Biordered Sets and Fundamental Semigroups

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Abstract

Given any biordered set E, a natural construction yields a semigroup T_E that is always fundamental, in the sense that T_E possesses no nontrivial idempotent-separating congruence. In the case that $E = E(S)$ is the biordered set of idempotents of a semigroup S generated by regular elements, there is a natural representation of S by T_E, such that S becomes a biorder-preserving coextension of a fundamental and symmetric subsemigroup of T_E. If further S is regular then this yields the fundamental constructions of Nambooripad, Grillet and Hall, which in turn generalise the construction of Munn of a maximum fundamental inverse semigroup from its semilattice of idempotents.

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1 Introduction

The biordered set of a semigroup S is the partial algebra $E(S)$ of idempotents of S where multiplication is restricted to those pairs of idempotents such that one idempotent is a left or right zero for the other. Remarkably, such partial algebras have been characterised by biordered set axioms [21, 22, 4]. An abstract biordered set E (here abbreviated to boset) is a generalisation of the notion of a partially ordered set (or poset) and exploits two intertwined quasi-orders whose intersection forms a partial order (so that when the quasi-orders coincide, E becomes a poset). When each pair of elements has a non-empty sandwich set, the boset is called regular, and a regular poset is just a semilattice. Nambooripad [22] developed boset axioms in order to successfully generalise Munn's construction [19, 20] of fundamental inverse semigroups from semilattices to the class of regular semigroups, but using a construction based on regular bosets. Equivalent constructions were found also by Grillet [12, 13, 14], based on cross-connections, and Hall [15], based on idempotent-generated semigroups. In this paper we construct a semigroup T_E from an arbitrary (not necessarily regular) boset E, and prove a number of properties. In particular, we show that T_E is always fundamental, in the sense of having no nontrivial idempotent-separating congruences. In the case that $E = E(S)$ for some semigroup S generated by regular elements, we show that S is represented by a symmetric subsemigroup of T_E, so that S becomes a biorder-preserving coextension of a fundamental semigroup. When E is regular, the construction reduces to Nambooripad's fundamental regular semigroup on a regular boset. When E is a semilattice, the construction reduces to Munn's fundamental inverse semigroup on a semilattice.

2 Preliminaries

Basic terminology and facts about semigroups and Green's relations, as given in say [1], [17] or [16], will be assumed. Let S be a semigroup. Denote its set of idempotents by $E(S)$ and its set of regular elements by $\mathrm{Reg}(S)$. We say that a congruence σ on S is *idempotent-separating* if $e\,\sigma\,f$ implies $e = f$ for any $e, f \in E(S)$, and say that S is *fundamental* if S possesses no nontrivial idempotent-separating congruences. Call a congruence σ on S an $\mathcal{H}$-*congruence* if

$$(\forall e \in E(S))(\forall x \in S) \quad e\,\sigma\,x \implies H_e \leq H_x \, .$$

In particular, $\mathcal{H}$-congruences are idempotent-separating. Surprisingly (Theorem 2.2 below), the absence of nontrivial $\mathcal{H}$-congruences implies that S is fundamental.

Denote the full transformation semigroup on a set X by $\mathcal{T}_X$, and its dual by $\mathcal{T}_X^*$. In order to avoid confusion in correctly interpreting the order of composition of transformations, if σ is an element of $\mathcal{T}_X$, then we write σ^* when it is to be considered as an element of $\mathcal{T}_X^*$. We often adjoin a new symbol ∞ to X, which stands for 'undefined' and is always mapped to itself by any transformation.

Now let X be the set of all regular $\mathcal{L}$-classes of a semigroup S and Y the set of all regular $\mathcal{R}$-classes. Define a representation

$$\phi^\circ = (\rho^\circ, \lambda^{\circ *}) : S \to \mathcal{T}_{X\cup\{\infty\}} \times \mathcal{T}^*_{Y\cup\{\infty\}} \, , \quad s \mapsto (\rho_s^\circ, \lambda_s^{\circ *})$$

by

$$\rho_s^\circ : L_x \mapsto \begin{cases} L_{xs} & \text{if } x\mathcal{R}xs \\ \infty & \text{otherwise} \end{cases} \quad \text{and} \quad \lambda_s^\circ : R_x \mapsto \begin{cases} R_{sx} & \text{if } x\mathcal{L}sx \\ \infty & \text{otherwise.} \end{cases}$$

This representation first appeared in [6] and [7], though similar representations had been used earlier in the literature (see, for example, [15] or [12]). The kernel of this representation,

$$\begin{aligned} \mu \;=\; \mu(S) \;&=\; \ker \phi^\circ \\ &=\; \{\, (a,b) \in S \times S \;\mid\; (\forall x \in \mathrm{Reg}(S)) \quad (x\mathcal{R}xa \text{ or } x\mathcal{R}xb) \implies xa\mathcal{H}xb \\ &\qquad\qquad \text{and} \quad (x\mathcal{L}ax \text{ or } x\mathcal{L}bx) \implies ax\mathcal{H}bx \,\} \, , \end{aligned}$$

has been studied extensively by Edwards [7, 8, 9, 10, 11] and Easdown [2, 5].

Theorem 2.1 [8, 2] *For any semigroup S, the congruence μ is the maximum $\mathcal{H}$-congruence on S and $\mu(S/\mu)$ is the trivial congruence on S/μ.* $\square$

Theorem 2.2 [5] *Any semigroup S is fundamental if and only if μ is the trivial congruence on S.* $\square$

Let E be a set with a partial multiplication (denoted by juxtaposition) with domain D_E (allowing for the possibility that $D_E = E \times E$). We call E a *partial algebra.* Define relations $\succ\!\!\!\relbar$ and $\longrightarrow$ on E by

$$e \succ\!\!\!\relbar f \quad \text{if} \quad (e,f) \text{ is in } D_E \text{ and } ef = e$$

and

$$e \longrightarrow f \quad \text{if} \quad (f,e) \text{ is in } D_E \text{ and } fe = e\,.$$

Call $\succ\!\!\!\relbar$ the *left arrow* and $\longrightarrow$ the *right arrow* on E. As usual, put $\succ\!\!\!\relbar\!\!\!\prec \; = \; \succ\!\!\!\relbar \cap \relbar\!\!\!\prec$, $\longleftrightarrow \; = \; \longleftarrow \cap \longrightarrow$ and $\rightarrowtail \; = \; \succ\!\!\!\relbar \cap \longrightarrow$. Then (following Easdown's slight reformulation of Nambooripad's original axioms, to avoid sandwich sets), E is a *biordered set* (abbreviated to *boset*) when the following axioms are satisfied, where e, f, g are arbitrary elements of E.

(B1) The left and right arrows are preorders and

$$D_E = \; \succ\!\!\!\relbar \cup \longrightarrow \cup \longleftarrow \cup \relbar\!\!\!\prec \,.$$

(B2.1)

$$e \longrightarrow f \quad \text{implies} \quad \begin{tikzcd} e \arrow[r] \arrow[d, leftrightarrow] & f \\ ef \arrow[ur, tail] & \end{tikzcd}$$

(B2.1)*

$$e \succ\!\!\!\relbar f \quad \text{implies} \quad \begin{tikzcd} e \arrow[r, tail, no head] \arrow[d, tail, no head] \arrow[d, tail reversed, no head] & f \\ fe \arrow[ur, tail] & \end{tikzcd}$$

(B2.2)

$$\begin{tikzcd} & g & \\ e \arrow[ur] \arrow[rr, tail, no head] & & f \arrow[ul] \end{tikzcd} \quad \text{implies} \quad eg \succ\!\!\!\relbar fg$$

(B2.2)*

$$\begin{tikzcd} & g & \\ e \arrow[ur, tail, no head] \arrow[rr] & & f \arrow[ul, tail, no head] \end{tikzcd} \quad \text{implies} \quad ge \longrightarrow gf$$

(B3.1)

$$e \longrightarrow f \longrightarrow g \quad \text{implies} \quad (eg)f = ef$$

(B3.1)*

$$e \succ\!\!\!\longrightarrow\!\!\!\!\!\!\!- \; f \succ\!\!- \; g \quad \text{implies} \quad f(ge) = fe$$

(B3.2)

$$\begin{tikzcd} & g & \\ e \arrow[ur] \arrow[rr, tail, no head] & & f \arrow[ul] \end{tikzcd} \quad \text{implies} \quad (fe)g = (fg)(eg)$$

(B3.2)*

$$\begin{tikzcd} & g & \\ e \arrow[ur, tail, no head] \arrow[rr] & & f \arrow[ul, tail, no head] \end{tikzcd} \quad \text{implies} \quad g(ef) = (ge)(gf)$$

(B4)

$$\begin{tikzcd} & g & \\ e \arrow[ur] \arrow[d, leftrightarrow] & & f \arrow[ul] \arrow[d, leftrightarrow] \\ eg \arrow[rr, tail, no head] & & fg \end{tikzcd} \quad \text{implies} \quad \begin{tikzcd} & g & \\ e \arrow[ur] \arrow[dd, leftrightarrow] & & f \arrow[ul] \arrow[dd, leftrightarrow] \\ & e' \arrow[uu] \arrow[ur, tail, no head] \arrow[dl, leftrightarrow] & \\ e'g = eg \arrow[rr, tail, no head] & & fg \end{tikzcd} \quad \text{for some } e' \in E$$

(B4)*

$$\begin{tikzcd} & g & \\ e \arrow[ur, tail, no head] \arrow[d, tail, tail reversed] & & f \arrow[ul, tail, no head] \arrow[d, tail, tail reversed] \\ ge \arrow[rr] & & gf \end{tikzcd} \quad \text{implies} \quad \begin{tikzcd} & g & \\ e \arrow[ur, tail, no head] \arrow[dd, tail, tail reversed] & & f \arrow[ul, tail, no head] \arrow[dd, tail, tail reversed] \\ & e' \arrow[uu, tail, no head] \arrow[ur] \arrow[dl, tail, tail reversed] & \\ ge' = ge \arrow[rr] & & gf \end{tikzcd} \quad \text{for some } e' \in E$$

Frequently, below, we will invoke (B4) when the arrow $eg \succ\!\!- fg$ in the hypothesis is double, that is, $eg \succ\!\!-\!\!\prec fg$. In this case, it follows quickly from boset associativity (see the next paragraph), or more directly by Lemma 2 of [3], that the arrow $e' \succ\!\!- f$ also becomes double, that is, $e' \succ\!\!-\!\!\prec f$. We will use this fact without comment, and also the corresponding fact for the dual axiom (B4)*.

Let E be an arbitrary boset. Call a subset F of E a *subboset* if F becomes a boset with respect to the restriction of the partial multiplication of E. A *morphism* from E to a boset F is a mapping $\theta\colon E \to F$ such that $(e\theta, f\theta) \in D_F$ and $(ef)\theta = (e\theta)(f\theta)$ for all $(e, f) \in D_E$. A *boset representation* is a boset morphism into a semigroup (typically consisting of pairs of transformations and dual transformations). A boset *embedding* is an injective morphism whose inverse (with respect to the image) is a morphism. An *isomorphism* is a surjective embedding. By a theorem of Easdown [4], there exists a semigroup S such that E is isomorphic to the boset $E(S)$ of idempotents of S with domain of multiplication consisting of pairs (e, f) such that e is a left or right zero for f, or f is a left or right zero for e. Thus we may reassociate brackets arbitrarily in expressions involving boset elements, provided the expressions are defined in the boset.

We say that a congruence σ on a semigroup S is *biorder-preserving* if $E = E(S)$ is a subboset of $E(S/\sigma)$ and the natural map $\sigma^\natural : S \to S/\sigma$ induces a boset isomorphism from E onto its image $E\sigma$. In this case we say S is a *biorder-preserving coextension* of S/σ.

Theorem 2.3 [6] *The congruence μ is biorder-preserving for any semigroup S.* $\square$

This and the preceding theorems then yield immediately the following result.

Corollary 2.4 [5] *If S is any semigroup then S/μ is fundamental and $E(S)$ is a biordered subset of $E(S/\mu)$. Thus every semigroup is a biorder-preserving coextension of a fundamental semigroup.* $\square$

It is then natural to ask if there are 'synthetic' constructions on arbitrary bosets that produce candidates for such fundamental images, generalising the classical constructions of Munn, for semilattices, and of Hall, Grillet and Nambooripad, in the regular setting. Such a candidate, with many nice properties, is offered in the next section.

We finish the preliminaries by recalling the definition of a sandwich set. Let E be a boset and $e, f \in E$. Define

$$\mathcal{M}(e, f) \;=\; \{\, g \in E \mid e \longrightarrow\!\!\!\!< g \longrightarrow\!\!> f \,\}$$

and

$$\mathcal{S}(e,f) \ = \ \{\, g \in \mathcal{M}(e,f) \mid (\forall h \in \mathcal{M}(e,f)\ eh \longrightarrow\!\!> eg \ \text{ and } \ hf >\!\!\longrightarrow gf \,\}$$

Recall that $\mathcal{S}(e,f)$ is the *sandwich set* of the pair $(e,f) \in E \times E$ and E is *regular* if sandwich sets are always nonempty. Nambooripad [22] proved that a boset arises as the boset of a regular semigroup if and only if it is regular, and he used regular bosets as the basis for his generalisation of the Munn inverse semigroup. We recover Nambooripad's construction up to isomorphism in the final section. The following observation follows routinely from the axioms and is useful also in the next section.

Lemma 2.5 [22] *If $e >\!\!\longrightarrow\!\!< x \longrightarrow\!\!> f$ in a boset then $x \in \mathcal{S}(e,f)$.* $\square$

3 The construction for an arbitrary boset

Throughout this section, E denotes an arbitrary boset. Put $\mathcal{L} = \mathcal{L}_E = >\!\!\longrightarrow\!\!<$, $\mathcal{R} = \mathcal{R}_E = <\!\!\longrightarrow\!\!>$, which are equivalence relations on E, and put $\leq = \leq_E = >\!\!\longrightarrow\!\!>$, which is a partial order on E. For $e \in E$, denote its $\mathcal{L}$-class by L_e, its $\mathcal{R}$-class by R_e, and put $\omega(e) = \{f \in E \,|\, f \leq e\}$, called the *principal ideal generated by* e. Principal ideals of E are subbosets and we exploit boset isomorphisms between principal ideals in a construction below that generalises the Munn inverse semigroup on a semilattice. Throughout, whenever we write $\alpha : \omega(e) \twoheadrightarrow \omega(f)$, for $e, f \in E$, we mean that α is a principal ideal isomorphism. For such α, put

$$\phi_\alpha \ = \ (\rho_\alpha, \lambda^*_\alpha) \in \mathcal{T}_{E/\mathcal{L}\cup\{\infty\}} \times \mathcal{T}^*_{E/\mathcal{R}\cup\{\infty\}}$$

where

$$\rho_\alpha : L \mapsto \begin{cases} L_{(xe)\alpha} & \text{if } x \longrightarrow\!\!> e \text{ for some } x \in L \\ \infty & \text{otherwise,} \end{cases}$$

and

$$\lambda_\alpha : R \mapsto \begin{cases} R_{(fx)\alpha^{-1}} & \text{if } x >\!\!\longrightarrow f \text{ for some } x \in R \\ \infty & \text{otherwise.} \end{cases}$$

The identity mapping $1_{\omega(e)}$ is always a principal ideal automorphism for any $e \in E$, and we write ϕ_e for $\phi_{1_{\omega(e)}}$, in which case the definition here coincides with the definition of ϕ_e in [3], where it is proved that the restriction of ϕ to E is a boset isomorphism onto $E\phi$. Write

$$\mathcal{U} \;=\; \{\, (e,f) \in E \times E \mid \omega(e) \cong \omega(f)\,\},$$

and, for $(e,f) \in \mathcal{U}$,

$$T_{e,f} \;=\; \{\,\text{principal ideal isomorphisms}:\ \omega(e) \twoheadrightarrow \omega(f)\,\}\;.$$

Now define

$$T_E \;=\; \Big\langle \bigcup_{(e,f)\in\mathcal{U}} \{\phi_\alpha \mid \alpha \in T_{e,f}\,\} \Big\rangle$$

the subsemigroup of $\mathcal{T}_{E/\mathcal{L}\cup\{\infty\}} \times \mathcal{T}^*_{E/\mathcal{R}\cup\{\infty\}}$ generated by pairs $\phi_\alpha = (\rho_\alpha, \lambda^*_\alpha)$ as α ranges over all principal ideal isomorphisms of E.

We say that a subsemigroup S of a semigroup T is *full* if S contains $E(T)$. A subsemigroup S of T_E is called *symmetric* if S contains ϕ_e for each $e \in E$ and

$$S \;=\; \Big\langle \bigcup_{(e,f)\in\mathcal{U}} \{\phi_\alpha \mid \alpha \in T'_{e,f}\,\} \Big\rangle$$

where, for each $(e,f) \in \mathcal{U}$, we have $T'_{e,f} \;\subseteq\; T_{e,f}$ and

$$\alpha \in T'_{e,f} \;\Longrightarrow\; \alpha^{-1} \in T'_{f,e}\;.$$

For example, $\langle E\phi\rangle$ is symmetric, taking each $T'_{e,f}$ to be the singleton set $\{\phi_e\}$, if $e = f$, and empty otherwise. A symmetric subsemigroup of T_E need not be full, but will be full if $E(T_E) = E\phi$ (which occurs, for example, when E is regular, as explained in the final section).

In this section we prove the following two theorems after developing some technical lemmas.

Theorem 3.1 *Let E be any boset. Then any symmetric subsemigroup of T_E is fundamental and generated by regular elements. In particular, T_E is fundamental and generated by regular elements.*

Theorem 3.2 *Let S be any semigroup generated by regular elements and put $E = E(S)$. Then there exists a representation $\Phi : S \to T_E$ such that* $\ker\Phi$

is the maximum $\mathcal{H}$-congruence on S and $S\Phi$ is a symmetric subsemigroup of T_E.

Immediately then, by Theorem 2.3, we have the following result that suggests T_E might play a central role in the study of semigroups generated by regular elements.

Corollary 3.3 *Any semigroup generated by regular elements with boset of idempotents E is a biorder-preserving coextension of a symmetric subsemigroup of T_E.*

One might ask whether there is any chance of relaxing the hypothesis to obtain a representation theorem that includes semigroups that need not be generated by regular elements. The following simple example suggests that to successfully generalise these ideas to even wider classes of semigroups may involve embeddings.

Example 3.4 Let $F = \langle x \rangle$ be the free semigroup on a single generator x, and put $\overline{F} = \{\overline{x^i} \,|\, i \geq 1\}$, a set in a one-one correspondence with F. Now put $S = F \cup \overline{F}$ and extend the multiplication of F by the rules

$$\overline{x^i}\,\overline{x^j} \;=\; x^i\,\overline{x^j} \;=\; \overline{x^j} \quad \text{and} \quad \overline{x^i}\,x^j \;=\; \overline{x^{i+j}}$$

for any positive integers i and j. Then S is a semigroup that is not generated by regular elements and its boset $E = E(S) = \overline{F}$ is a single $\mathcal{R}$-class of mutual right zeros. Clearly $T_E \cong \overline{F}$, yet S is easily checked to be fundamental. Certainly S does not embed in T_E. However if we modify this example, just slightly, by allowing F to be the free monoid on a single generator, then T_E then expands to include a group of units which is the full symmetric group on a countably infinite set (the automorphism group of an infinite right zero boset with identity adjoined), and then the new S embeds easily (and of course contains the old S).

Lemma 3.5 *Suppose $\alpha : \omega(e) \twoheadrightarrow \omega(f)$ and $\beta : \omega(g) \twoheadrightarrow \omega(h)$ are principal ideal isomorphisms and $i \in \mathcal{S}(f, g)$. Then $\phi_\alpha \phi_\beta = \phi_\gamma$ for the principal ideal isomorphism*

$$\gamma : \omega[(fi)\alpha^{-1}] \twoheadrightarrow \omega[(ig)\beta]\,, \qquad x \mapsto [[i(x\alpha)](ig)]\beta\;.$$

Proof. Put $j = (fi)\alpha^{-1}$ and $k = (ig)\beta$. Certainly γ is a principal ideal isomorphism, being the composition of α restricted to $\omega(j)$, left translation by i, right translation by ig and β restricted to $\omega(ig)$, which are respective isomorphisms:

$$\omega(j) \twoheadrightarrow \omega(fi) \twoheadrightarrow \omega(i) \twoheadrightarrow \omega(ig) \twoheadrightarrow \omega(k) \ .$$

By duality, it suffices to verify that $\rho_\alpha \rho_\beta = \rho_\gamma$. Suppose first that $L\rho_\alpha\rho_\beta \neq \infty$, so $x \longrightarrow e$ for some $x \in L$, and $y \longrightarrow g$ for some $y \in L_{(xe)\alpha}$. Hence $fy \longrightarrow fi$, since $i \in \mathcal{S}(f,g)$, so, by (B4)*, there exists y' such that $y \succ\!\!\!-\!\!\!\prec y' \longrightarrow i$ and $fy' = fy$. Put $x' = (fy)\alpha^{-1}$, so $x' \succ\!\!\!-\!\!\!\prec xe$, since α^{-1} is a morphism. By (B4), there exists x'' such that $x' = x''e \longleftrightarrow x'' \succ\!\!\!-\!\!\!\prec x$. Note further that $x' \longrightarrow j$ since $x'\alpha \longrightarrow j\alpha = fi$, and also $y'g \succ\!\!- ig$ since $i \in \mathcal{S}(f,g)$. But $L\rho_\alpha\rho_\beta = L_{(yg)\beta}$ and $L\rho_\gamma = L_{(x''j)\gamma}$, where

$$(x''j)\gamma \ = \ [(i[(x''j)\alpha])(ig)]\beta \ .$$

By boset associativity, and since α is a morphism,

$$(x''j)\alpha \ = \ ((x''e)j)\alpha \ = \ [(x''e)\alpha][j\alpha] \ = \ (fy)(fi) \ = \ (fy')(fi) \ = \ f(y'i) \ ,$$

so that

$$\begin{aligned}(x''j)\gamma \ &= \ [(i[f(y'i)])(ig)]\beta \ = \ [(i(y'i))(ig)]\beta \\ &= \ [(y'i)(ig)]\beta \ = \ [y'(ig)]\beta \ = \ [(y'g)(ig)]\beta \ = \ (y'g)\beta \ ,\end{aligned}$$

yielding $L\rho_\gamma = L_{(y'g)\beta} = L_{(yg)\beta} = L\rho_\alpha\rho_\beta$.

Suppose now $L\rho_\alpha\rho_\beta = \infty$. If $L\rho_\gamma \neq \infty$ then $x \longrightarrow j$ for some $x \in L$, so that, by (B4)*, there exists y such that $fy = (xe)\alpha \succ\!\!\!-\!\!\!\prec y \longrightarrow i \longrightarrow g$, so that $L\rho_\alpha\rho_\beta = L_{(yg)\beta} \neq \infty$, which is a contradiction. Hence $L\rho_\gamma = \infty$ This completes the proof that $\rho_\alpha\rho_\beta = \rho_\gamma$. □

Lemma 3.6 *Suppose $\alpha : \omega(e) \twoheadrightarrow \omega(f)$ is a principal ideal isomorphism. Then*

$$x \longrightarrow e \quad \Longrightarrow \quad \phi_{\alpha^{-1}}\phi_x\phi_\alpha \ = \ \phi_{(xe)\alpha}$$

and dually

$$x \succ\!\!- f \quad \Longrightarrow \quad \phi_\alpha\phi_x\phi_{\alpha^{-1}} \ = \ \phi_{(fx)\alpha^{-1}} \ .$$

Proof. Throughout this proof we use boset associativity and the fact that α is a boset morphism without comment. Suppose that $x \longrightarrow e$. We first show $\rho_{\alpha^{-1}}\rho_x\rho_\alpha = \rho_{(xe)\alpha}$. If $L\rho_{(xe)\alpha} \neq \infty$ then $y \longrightarrow (xe)\alpha$ for some $y \in L$, so that

$$L\rho_{\alpha^{-1}}\rho_x\rho_\alpha \;=\; L_{[([(yf)\alpha^{-1}]x)e]\alpha} \;=\; L_{(yf)[(xe)\alpha]} \;=\; L_{y[(xe)\alpha]} \;=\; L\rho_{(xe)\alpha}\ .$$

If, on the other hand, $L\rho_{\alpha^{-1}}\rho_x\rho_\alpha \neq \infty$ then $y \longrightarrow f$ for some $y \in L$ and $z \longrightarrow x$ for some $z \in L_{(yf)\alpha^{-1}}$, so that, by (B4), there exists z' such that $z'f = z\alpha \longleftrightarrow z' \succ\!\!\prec y$, whence $z' \in L_y = L$ and $z' \longleftrightarrow z\alpha \longrightarrow (xe)\alpha$, yielding $z' \longrightarrow (xe)\alpha$, so that $L\rho_{(xe)\alpha} \neq \infty$. This verifies that $\rho_{\alpha^{-1}}\rho_x\rho_\alpha = \rho_{(xe)\alpha}$.

Now we show that $\lambda^*_{\alpha^{-1}}\lambda^*_x\lambda^*_\alpha = \lambda^*_{(xe)\alpha}$, that is, $\lambda_\alpha\lambda_x\lambda_{\alpha^{-1}} = \lambda_{(xe)\alpha}$. If $R\lambda_{(xe)\alpha} \neq \infty$ then $y \succ\!\!- (xe)\alpha$ for some $y \in R$, so, by (B4), there exists z such that $ze = (fy)\alpha^{-1} \longleftrightarrow z \succ\!\!- x$, so that

$$\begin{aligned} R\lambda_\alpha\lambda_x\lambda_{\alpha^{-1}} \;&=\; R_{[(xz)e]\alpha} \;=\; R_{[(xe)\alpha][(ze)\alpha]} \\ &=\; R_{[(xe)\alpha](fy)} \;=\; R_{[(xe)\alpha]y} \;=\; R\lambda_{(xe)\alpha}\ . \end{aligned}$$

If, on the other hand, $R\lambda_\alpha\lambda_x\lambda_{\alpha^{-1}} \neq \infty$ then $y \succ\!\!- f$ for some $y \in R$ and $z \succ\!\!- x$ for some $z \in R_{(fy)\alpha^{-1}}$, so that, by (B4)*, there exists y' such that $fy' = (ze)\alpha \succ\!\!\prec y' \longleftrightarrow y$, whence $y' \in R_y = R$ and $y' \succ\!\!\prec (ze)\alpha \succ\!\!- (xe)\alpha$, yielding $y' \succ\!\!- (xe)\alpha$, so that $R\lambda_{(xe)\alpha} \neq \infty$. This completes the proof that $\phi_{\alpha^{-1}}\phi_x\phi_\alpha = \phi_{(xe)\alpha}$. $\square$

Lemma 3.7 *Let $x \in E$. Suppose that S is a subsemigroup of T_E containing $E\phi$ and $\phi_{\alpha_1}, \phi_{\alpha_1^{-1}}, \ldots, \phi_{\alpha_n}, \phi_{\alpha_n^{-1}}$ where each $\alpha_i : \omega(e_i) \twoheadrightarrow \omega(e_i\alpha_i)$ is a principal ideal isomorphism. Then $\phi_x\ \mathcal{R}_S\ \phi_x\phi_{\alpha_1}\ldots\phi_{\alpha_n}$ if and only if there exists a sequence $x_1, \ldots, x_n \in E$ such that*

$$\begin{tikzcd}
x \arrow[d, tail, tail reversed] & e_1 & e_1\alpha_1 & & & & & & \\
x_1 \arrow[r, leftrightarrow] \arrow[ur] & x_1e_1 \arrow[u, tail] & (x_1e_1)\alpha_1 \arrow[u, tail] \arrow[d, tail, tail reversed] & e_2 & e_2\alpha_2 & & & & \\
 & & x_2 \arrow[r, leftrightarrow] \arrow[ur] & x_2e_2 \arrow[u, tail] & (x_2e_2)\alpha_2 \arrow[u, tail] \arrow[d, tail, tail reversed] & & & & \\
 & & & & \vdots \arrow[d, tail, tail reversed] & e_i & e_i\alpha_i & & \\
 & & & & x_i \arrow[r, leftrightarrow] \arrow[ur] & x_ie_i \arrow[u, tail] & (x_ie_i)\alpha_i \arrow[u, tail] \arrow[d, tail, tail reversed] & & \\
 & & & & & & \vdots \arrow[d, tail, tail reversed] & e_n & e_n\alpha_n \\
 & & & & & & x_n \arrow[r, leftrightarrow] \arrow[ur] & x_ne_n \arrow[u, tail] & (x_ne_n)\alpha_n \arrow[u, tail]
\end{tikzcd}$$

in which case there is a principal ideal isomorphism $\beta : \omega(x) \to \omega[(x_ne_n)\alpha_n]$ *such that* $\phi_x\phi_{\alpha_1}\dots\phi_{\alpha_n} = \phi_\beta$ *.*

Proof. If $\phi_x \; \mathcal{R}_S \; \phi_x\phi_{\alpha_1}\dots\phi_{\alpha_n}$ then $\phi_x = \phi_x\phi_{\alpha_1}\dots\phi_{\alpha_n}\phi_{\beta_1}\dots\phi_{\beta_m}$ for some $\phi_{\beta_1},\dots,\phi_{\beta_m}$ in T_E, so that $\rho_x = \rho_x\rho_{\alpha_1}\dots\rho_{\alpha_n}\rho_{\beta_1}\dots\rho_{\beta_m}$, yielding, in particular,

$$L_x\rho_x\rho_{\alpha_1}\dots\rho_{\alpha_n}\rho_{\beta_1}\dots\rho_{\beta_m} \;=\; L_x\rho_x \;=\; L_x \;\neq\; \infty \;,$$

so that, by definition of ρ, there exist $x_1,\dots,x_n$ with the desired property.

Suppose conversely that $x_1,\dots,x_n$ exist with the desired property. We obtain a new sequence $x_1',\dots,x_n'$ inductively as follows. Put $x_n' = x_n$ and suppose x_{i+1}' has been defined such that

$$x_{i+1}' \;\succ\!\!-\!\!\prec\; (x_ie_i)\alpha_i \;\succ\!\!\longrightarrow\; e_i\alpha_i \;.$$

By axiom (B4), there exists x_i' such that

$$x_i'e_i = [(e_i\alpha_i)x_{i+1}']\alpha_i^{-1} \;\longleftrightarrow\; x_i' \;\succ\!\!-\!\!\prec\; x_i \;\succ\!\!-\!\!\prec\; (x_{i-1}e_{i-1})\alpha_{i-1} \;,$$

so that

$$x_i' \;\succ\!\!-\!\!\prec\; (x_{i-1}e_{i-1})\alpha_{i-1} \;\succ\!\!\longrightarrow\; e_{i-1}\alpha_{i-1} \;.$$

By induction we obtain a sequence $x_1',\dots,x_n'$ such that the following diagram holds:

$$
\begin{tikzcd}[column sep=small]
x & e_1 & e_1\alpha_1 & & & & \\
x'_1 \arrow[u, >-<] \arrow[ur] \arrow[r, leftrightarrow] & x'_1e_1 \arrow[u] & (x'_1e_1)\alpha_1 = (e_1\alpha_1)x'_2 \arrow[u] & e_2 & e_2\alpha_2 & & \\
 & & x'_2 \arrow[u, >-<] \arrow[ur] \arrow[r, leftrightarrow] & x'_2e_2 \arrow[u] & (x'_2e_2)\alpha_2 = (e_2\alpha_2)x'_3 \arrow[u] & e_3 & \cdots \\
 & & & & x'_3 \arrow[u, >-<] \arrow[ur] \arrow[r, leftrightarrow] & x'_3e_3 \arrow[u] & \cdots
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=small]
\cdots & e_{n-1}\alpha_{n-1} & & \\
\cdots & (x'_{n-1}e_{n-1})\alpha_2 = (e_{n-1}\alpha_{n-1})x'_n \arrow[u] & e_n & e_n\alpha_n \\
 & x_n = x'_n \arrow[u, >-<] \arrow[ur] \arrow[r, leftrightarrow] & x_ne_n \arrow[u] & (x_ne_n)\alpha_n \arrow[u]
\end{tikzcd}
$$

Put $x'_{n+1} = (x_ne_n)\alpha_n$. We verify by induction, for $i = 0$ to n, that

$$\phi_x \;=\; (\phi_x\phi_{\alpha_1}\ldots\phi_{\alpha_i})(\phi_{x'_{i+1}}\phi_{\alpha_i^{-1}}\phi_{x'_i}\phi_{\alpha_{i-1}^{-1}}\ldots\phi_{x'_2}\phi_{\alpha_1^{-1}}\phi_{x'_1})\ ,$$

where we interpret this for $i = 0$ as saying $\phi_x\phi_{x'_1} = \phi_x$, which holds since $x \succ\!\!-\!\!\prec x'_1$ and ϕ is a boset morphism, which starts the induction. The inductive step follows from the previous lemma, the fact that $x'_i \longleftrightarrow x'_ie_i = [(e_i\alpha_i)x'_{i+1}]\alpha_i^{-1}$, and an inductive hypothesis:

$$
\begin{aligned}
&\phi_x\phi_{\alpha_1}\ldots(\phi_{\alpha_i}\phi_{x'_{i+1}}\phi_{\alpha_i^{-1}})\phi_{x'_i}\phi_{\alpha_{i-1}^{-1}}\ldots\phi_{x'_2}\phi_{\alpha_1^{-1}}\phi_{x'_1} \\
&\qquad = \;\phi_x\phi_{\alpha_1}\ldots\phi_{\alpha_{i-1}}\phi_{[(e_i\alpha_i)x'_{i+1}]\alpha_i^{-1}}\phi_{x'_i}\phi_{\alpha_{i-1}^{-1}}\ldots\phi_{x'_2}\phi_{\alpha_1^{-1}}\phi_{x'_1} \\
&\qquad = \;\phi_x\phi_{\alpha_1}\ldots\phi_{\alpha_{i-1}}\phi_{x'_i}\phi_{\alpha_{i-1}^{-1}}\ldots\phi_{x'_2}\phi_{\alpha_1^{-1}}\phi_{x'_1} \;=\; \phi_x\ .
\end{aligned}
$$

The case $i = n$ verifies that $\phi_x\ \mathcal{R}_S\ \phi_x\phi_{\alpha_1}\ldots\phi_{\alpha_n}$. The last claim of the lemma follows by induction from Lemma 3.5, noting, by Lemma 2.5, that $x_1 \in \mathcal{S}(x, e_1)$ and $x_i \in \mathcal{S}((x_{i-1}e_{i-1})\alpha_{i-1}, e_i)$ for $i = 2$ to n. $\square$

Lemma 3.8 *Suppose $\alpha : \omega(e) \twoheadrightarrow \omega(f)$ and $\beta : \omega(g) \twoheadrightarrow \omega(h)$ are principal ideal isomorphisms. Then $\phi_\alpha\ \mathcal{H}_{T_E}\ \phi_\beta$ if and only if $e \longleftrightarrow g$ and $f \succ\!\!-\!\!\prec h$.*

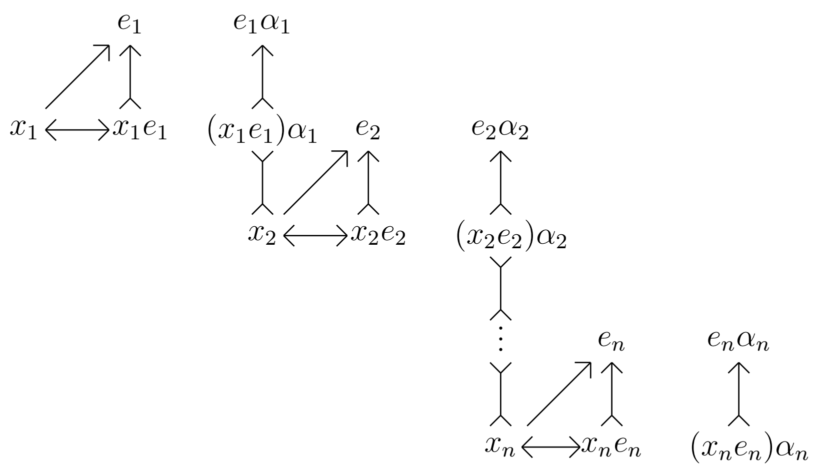
Proof. If $\phi_\alpha \, \mathcal{H}_{T_E} \, \phi_\beta$ then $\phi_e = \phi_\alpha \phi_{\alpha^{-1}} \, \mathcal{R}_{T_E} \, \phi_\beta \phi_{\beta^{-1}} = \phi_g$, so that $\phi_e \longleftrightarrow \phi_g$, yielding $e \longleftrightarrow g$, since ϕ^{-1} is a boset morphism, and dually $f \succ\!\!\!-\!\!\!\prec h$. Conversely if $e \longleftrightarrow g$ and $f \succ\!\!\!-\!\!\!\prec h$ then $\phi_\beta \phi_{\beta^{-1}} \phi_e \phi_\alpha = \phi_g \phi_e \phi_\alpha = \phi_e \phi_\alpha = \phi_\alpha$, since ϕ is a boset morphism, and similarly $\phi_\alpha \phi_{\alpha^{-1}} \phi_g \phi_\beta = \phi_\beta$, yielding $\phi_\alpha \, \mathcal{R}_{T_E} \, \phi_\beta$, and dually $\phi_\alpha \, \mathcal{L}_{T_E} \, \phi_\beta$. $\square$

Proof of Theorem 3.1. Let S be a symmetric subsemigroup of T_E. A typical generator ϕ_α of S is regular since we may assume also that $\phi_{\alpha^{-1}} \in S$ and clearly $\phi_\alpha \phi_{\alpha^{-1}} \phi_\alpha = \phi_\alpha$. It remains then to prove that S is fundamental. By Theorem 2.2, it suffices to show that the kernel of the representation ϕ° of S is trivial. Suppose then that

$$\phi^\circ_{\phi_{\alpha_1} \dots \phi_{\alpha_n}} = \phi^\circ_{\phi_{\beta_1} \dots \phi_{\beta_m}}$$

for some typical generators $\phi_{\alpha_1}, \dots, \phi_{\alpha_n}, \phi_{\beta_1}, \dots, \phi_{\beta_m}$ of S coming from principal ideal isomorphisms $\alpha_i : \omega(e_i) \twoheadrightarrow \omega(e_i \alpha_i)$ for $i = 1$ to n and $\beta_j : \omega(f_j) \twoheadrightarrow \omega(f_j \beta_j)$ for $j = 1$ to m. Since S is symmetric we may assume $\phi_{{\alpha_i}^{-1}}, \phi_{{\beta_j}^{-1}} \in S$ for each i, j.

Our task is to show that $\phi_{\alpha_1} \dots \phi_{\alpha_n} = \phi_{\beta_1} \dots \phi_{\beta_m}$. By duality it suffices to show $\rho_{\alpha_1} \dots \rho_{\alpha_n} = \rho_{\beta_1} \dots \rho_{\beta_m}$. Suppose then that $L\rho_{\alpha_1} \dots \rho_{\alpha_n} \neq \infty$, so there exists $x_1 \in L$ and $x_2, \dots, x_n \in E$ such that the following diagram holds:



By Lemma 3.7, since S contains $E\phi$ and $\phi_{\alpha_1}, \phi_{{\alpha_1}^{-1}}, \dots, \phi_{\alpha_n}, \phi_{{\alpha_n}^{-1}}$, we have that

$$\phi_{x_1} \, \mathcal{R}_S \, \phi_{x_1} \phi_{\alpha_1} \dots \phi_{\alpha_n} .$$

But $\rho^{\circ}_{\phi_{\alpha_1}\dots\phi_{\alpha_n}} = \rho^{\circ}_{\phi_{\beta_1}\dots\phi_{\beta_m}}$, so, by definition of ρ°,

$$\phi_{x_1} \mathcal{R}_S \phi_{x_1}\phi_{\beta_1}\dots\phi_{\beta_m} \qquad \text{and} \qquad \phi_{x_1}\phi_{\alpha_1}\dots\phi_{\alpha_n} \mathcal{L}_S \phi_{x_1}\phi_{\beta_1}\dots\phi_{\beta_m} .$$

By Lemma 3.7 again, there exist $y_1, \dots y_m$ such that the following diagram holds:

$$\begin{tikzcd}
x_1 \arrow[d, tail-tail] & f_1 & f_1\beta_1 & & & & \\
y_1 \arrow[r, leftrightarrow] \arrow[ur] & y_1f_1 \arrow[u, tail] & (y_1f_1)\beta_1 \arrow[u, tail] \arrow[d, tail-tail] & f_2 & f_2\beta_2 & & \\
& & y_2 \arrow[r, leftrightarrow] \arrow[ur] & y_2f_2 \arrow[u, tail] & (y_2f_2)\beta_2 \arrow[u, tail] \arrow[d, tail-tail] & & \\
& & & & \vdots \arrow[d, tail-tail] & f_m & f_m\beta_m \\
& & & & y_m \arrow[r, leftrightarrow] \arrow[ur] & y_mf_m \arrow[u, tail] & (y_mf_m)\beta_m \arrow[u, tail]
\end{tikzcd}$$

Also, by Lemma 3.7,

$$\phi_{x_1}\phi_{\alpha_1}\dots\phi_{\alpha_n} = \phi_{\alpha} \qquad \text{and} \qquad \phi_{x_1}\phi_{\beta_1}\dots\phi_{\beta_m} = \phi_{\beta}$$

for some $\alpha : \omega(x_1) \twoheadrightarrow \omega((x_ne_n)\alpha_n)$ and $\beta : \omega(x_1) \twoheadrightarrow \omega((y_mf_m)\beta_m)$. But

$$\phi_{\alpha} = \phi_{x_1}\phi_{\alpha_1}\dots\phi_{\alpha_n} \mathcal{H}_S \phi_{x_1}\phi_{\beta_1}\dots\phi_{\beta_m} = \phi_{\beta} ,$$

so that

$$(x_ne_n)\alpha_n \succ\!\!-\!\!\prec (y_mf_m)\beta_m ,$$

by Lemma 3.8, whence

$$L\rho_{\alpha_1}\dots\rho_{\alpha_n} = L_{(x_ne_n)\alpha_n} = L_{(y_mf_m)\beta_m} = L\rho_{\beta_1}\dots\rho_{\beta_m} .$$

Similarly, if $L\rho_{\beta_1}\dots\rho_{\beta_m} \neq \infty$ then $L\rho_{\alpha_1}\dots\rho_{\alpha_n} = L\rho_{\beta_1}\dots\rho_{\beta_m}$, which verifies that $\rho_{\alpha_1}\dots\rho_{\alpha_n} = \rho_{\beta_1}\dots\rho_{\beta_m}$. This completes the proof of the theorem. $\square$

Before proving Theorem 3.2, we introduce another representation of an arbitrary semigroup S that essentially reconstructs the earlier representation ϕ° of S, except that we now use transformations and dual transformations of $\mathcal{L}$ and $\mathcal{R}$-classes respectively of the boset $E = E(S)$. In the following definition,

L and R denote typical boset $\mathcal{L}$ and $\mathcal{R}$-classes respectively, while if $x \in S$ then L_x^S and R_x^S denote semigroup $\mathcal{L}_S$ and $\mathcal{R}_S$-classes respectively. Note that if $x \in \operatorname{Reg} S$, then $L_x^S \cap E$ and $R_x^S \cap E$ are boset $\mathcal{L}$ and $\mathcal{R}$-classes respectively. Define a mapping

$$\Phi = (\mathrm{P}, \Lambda^*) : S \to \mathcal{T}_{E/\mathcal{L} \cup \{\infty\}} \times \mathcal{T}^*_{E/\mathcal{R} \cup \{\infty\}}\ , \quad s \mapsto (\mathrm{P}_s, \Lambda_s^*)$$

by

$$\mathrm{P}_s : L \mapsto \begin{cases} L_{xs}^S \cap E & \text{if } x \,\mathcal{R}_S\, xs \text{ for some } x \in L \\ \infty & \text{otherwise} \end{cases}$$

and

$$\Lambda_s : R \mapsto \begin{cases} R_{sx}^S \cap E & \text{if } x \,\mathcal{L}_S\, sx \text{ for some } x \in R \\ \infty & \text{otherwise.} \end{cases}$$

Lemma 3.9 *The mapping Φ is a representation of S with kernel μ. For $s \in$* $\operatorname{Reg}(S)$ *with inverse s' the map*

$$\alpha : \omega(ss') \to \omega(s's), \quad x \mapsto s'xs$$

is a principal ideal isomorphism of $E = E(S)$ and $\Phi_s = \phi_\alpha$. In particular, $\Phi_e = \phi_e$ for each $e \in E$, and if S is generated by regular elements then Φ is a representation of S by T_E.

Proof. That P is well-defined follows because $\mathcal{L}_S$ is a right congruence on S, and the homomorphic property follows easily from Green's Lemma. Hence P is a representation. Dually Λ is an anti-representation, so Φ is a representation. Let X denote the set of regular $\mathcal{L}_S$-classes of S. Intersection with E is a bijection between X and $E/\mathcal{L}$. If $L^S \in X$ and $s \in S$, then $L^S\rho_s^\circ \neq \infty$ if and only if there exists an idempotent $x \in L^S$ such that $x \,\mathcal{R}_S\, xs$, and this occurs if and only if $(L^S \cap E)\mathrm{P}_s \neq \infty$, in which case the following diagram commutes:

$$\begin{tikzcd} L_x^S \arrow[r, mapsto, "\rho_s^\circ"] \arrow[d, mapsto, "\cap E"'] & L_{xs}^S \arrow[d, mapsto, "\cap E"] \\ L_x \arrow[r, mapsto, "\mathrm{P}_s"'] & L_{xs}^S \cap E \end{tikzcd}$$

Immediately then $\ker \mathrm{P} = \ker \rho^\circ$, and dually $\ker \Lambda = \ker \lambda^\circ$, so that

$$\ker \Phi \;=\; \ker \phi^\circ \;=\; \mu\ .$$

It follows quickly from the definitions that $\alpha : \omega(ss') \to \omega(s's), \quad x \mapsto s'xs$ is a principal ideal isomorphism, so, by duality, to complete the proof of the lemma, it suffices to verify that $\mathrm{P}_s = \rho_\alpha$.

Suppose first that $L\rho_\alpha \neq \infty$. Then $x \longrightarrow ss'$ for some $x \in L$, so that $x \longleftrightarrow x(ss')$, yielding $x \,\mathcal{R}_S\, xss'$, whence $x \,\mathcal{R}_S\, xs$ and $L\,\mathrm{P}_s = L^S_{xs} \cap E$. But $s(s'xs) = xs$ since $x \longrightarrow ss'$, so that $xs \,\mathcal{L}_S\, s'xs$ and

$$L\rho_\alpha \;=\; L_{[x(ss')]\alpha} \;=\; L_{s'[x(ss')]s} \;=\; L_{s'xs} \;=\; L^S_{s'xs} \cap E \;=\; L^S_{xs} \cap E \;=\; L\,\mathrm{P}_s\ .$$

Suppose now that $L\,\mathrm{P}_s \neq \infty$. Then $x \,\mathcal{R}_S\, xs$ for some $x \in L$, so $x = xst$ for some $t \in S$. Put $e = stx$. Then one readily checks that $e \in E$ and $ss' \longleftarrow e \succ\!\!\!-\!\!\!\prec x$, so that $L\rho_\alpha \neq \infty$. This completes the proof that $\mathrm{P}_s = \rho_\alpha$. $\square$

Proof of Theorem 3.2. We are supposing that S is generated by regular elements and $E = E(S)$. By Lemma 3.9, Φ is a representation of S with kernel μ and $S\Phi$ is contained in T_E. It remains to check that $S\Phi$ is symmetric. Since $S = \langle \mathrm{Reg}(S) \rangle$,

$$S\Phi \;=\; \langle (\mathrm{Reg}(S))\Phi \rangle \;=\; \langle \Phi_s \mid s \in \mathrm{Reg}(S) \rangle \;=\; \Big\langle \bigcup_{(e,f)\in\mathcal{U}} \{\phi_\alpha \,|\, \alpha \in T'_{e,f}\ \} \Big\rangle$$

by Lemma 3.9, where, for each $(e,f) \in \mathcal{U}$,

$$T'_{e,f} \;=\; \{\, \alpha : \omega(e) \twoheadrightarrow \omega(f) \mid (\exists s \in \mathrm{Reg}(S))(\exists s' \in \mathrm{V}(x))\ \ e = ss', f = s's \\ \text{and} \quad (\forall z \in \omega(e))\ \alpha : z \mapsto s'zs\ \}\ .$$

Certainly $T'_{e,f} \subseteq T_{e,f}$ and $\phi_e = \mathrm{P}_e \in T_{e,e}$. Further, if $\alpha \in T'_{e,f}$ so that $e = ss'$, $f = s's$ for some $s \in \mathrm{Reg}(S)$ with inverse s' and $\alpha : z \mapsto s'zs$ for $z \in \omega(e)$, then it is straightforward to check that $\alpha^{-1} = \beta \in T_{f,e}$ where

$$\beta : \omega(f) \to \omega(e), \quad z \mapsto szs'\ .$$

Thus $S\Phi$ is symmetric and the proof is complete. $\square$

Corollary 3.10 *Let S be a fundamental semigroup generated by regular elements and S' a full subsemigroup of S generated by regular elements. Then S' is fundamental.*

Proof. By Theorem 3.2, $\Phi : S \to T_{E(S)}$ is a faithful representation. By the method in the last part of the proof of Theorem 3.2, the image of S' is a symmetric subsemigroup of $T_{E(S)}$, so is fundamental by Theorem 3.1. $\square$

Corollary 3.11 *Any full subsemigroup of T_E, for any boset E, generated by regular elements is fundamental.* $\square$

This result is surprising, because whilst the full subsemigroup of T_E may not be symmetric, the method of proof utilises a faithful representation into $T_{E(T_E)}$, where the image then becomes symmetric. This phenomenon is obscured in the regular case (see the final section), because if E is a regular boset then $E \cong E(T_E)$ and all full subsemigroups of T_E are automatically symmetric.

The next two examples are special cases of large classes of (typically non-regular) bosets, studied by Jordan [18] and Roberts [23].

Example 3.12 The following diagram uniquely defines a five element boset E. Both E and its dual are the smallest examples of bosets of finite semigroups that are not regular. However T_E is regular and is described by the eggbox diagram listed below. The subsemigroup S consisting of the two lower $\mathcal{D}$-classes of T_E together with ϕ_e and ϕ_f is a full subsemigroup of T_E, so is fundamental by the previous corollary. (It is clear S is fundamental also because S is regular and it is routine to check that there is no nontrivial congruence contained in $\mathcal{H}$.) However S is not symmetric.

$$E \qquad \begin{tikzcd}
 & e & & f & \\
h \arrow[ur, tail] \arrow[rr, leftrightarrow] & & g \arrow[ul, tail] \arrow[ur, tail] \arrow[rr, leftrightarrow] & & k \arrow[ul, tail] \\
 & & \rotatebox{90}{$=$} & & \rotatebox{90}{$=$} \\
 & & ke & & hf
\end{tikzcd}$$

In the following diagram, α and β are principal ideal isomorphisms where $\alpha : e \mapsto e,\ h \mapsto g \mapsto h$ and $\beta : e \mapsto f,\ h \mapsto g \mapsto k$, and the idempotents are indicated by asterisks. The original boset E embeds in $E(T_E)$, which is regular with three additional idempotents in the middle $\mathcal{D}$-class.

T_E

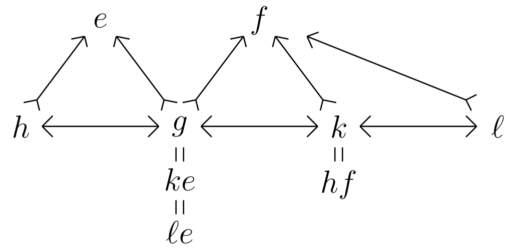
${}^*\phi_e$ ϕ_α	ϕ_β $\phi_{\alpha\beta}$
$\phi_{\beta^{-1}}$ $\phi_{\beta^{-1}\alpha}$	${}^*\phi_f$ $\phi_{\beta^{-1}\alpha\beta}$

$\phi_e\phi_{\beta^{-1}}$ ${}^*\phi_\alpha\phi_{\beta^{-1}}$	$\phi_e\phi_f$ $\phi_\alpha\phi_f$
${}^*\phi_{\beta^{-1}}\phi_{\beta^{-1}}$ $\phi_{\beta^{-1}\alpha}\phi_{\beta^{-1}}$	$\phi_{\beta^{-1}}\phi_f$ ${}^*\phi_{\beta^{-1}\alpha}\phi_f$

${}^*\phi_h$	${}^*\phi_g$	${}^*\phi_k$

Example 3.13 The previous example can be modified slightly to yield another boset F which is not regular, and for which T_F is also not regular.

F



Now the symmetry is broken and $\omega(e)$ and $\omega(f)$ are no longer isomorphic. In the following diagram, α, γ and δ are principal ideal automorphisms where $\alpha : e \mapsto e,\ h \mapsto g \mapsto h$, $\gamma : f \mapsto f,\ g \mapsto k \mapsto \ell \mapsto g$ and $\delta : f \mapsto f,\ g \mapsto g,\ k \mapsto \ell \mapsto k$. The bosets F and $E(T_F)$ are isomorphic and the middle $\mathcal{D}$-class of T_F is not regular.

T_F

$^*\phi_e$	
ϕ_α	

$^*\phi_f$	ϕ_γ	ϕ_{γ^2}
ϕ_δ	$\phi_{\delta\gamma}$	$\phi_{\delta\gamma^2}$

$\phi_e\phi_f$	$\phi_e\phi_\gamma$	$\phi_e\phi_{\gamma^2}$
$\phi_\alpha\phi_f$	$\phi_\alpha\phi_\gamma$	$\phi_\alpha\phi_{\gamma^2}$

$^*\phi_h$	$^*\phi_g$	$^*\phi_k$	$^*\phi_\ell$

4 The regular case

In the regular case, we can dispense with angular brackets in the definition of T_E and recover all of the properties of Nambooripad's original formulation [21, 22] of the maximum fundamental regular semigroup on a regular boset.

Theorem 4.1 *Let E be any regular boset. Then*

$$T_E = \bigcup_{(e,f)\in\mathcal{U}} \{\phi_\alpha \,|\, \alpha \in T_{e,f}\}$$

is regular and $E \cong E\phi = E(T_E)$.

Proof. The fact that angular brackets can be ignored in the definition of T_E is immediate by Lemma 3.5, since all sandwich sets are nonempty. Certainly each ϕ_α is regular for any principal ideal isomophism α, so T_E is regular. It remains only to verify that $E(T_E) \subseteq E\phi$. Suppose that ϕ_α is idempotent and $\alpha : \omega(e) \twoheadrightarrow \omega(f)$. In particular, ρ_α is idempotent and $L_e\rho_\alpha = L_f$, so $L_f\rho_\alpha = L_f$, yielding some $x \in L_f$ such that $x \longrightarrow\!\!> e$ and $L_f = L_{(xe)\alpha}$. But $(xe)\alpha >\!\!\longrightarrow\!\!> f$ so that $(xe)\alpha = f = e\alpha$, yielding also $xe = e$. Hence $e <\!\!\longrightarrow\!\!> x >\!\!\!-\!\!\!< f$, so that $\phi_\alpha \,\mathcal{H}_{T_E}\, \phi_x$, whence $\phi_\alpha = \phi_x \in E\phi$, and the proof is complete. $\square$

Corollary 4.2 *Let E be any regular boset. Then every regular semigroup S with boset E can be represented by T_E by a homomorphism whose kernel is the maximum idempotent-separating congruence on S. In particular T_E is, up to isomorphism, the maximum fundamental regular semigroup with boset E. All full subsemigroups of T_E are fundamental.*

REFERENCES

[1] Clifford, A.H. and Preston, G.B. *The Algebraic Theory of Semigroups, vol. I.* Math. Surveys of the American Math. Soc. 7, Providence, R.I., 1961.

[2] Easdown, D. *Biordered Sets of Semigroups.* Doctoral Thesis, Monash University, 1983.

[3] Easdown, D. Biordered sets are biordered subsets of idempotents of semigroups. *J. Austral. Math. Soc.* 37 (1984), 258–268.

[4] Easdown, D. Biordered sets come from semigroups. *J. Algebra.* 96 (1985), 581–591.

[5] Easdown, D. Biorder preserving coextensions of fundamental semigroups. *Proc. Edinb. Math. Soc.* 31 (1988), 463–467.

[6] Easdown, D., Hall, T.E. Reconstructing some idempotent-generated semigroups from their biordered sets. *Semigroup Forum.* 29 (1984), 207–216.

[7] Edwards, P.M. Eventually regular semigroups. *Bull. Austral. Math. Soc.* 28 (1983), 23-38.

[8] Edwards, P.M. Fundamental semigroups. *Proc. R. Soc. Edinb. Sec. A.* 99 (1985), 313-317.

[9] Edwards, P.M. On the lattice of congruences on an eventually regular semigroup. *J. Austral. Math. Soc.* 38 (1985), 281-286.

[10] Edwards, P.M. Eventually regular semigroups that are group-bound. *Bull. Austral. Math. Soc.* 34 (1986), 127-132.

[11] Edwards, P.M. Maximising a congruence with respect to its partition of idempotents. *Semigroup Forum.* 39 (1989), 257-262.

[12] Grillet, P.A. The structure of regular semigroups. I. A representation. *Semigroup Forum.* 8 (1974), 177–183.

[13] Grillet, P.A. The structure of regular semigroups. II. Cross-connections. *Semigroup Forum.* 8 (1974), 254-259.

[14] Grillet, P.A. The structure of regular semigroups. III. The reduced case. *Semigroup Forum.* 8 (1974), 250-265.

[15] Hall, T.E. On regular semigroups. *J. Algebra.* 24 (1973), 1–24.

[16] Higgins, P.M. *Techniques of Semigroup Theory.* Oxford University Press, 1992.

[17] Howie, J.M. *An Introduction to Semigroup Theory.* London: Academic Press, 1976.

[18] Jordan, P.K. *On Biordered Sets, Including an Alternative Approach to Fundamental Regular Semigroups.* Masters Thesis, University of Sydney, 2002.

[19] Munn, W.D. Uniform semilattices and bisimple inverse semigroups. *Quart. J. Math. Oxford Ser. (2).* 17 (1966), 151–159.

[20] Munn, W.D. Fundamental inverse semigroups. *Quart. J. Math. Oxford Ser. (2).* 21 (1970), 157–170.

[21] Nambooripad, K.S.S. Structure of regular semigroups. I. Fundamental regular semigroups. *Semigroup Forum.* 9 (1975), 354–363.

[22] Nambooripad, K.S.S. Structure of regular semigroups. I. *Mem. Amer. Math. Soc.* 22 (1979).

[23] Roberts, B. *On Bosets and Fundamental Semigroups.* Doctoral Thesis, University of Sydney, 2007.