a) + s_{(1^3)}(x\,\|\,a) + (a_{-1} - a_0)\,s_{(1^2)}(x\,\|\,a).\end{aligned}$$ □

The following formula for the dual character polynomials is implied by Corollary 5.6.

Corollary 5.12. *We have*

$$\widehat{\chi}^{\lambda}_{\mu}(a) = \sum_{\rho} (-1)^{m(\lambda/\rho)}\,\chi^{\rho}_{\mu}\,\psi_{\lambda/\rho}(a),$$

summed over diagrams ρ with $|\rho| = |\mu|$. □

6. Interpolation formulas

6.1. **Rational expressions for the transition coefficients.** Applying Proposition 2.8, we can get expressions for the polynomials $\widehat{K}_{\lambda\mu}(a)$, $\chi^{\lambda}_{\mu}(a)$, $\widehat{c}^{\,\nu}_{\lambda\mu}(a)$ and $c^{\nu}_{\lambda\mu}(a)$ as rational functions in the variables a_i.

Proposition 6.1. *We have the expressions*

$$\widehat{K}_{\lambda\mu}(a) = \sum_{R} \sum_{k=0}^{l} \frac{h_{\mu}(a_{\rho^{(k)}}\,\|\,a)}{(|a_{\rho^{(k)}}| - |a_{\rho^{(0)}}|)\dots\wedge\dots(|a_{\rho^{(k)}}| - |a_{\rho^{(l)}}|)}, \tag{6.1}$$

$$\chi^{\lambda}_{\mu}(a) = \sum_{R} \sum_{k=0}^{l} \frac{p_{\mu}(a_{\rho^{(k)}}\,\|\,a)}{(|a_{\rho^{(k)}}| - |a_{\rho^{(0)}}|)\dots\wedge\dots(|a_{\rho^{(k)}}| - |a_{\rho^{(l)}}|)}, \tag{6.2}$$

$$\widehat{c}^{\,\nu}_{\lambda\mu}(a) = \sum_{R} \sum_{k=0}^{l} \frac{s_{\nu/\mu}(a_{\rho^{(k)}}\,\|\,a)}{(|a_{\rho^{(k)}}| - |a_{\rho^{(0)}}|)\dots\wedge\dots(|a_{\rho^{(k)}}| - |a_{\rho^{(l)}}|)}, \tag{6.3}$$

summed over all sequences of partitions R of the form

$$\varnothing = \rho^{(0)} \to \rho^{(1)} \to \cdots \to \rho^{(l-1)} \to \rho^{(l)} = \lambda.$$

Moreover,

$$c_{\lambda\mu}^{\nu}(a) = \sum_R \sum_{k=0}^{l} \frac{s_\lambda(a_{\rho^{(k)}} \,\|\, a)}{(|a_{\rho^{(k)}}| - |a_{\rho^{(0)}}|) \dots \wedge \dots (|a_{\rho^{(k)}}| - |a_{\rho^{(l)}}|)}, \tag{6.4}$$

summed over all sequences of partitions R of the form

$$\mu = \rho^{(0)} \to \rho^{(1)} \to \cdots \to \rho^{(l-1)} \to \rho^{(l)} = \nu. \qquad \square$$

The last formula was given in [19] for polynomials closely related to $c_{\lambda\mu}^{\nu}(a)$. Due to (5.15), the Kostka numbers $K_{\lambda\mu}$ and the values of the irreducible characters χ_μ^λ of the symmetric group can be found from (6.1) and (6.2).

Example 6.2. If $|\lambda| = n$, then

$$\chi_{(1^n)}^{\lambda} = \sum_R \sum_{k=1}^{n} \frac{(|a_{\rho^{(k)}}| - |a_{\rho^{(0)}}|)^{n-1}}{(|a_{\rho^{(k)}}| - |a_{\rho^{(1)}}|) \dots \wedge \dots (|a_{\rho^{(k)}}| - |a_{\rho^{(n)}}|)} = \sum_R 1,$$

which coincides with the number of standard λ-tableaux. $\square$

6.2. **Identities with dimensions of skew diagrams.** Specializing the variables by setting $a_i = -i + 1$ for all $i \in \mathbb{Z}$ in the expressions of Proposition 6.1, we obtain some identities for the Kostka numbers, the values of the irreducible characters and the Littlewood–Richardson coefficients involving dimensions of skew diagrams. Under this specialization, the double symmetric functions become the *shifted symmetric functions* of [22], so that some of the combinatorial results concerning the ring $\Lambda(x \,\|\, a)$ discussed above in the paper reduce to the respective results of [22] for the ring Λ^* of shifted symmetric functions; see also [8] for an alternative description of the ring Λ^*.

For any skew diagram θ denote by $\dim \theta$ the number of standard θ-tableaux (i.e., row and column strict) with entries in $\{1, 2, \dots, |\theta|\}$ and set

$$H_\theta = \frac{|\theta|!}{\dim \theta}.$$

If θ is normal (non-skew), then H_θ coincides with the product of the hooks of θ due to the hook formula. Under the specialization $a_i = -i + 1$, for any partition μ we have

$$a_\mu = (\mu_1, \mu_2 - 1, \dots).$$

The following formula for the values of the double Schur functions was proved in [22]: if $\mu \subseteq \nu$, then

$$s_\mu(a_\nu \,\|\, a) = \frac{H_\nu}{H_{\nu/\mu}}.$$

This formula can be deduced from Proposition 2.8 with the use of (2.20) which takes the form $s_\lambda(a_\lambda \| a) = H_\lambda$. Then (6.4) implies the identity for the Littlewood–Richardson coefficients $c_{\lambda\mu}^{\nu}$ which was proved in [19]:

$$c_{\lambda\mu}^{\nu} = \sum_{\rho} (-1)^{|\nu/\rho|} \frac{H_\rho}{H_{\nu/\rho}\, H_{\rho/\lambda}\, H_{\rho/\mu}},$$

summed over diagrams ρ which contain both λ and μ, and are contained in ν.

We also have the respective consequences of (6.1) and (6.2). For partitions $\mu = (1^{m_1} 2^{m_2} \dots r^{m_r})$ and $\rho = (\rho_1, \dots, \rho_l)$ set

$$\pi_\mu(\rho) = \prod_{k=1}^{r} \Big((1-\rho_1)^k + \dots + (l-\rho_l)^k - 1^k - \dots - l^k \Big)^{m_k}$$

and

$$\varkappa_\mu(\rho) = \prod_{k=1}^{r} \Big(\sum_{i_1 \geqslant \dots \geqslant i_k \geqslant 1} \rho_{i_1}(\rho_{i_2}-1)\dots(\rho_{i_k}-k+1) \Big)^{m_k}.$$

The the following formulas are obtained by specializing $a_i = i$ and $a_i = -i+1$, respectively, in (6.2) and (6.1).

Corollary 6.3. *Let λ and μ be partitions of n. Then*

$$\chi_\mu^\lambda = \sum_{\rho \subseteq \lambda} \frac{(-1)^{|\rho|}\, \pi_\mu(\rho)}{H_\rho\, H_{\lambda/\rho}}$$

and

$$K_{\lambda\mu} = \sum_{\rho \subseteq \lambda} \frac{(-1)^{|\lambda/\rho|}\, \varkappa_\mu(\rho)}{H_\rho\, H_{\lambda/\rho}}.$$

□

Example 6.4. Let $\lambda = (3\,2)$ and $\mu = (2\,1^3)$. Then

$$\pi_\mu(\rho) = -(\rho_1+\rho_2)^3\,(\rho_1^2+\rho_2^2-2\,\rho_1-4\,\rho_2),$$

and

$$H_{(3\,2)/(1)} = 24/5, \qquad H_{(3\,2)/(2)} = 2, \qquad H_{(3\,2)/(1^2)} = 3, \qquad H_{(3\,2)/(3)} = 2,$$
$$H_{(3\,2)/(2\,1)} = 1, \qquad H_{(3\,2)/(2\,2)} = 1, \qquad H_{(3\,2)/(3\,2)} = 1.$$

Hence,

$$\chi_{(2\,1^3)}^{(3\,2)} = -\frac{5}{24} + \frac{32}{6} + \frac{81}{12} - \frac{81}{3} + \frac{256}{12} - \frac{125}{24} = 1.$$

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