A SPECHT FILTRATION OF AN INDUCED SPECHT MODULE

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To John Cannon and Derek Holt on the occasions of their significant birthdays, in recognition of their distinguished contributions to mathematics.

ABSTRACT. Let $\mathscr{H}_n$ be a (degenerate or non-degenerate) Hecke algebra of type $G(\ell,1,n)$, defined over a commutative ring R with one, and let $S(\boldsymbol{\mu})$ be a Specht module for $\mathscr{H}_n$. This paper shows that the induced Specht module $S(\boldsymbol{\mu})\otimes_{\mathscr{H}_n}\mathscr{H}_{n+1}$ has an explicit Specht filtration.

1. INTRODUCTION

The Ariki-Koike algebras, and their rational degenerations, are interesting algebras which appear naturally in the representation theory of affine Hecke algebras, quantum groups, symmetric groups and general linear groups; see [**14**, **18**] for details. They include as special cases the group algebras of the Coxeter groups of type A (the symmetric groups) and the Coxeter groups of type B (the hyperoctahedral groups).

Let $\mathscr{H}_n$ be an Ariki-Koike algebra, or a degenerate cyclotomic Hecke algebra, of type $G(\ell,1,n)$, for integers $\ell, n\ge 1$. For each multipartition $\boldsymbol{\mu}$ of n there is a **Specht module** $S(\boldsymbol{\mu})$, which is a right $\mathscr{H}_n$-module. (All of the undefined terms and notation, here and below, can be found in section 2.) When $\mathscr{H}_n$ is semisimple the Specht modules give a complete set of pairwise non-isomorphic irreducible $\mathscr{H}_n$-modules as $\boldsymbol{\mu}$ runs through the multipartitions of n. In general, the Specht modules are not irreducible however every irreducible $\mathscr{H}_n$-module arises, in a unique way, as the simple head of some Specht module.

The Hecke algebra $\mathscr{H}_n$ embeds into $\mathscr{H}_{n+1}$ so there are natural induction and restriction functors, Ind and Res, between the categories of finite dimensional $\mathscr{H}_n$-modules and $\mathscr{H}_{n\pm1}$-modules. By [**2**, Proposition 1.9], in the Ariki-Koike case the restriction of the Specht module $S(\boldsymbol{\mu})$ to $\mathscr{H}_{n-1}$ has a Specht filtration of the form

$$0=R_0\subset R_1\subset\cdots\subset R_r=\operatorname{Res}S(\boldsymbol{\mu}),$$

such that $R_j/R_{j-1}\cong S(\boldsymbol{\mu}-\rho_j)$, where $\rho_1>\rho_2>\cdots>\rho_r$ are the removable nodes of $\boldsymbol{\mu}$. Consequently, if $\mathscr{H}_{n+1}$ is semisimple then by Frobenius reciprocity

$$\operatorname{Ind}S(\boldsymbol{\mu})\cong S(\boldsymbol{\mu}\cup\alpha_1)\oplus\cdots\oplus S(\boldsymbol{\mu}\cup\alpha_a),$$

where $\alpha_1,\dots,\alpha_a$ are the addable nodes of $\boldsymbol{\mu}$. This note generalizes this result to the case when $\mathscr{H}_n$ is not necessarily semisimple. More precisely, we prove the following:

Main Theorem. *Suppose that $\mathscr{H}_n$ is an Ariki-Koike algebra or a degenerate cyclotomic Hecke algebra of type $G(\ell,1,n)$ and let $\boldsymbol{\mu}$ be a multipartition of n. Then, as an $\mathscr{H}_{n+1}$-module, the induced module $\operatorname{Ind}S(\boldsymbol{\mu})$ has a filtration*

$$0=I_0\subset I_1\subset\cdots\subset I_a=\operatorname{Ind}S(\boldsymbol{\mu}),$$

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such that $I_j/I_{j-1} \cong S(\boldsymbol{\mu} \cup \alpha_j)$*, where* $\alpha_1 > \alpha_2 > \cdots > \alpha_a$ *are the addable nodes of* $\boldsymbol{\mu}$*.*

This result is part of the folklore for the representation theory of these algebras, however, we have been unable to find a proof of it in the literature when $\ell > 1$. If $\ell = 1$ then our Main Theorem is an old result of James [**12**, §17] in the degenerate case (that is, for the symmetric group), and it can be deduced from [**9**, Theorem 7.4] in the non-degenerate case (the Hecke algebra of the symmetric group). We prove our Main Theorem by giving an explicit construction of $\operatorname{Ind} S(\boldsymbol{\lambda})$; see Corollary **(3.7)**. Our argument is similar in spirit to that originally used by James [**12**] for the symmetric groups in that we identify the induced module as a quotient of the corresponding permutation module. Our approach, which uses cellular basis techniques, gives an explicit Specht filtration of the induced module; in contrast, James' approach is recursive.

Suppose now that $\mathscr{H}_n$ is defined over a field of characteristic $p \geq 0$, or a suitable discrete valuation ring. Then by projecting onto the blocks of $\mathscr{H}_n$ the induction functor Ind can be decomposed as a direct sum of subfunctors

$$\operatorname{Ind} = \bigoplus_{i \in I} i\text{-}\operatorname{Ind},$$

where $I = \mathbb{Z}/p\mathbb{Z}$, in the degenerate case, and $I = \{\, q^a Q_s \mid a \in \mathbb{Z} \text{ and } 1 \leq s \leq r \,\}$ in the non-degenerate case. (If the parameters $Q_1, \dots, Q_r$ are all non-zero then, up to Morita equivalence, it is enough to consider the cases where $Q_1, \dots, Q_r$ are all powers of q by the main result of [**11**]. In this case we can take $I = \mathbb{Z}/e\mathbb{Z}$ where e is the smallest positive integer such that $1 + q + \cdots + q^{e-1} = 0$.) The functor i-Ind is a natural generalization of Robinson's i-induction functor; see [**2**, 1.11] and [**14**, §8] for the precise definitions.

(1.1). **Corollary.** *Suppose that* $\boldsymbol{\mu}$ *is a multipartition of* n *and* $i \in I$*. Then* i-$\operatorname{Ind} S(\boldsymbol{\mu})$ *has a filtration*

$$0 = I_0 \subset I_1 \subset \cdots \subset I_b = i\text{-}\operatorname{Ind} S(\boldsymbol{\mu}),$$

such that $I_j/I_{j-1} \cong S(\boldsymbol{\mu} \cup \alpha_j)$*, where* $\alpha_1 > \alpha_2 > \cdots > \alpha_b$ *are the addable* i*-nodes of* $\boldsymbol{\mu}$*.*

Proof. By [**15**] and [**4**], the Specht modules $S(\boldsymbol{\mu} \cup \alpha)$ and $S(\boldsymbol{\mu} \cup \beta)$ are in the same block if and only if α and β have the same residue. By the Main Theorem and the definition of the functor i-Ind, the Specht module $S(\boldsymbol{\mu} \cup \alpha)$ is a subquotient of i-$\operatorname{Ind} S(\boldsymbol{\mu})$ if and only if α is an i-node (*cf.* [**2**, Cor. 1.12]). This implies the result. □

Recently Brundan and Kleshchev [**5**] have shown that $\mathscr{H}_n$ is naturally $\mathbb{Z}$-graded and Brundan, Kleshchev and Wang [**8**] have shown that $S(\boldsymbol{\mu})$ admits a natural grading. There should be a graded analogue of our induction theorem; see [**8**, Remark 4.12] for a precise conjecture. Unfortunately, the arguments of this paper do not automatically lift to the graded setting because it is not clear how to use our results to find a homogeneous basis of the induced module.

2. ARIKI-KOIKE ALGEBRAS

In order to make this note self-contained, this section quickly recalls the definitions and results that we need from the literature and, at the same time, sets our notation. We concentrate on the non-degenerate case as the degenerate case follows in exactly the same way, with only minor changes of notation, using the results of [**3**, §6]. See the remarks at the end of this section for more details.

Throughout this note we fix positive integers ℓ and n and let $\mathfrak{S}_n$ be the symmetric group of degree n. For $1 \leq i < n$ let $s_i = (i, i+1) \in \mathfrak{S}_n$. Then $s_1, \dots, s_{n-1}$ are the standard Coxeter generators of $\mathfrak{S}_n$.

Let R be a commutative ring with 1 and let $q, Q_1, \dots, Q_\ell$ be elements of R with q invertible. The Ariki–Koike algebra $\mathscr{H}_n = \mathscr{H}_{R,\ell,n}(q, Q_1, \dots, Q_\ell)$ is the associative unital R–algebra with generators $T_0, T_1, \dots, T_{n-1}$ and relations

$$\begin{aligned}
(T_0 - Q_1)\dots(T_0 - Q_\ell) &= 0, \\
(T_i - q)(T_i + 1) &= 0, && \text{for } 1 \le i \le n-1, \\
T_0T_1T_0T_1 &= T_1T_0T_1T_0, \\
T_{i+1}T_iT_{i+1} &= T_iT_{i+1}T_i, && \text{for } 1 \le i \le n-2, \\
T_iT_j &= T_jT_i, && \text{for } 0 \le i < j-1 \le n-2.
\end{aligned}$$

Using the relations it follows that there is a unique anti-isomorphism $*: \mathscr{H}_n \longrightarrow \mathscr{H}_n$ such that $T_i^* = T_i$, for $0 \le i < n$.

Ariki and Koike [**1**, Theorem 3.10] showed that $\mathscr{H}_n$ is free as an R-module with basis $\{ L_1^{a_1} \dots L_n^{a_n} T_w \mid 0 \le a_1, \dots, a_n < \ell \text{ and } w \in \mathfrak{S}_n \}$ where $L_1 = T_0$ and $L_{i+1} = q^{-1}T_iL_iT_i$ for $i = 1, \dots, n-1$, and $T_w = T_{i_1} \dots T_{i_k}$ if $w = s_{i_1} \dots s_{i_k} \in \mathfrak{S}_n$ is a reduced expression (that is, k is minimal).

The Ariki-Koike basis theorem implies that there is a natural embedding of $\mathscr{H}_n$ in $\mathscr{H}_{n+1}$ and that $\mathscr{H}_{n+1}$ is free as an $\mathscr{H}_n$-module of rank $\ell(n+1)$. If M is an $\mathscr{H}_n$-module let

$$\operatorname{Ind} M = M \otimes_{\mathscr{H}_n} \mathscr{H}_{n+1}$$

be the corresponding induced $\mathscr{H}_{n+1}$-module. Note that induction is an exact functor since $\mathscr{H}_{n+1}$ is free as an $\mathscr{H}_n$-module.

We will need to the following easily proved property of the basis elements [**10**, 2.1].

(2.1). *Suppose that* $1 \le k \le n$, $a \in R$ *and* $w \in \mathfrak{S}_k \times \mathfrak{S}_{n-k}$. *Then*

$$(L_1 - a)\dots(L_k - a)T_w = T_w(L_1 - a)\dots(L_k - a).$$

The algebra $\mathscr{H}_n$ has another basis which is crucial to this note. In order to describe it recall that a partition of n is a weakly decreasing sequence $\lambda = (\lambda_1 \ge \lambda_2 \ge \dots)$ of non-negative integers such that $|\lambda| = \sum_i \lambda_i = n$. A **multipartition**, or ℓ-partition, of n is an ordered ℓ-tuple $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(\ell)})$ of partitions such that $|\boldsymbol{\lambda}| = |\lambda^{(1)}| + \dots + |\lambda^{(\ell)}| = n$. Let $\Lambda^+_{\ell,n}$ be the set of multipartitions of n. If $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \Lambda^+_{\ell,n}$ then $\boldsymbol{\lambda}$ **dominates** $\boldsymbol{\mu}$, and we write $\boldsymbol{\lambda} \trianglerighteq \boldsymbol{\mu}$, if

$$\sum_{t=1}^{s-1} |\lambda^{(t)}| + \sum_{i=1}^{k} \lambda_i^{(s)} \ge \sum_{t=1}^{s-1} |\mu^{(t)}| + \sum_{i=1}^{k} \mu_i^{(s)},$$

for $1 \le s \le \ell$ and for all $k \ge 1$. Dominance is a partial order on $\Lambda^+_{\ell,n}$.

If $\boldsymbol{\lambda}$ is a multipartition let $\mathfrak{S}_{\boldsymbol{\lambda}} = \mathfrak{S}_{\lambda^{(1)}} \times \dots \times \mathfrak{S}_{\lambda^{(\ell)}}$ be the corresponding parabolic subgroup of $\mathfrak{S}_n$ and set $a_s^{\boldsymbol{\lambda}} = \sum_{t=1}^{s-1} |\lambda^{(t)}|$, for $1 \le s \le \ell$, and put $a_{\ell+1}^{\boldsymbol{\lambda}} = n-1$. Define $m_{\boldsymbol{\lambda}} = u_{\boldsymbol{\lambda}}^+ x_{\boldsymbol{\lambda}}$ where

$$u_{\boldsymbol{\lambda}}^+ = \prod_{s=2}^{\ell} \prod_{k=1}^{a_s^{\boldsymbol{\lambda}}} (L_k - Q_s) \quad \text{and} \quad x_{\boldsymbol{\lambda}} = \sum_{w \in \mathfrak{S}_{\boldsymbol{\lambda}}} T_w.$$

Then $u_{\boldsymbol{\lambda}}^+ x_{\boldsymbol{\lambda}} = m_{\boldsymbol{\lambda}} = x_{\boldsymbol{\lambda}} u_{\boldsymbol{\lambda}}^+$ by **(2.1)**.

Let $\boldsymbol{\lambda}$ be a multipartition (of n). The **diagram** of $\boldsymbol{\lambda}$ is the set of nodes

$$[\boldsymbol{\lambda}] = \{ (r, c, s) \mid 1 \le \lambda_r^{(s)} \le c \text{ and } 1 \le s \le \ell \}.$$

More generally a **node** is any element of $\mathbb{N} \times \mathbb{N} \times \{1, \dots, \ell\}$, which we consider as a partially ordered set where $(r, c, s) \ge (r', c', s')$ if either $s > s'$, or $s = s'$ and $r < r'$. For the sake of Corollary **(1.1)** only, define the **residue** of the node (r, c, s) to be $q^{c-r}Q_s$.

An **addable** node of $\boldsymbol{\lambda}$ is any node $\alpha \notin [\boldsymbol{\lambda}]$ such that $[\boldsymbol{\lambda}] \cup \{\alpha\}$ is the diagram of some multipartition. Let $\boldsymbol{\lambda} \cup \alpha$ be the multipartition such that $[\boldsymbol{\lambda} \cup \alpha] = [\boldsymbol{\lambda}] \cup \{\alpha\}$. Similarly, a **removable** node of $\boldsymbol{\lambda}$ is a node $\rho \in [\boldsymbol{\lambda}]$ such that $[\boldsymbol{\lambda}] - \{\rho\}$ is the diagram of a multipartition; let $\boldsymbol{\lambda} - \rho$ be this multipartition. Note that the set of addable and removable nodes for $\boldsymbol{\lambda}$ are both totally ordered by $>$.

If X is a set then an X-valued $\boldsymbol{\lambda}$-tableau is a function $\mathsf{T}\colon[\boldsymbol{\lambda}] \longrightarrow X$. If T is a $\boldsymbol{\lambda}$-tableau then we write $\operatorname{Shape}(\mathsf{T}) = \boldsymbol{\lambda}$. For convenience we identify $\mathsf{T} = (\mathsf{T}^{(1)}, \dots, \mathsf{T}^{(\ell)})$ with a labeling of the diagram $[\boldsymbol{\lambda}]$ by elements of X in the obvious way. Thus, we can talk of the rows, columns and components of T.

A **standard $\boldsymbol{\lambda}$-tableau** is a map $\mathfrak{t}\colon[\boldsymbol{\lambda}] \longrightarrow \{1, 2, \dots, n\}$ such that for $s = 1, \dots, \ell$ the entries in each row of $\mathfrak{t}^{(s)}$ increase from left to right and the entries in each column of $\mathfrak{t}^{(s)}$ increase from top to bottom. Let $\mathcal{T}^{\text{Std}}(\boldsymbol{\lambda})$ be the set of standard $\boldsymbol{\lambda}$-tableaux.

Let $\mathfrak{t}^{\boldsymbol{\lambda}}$ be the standard $\boldsymbol{\lambda}$-tableau such that the entries in $\mathfrak{t}^{\boldsymbol{\lambda}}$ increase from left to right along the rows of $\mathfrak{t}^{\lambda^{(1)}}, \dots, \mathfrak{t}^{\lambda^{(\ell)}}$ in order. If $\mathfrak{t}$ is a standard $\boldsymbol{\lambda}$-tableau let $d(\mathfrak{t}) \in \mathfrak{S}_n$ be the unique permutation such that $\mathfrak{t} = \mathfrak{t}^{\boldsymbol{\lambda}} d(\mathfrak{t})$. Define $m_{\mathfrak{s}\mathfrak{t}} = T^*_{d(\mathfrak{s})} m_{\boldsymbol{\lambda}} T_{d(\mathfrak{t})}$, for $\mathfrak{s}, \mathfrak{t} \in \mathcal{T}^{\text{Std}}(\boldsymbol{\lambda})$. By [**10**, Theorem 3.26], the set

$$\{\, m_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \mathcal{T}^{\text{Std}}(\boldsymbol{\lambda}) \text{ and } \boldsymbol{\lambda} \in \Lambda^+_{\ell,n} \,\}$$

is a cellular basis of $\mathscr{H}_n$. Consequently, if $\mathscr{H}_n(\boldsymbol{\lambda})$ is the R-module spanned by

$$\{\, m_{\mathfrak{s}\mathfrak{t}} \mid \mathfrak{s}, \mathfrak{t} \in \mathcal{T}^{\text{Std}}(\boldsymbol{\mu}) \text{ for some } \boldsymbol{\mu} \in \Lambda^+_{\ell,n} \text{ with } \boldsymbol{\mu} \rhd \boldsymbol{\lambda} \,\},$$

then $\mathscr{H}_n(\boldsymbol{\lambda})$ is a two-sided ideal of $\mathscr{H}_n$.

The **Specht module** $S(\boldsymbol{\lambda})$ is the submodule of $\mathscr{H}_n/\mathscr{H}_n(\boldsymbol{\lambda})$ generated by $m_{\boldsymbol{\lambda}} + \mathscr{H}_n(\boldsymbol{\lambda})$. It follows from the general theory of cellular algebras that $S(\boldsymbol{\lambda})$ is free as an R-module with basis $\{\, m_{\mathfrak{t}} \mid \mathfrak{t} \in \mathcal{T}^{\text{Std}}(\boldsymbol{\lambda}) \,\}$, where $m_{\mathfrak{t}} = m_{\mathfrak{t}^{\boldsymbol{\mu}}\mathfrak{t}} + \mathscr{H}_n(\boldsymbol{\lambda})$ for $\mathfrak{t} \in \mathcal{T}^{\text{Std}}(\boldsymbol{\lambda})$.

Let M be an $\mathscr{H}_n$-module. Then M has a **Specht filtration** if there exists a filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_k = M$$

and multipartitions $\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_k$ such that $M_i/M_{i-1} \cong S(\boldsymbol{\lambda}_i)$, for $i = 1, \dots, k$.

For each multipartition $\boldsymbol{\mu} \in \Lambda^+_{\ell,n}$ let $M(\boldsymbol{\mu}) = m_{\boldsymbol{\mu}}\mathscr{H}_n$. The final result that we will need gives an explicit Specht filtration of $M(\boldsymbol{\mu})$. The proof of our Main Theorem is inspired by this filtration.

Given two tuples (i, s) and (j, t) write $(i, s) \preceq (j, t)$ if either $s < t$, or $s = t$ and $i \le j$.

(2.2). **Definition** ([**10**, Definition 4.4]). *Suppose that $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \Lambda^+_{\ell,n}$ and let $\mathsf{T}\colon[\boldsymbol{\lambda}] \longrightarrow \mathbb{N} \times \{1, 2, \dots, \ell\}$ be a $\boldsymbol{\lambda}$-tableau. Then:*

a) *T is a tableau of **type** $\boldsymbol{\mu}$ if $\mu^{(s)}_i = \#\{\, x \in [\lambda] \mid \mathsf{T}(x) = (i, s) \,\}$, for all $i \ge 1$ and $1 \le s \le \ell$.*
b) *T is **semistandard** if the entries in each component $\mathsf{T}^{(s)}$, for $1 \le s \le \ell$, of T are:*
   i) *weakly increasing from left to right along each row (with respect to $\preceq$);*
   ii) *strictly increasing from top to bottom down columns; and,*
   iii) *(j, t) appears in $\mathsf{T}^{(s)}$ only if $t \ge s$.*

   Let $\mathcal{T}^{\text{SStd}}_{\boldsymbol{\mu}}(\boldsymbol{\lambda})$ be the set of semistandard $\boldsymbol{\lambda}$–tableau of type $\boldsymbol{\mu}$ and let $\mathcal{T}^{\text{SStd}}_{\boldsymbol{\mu}}(\Lambda^+_{\ell,n}) = \bigcup_{\boldsymbol{\lambda} \in \Lambda^+_{\ell,n}} \mathcal{T}^{\text{SStd}}_{\boldsymbol{\mu}}(\boldsymbol{\lambda})$ be the set of all semistandard tableaux of type $\boldsymbol{\mu}$.

Let $\mathfrak{t}$ be a standard $\boldsymbol{\lambda}$–tableau. Define $\boldsymbol{\mu}(\mathfrak{t})$ to be the tableau obtained from $\mathfrak{t}$ by replacing each entry j in $\mathfrak{t}$ with (i, s) if j appears in row i of $\mathfrak{t}^{\mu^{(s)}}$. The tableau $\boldsymbol{\mu}(\mathfrak{t})$ is a $\boldsymbol{\lambda}$–tableau of type $\boldsymbol{\mu}$; it is not necessarily semistandard. Finally, if $\mathsf{S} \in \mathcal{T}^{\mathrm{SStd}}_{\boldsymbol{\mu}}(\boldsymbol{\lambda})$ and $\mathfrak{t} \in \mathcal{T}^{\mathrm{Std}}(\boldsymbol{\lambda})$ set

$$m_{\mathsf{S}\mathfrak{t}} = \sum_{\substack{\mathfrak{s} \in \mathcal{T}^{\mathrm{Std}}(\boldsymbol{\lambda}) \\ \boldsymbol{\mu}(\mathfrak{s}) = \mathsf{S}}} m_{\mathfrak{s}\mathfrak{t}}.$$

(2.3) ([**10**, Theorem 4.14 and Corollary 4.15]). *Suppose that $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \Lambda^+_{\ell,n}$. Then:*

a) *$M(\boldsymbol{\mu})$ is free as an R–module with basis*

$$\{\, m_{\mathsf{S}\mathfrak{t}} \mid \mathsf{S} \in \mathcal{T}^{SStd}_{\boldsymbol{\mu}}(\boldsymbol{\lambda}), \mathfrak{t} \in \mathcal{T}^{Std}(\boldsymbol{\lambda}) \text{ for some } \boldsymbol{\lambda} \in \Lambda^+_{\ell,n} \,\}.$$

b) *Suppose that $\mathcal{T}^{SStd}_{\boldsymbol{\mu}}(\Lambda^+_{\ell,n}) = \{\mathsf{S}_1, \dots, \mathsf{S}_m\}$ ordered so that $i \le j$ whenever $\boldsymbol{\lambda}_i \trianglerighteq \boldsymbol{\lambda}_j$, where $\boldsymbol{\lambda}_i = \operatorname{Shape}(\mathsf{S}_i)$. Let M_i be the R-submodule of $M(\boldsymbol{\mu})$ spanned by the elements $\{\, m_{\mathsf{S}_j\mathfrak{t}} \mid j \le i \text{ and } \in \mathcal{T}^{Std}(\boldsymbol{\lambda}_j)\,\}$, Then*

$$0 = M_0 \subset M_1 \subset \dots \subset M_m = M(\boldsymbol{\mu})$$

is an $\mathscr{H}_n$-module filtration of $M(\boldsymbol{\mu})$ and $M_i/M_{i-1} \cong S(\boldsymbol{\lambda}_i)$, for $1 \le i \le m$.

(2.4). **Remark.** Very few changes need to be made to the results above in the degenerate case. The analogue of the cellular basis $\{m_{\mathfrak{s}\mathfrak{t}}\}$ in the degenerate case is constructed in [**3**, §6]. Using this basis of the degenerate Hecke algebra, the construction of the Specht filtration of the ideals $M(\boldsymbol{\mu})$ follows easily using the arguments of [**10**, §4]; *cf.* [**7**, Cor. 6.13]. The arguments in the next section, modulo minor differences in the meaning of the symbols, applies to both the degenerate and non-degenerate cases.

3. INDUCING SPECHT MODULES

We are now ready to start proving the Main Theorem. Fix a multipartition $\boldsymbol{\mu} \in \Lambda^+_{\ell,n}$. As in **(2.3)** we let $\mathcal{T}^{\mathrm{SStd}}_{\boldsymbol{\mu}}(\Lambda^+_{\ell,n}) = \{\mathsf{S}_1, \dots, \mathsf{S}_m\}$ be the set of semistandard tableau of type $\boldsymbol{\mu}$ ordered so that $i \le j$ whenever $\boldsymbol{\lambda}_i \trianglerighteq \boldsymbol{\lambda}_j$, where $\boldsymbol{\lambda}_i = \operatorname{Shape}(\mathsf{S}_i)$ for $1 \le i \le m$. So, in particular, $\mathsf{S}_m = \mathsf{T}^{\boldsymbol{\mu}} = \boldsymbol{\mu}(\mathfrak{t}^{\boldsymbol{\mu}})$ is the unique semistandard $\boldsymbol{\mu}$-tableaux of type $\boldsymbol{\mu}$.

Throughout this section we freely identify $\mathscr{H}_n$ with its image under the natural embedding $\mathscr{H}_n \hookrightarrow \mathscr{H}_{n+1}$. In particular, we will think of the basis element $m_{\mathfrak{s}\mathfrak{t}}$ as an element of $\mathscr{H}_{n+1}$, for standard $\boldsymbol{\lambda}$-tableaux $\mathfrak{s}, \mathfrak{t} \in \mathcal{T}^{\mathrm{Std}}(\boldsymbol{\lambda})$ with $\boldsymbol{\lambda} \in \Lambda^+_{\ell,n}$. This embedding also identifies $\operatorname{Ind} M(\mu)$ with a submodule of $\mathscr{H}_{n+1}$.

The following simple Lemma contains the idea which drives our proof.

(3.1). **Lemma.** *Suppose that $\boldsymbol{\mu}$ is a multipartition of n and let ω be the lowest addable node of $\boldsymbol{\mu}$ (that is, $\alpha \ge \omega$ whenever α is an addable node of $\boldsymbol{\mu}$). Then :*

a) $\operatorname{Ind} M(\boldsymbol{\mu}) = M(\boldsymbol{\mu} \cup \omega)$.

b) *The induced module $\operatorname{Ind} M(\boldsymbol{\mu})$ has a filtration*

$$0 = N_0 \subset N_1 \dots \subset N_m = \operatorname{Ind} M(\boldsymbol{\mu})$$

such that $N_i/N_{i-1} \cong \operatorname{Ind} S(\boldsymbol{\lambda}_i)$, where $\boldsymbol{\lambda}_i = \operatorname{Shape}(\mathsf{S}_i)$ for $1 \le i \le m$.

Proof. By definition, $m_{\boldsymbol{\mu}} = m_{\boldsymbol{\mu}\cup\omega}$ using the embedding $\mathscr{H}_n \hookrightarrow \mathscr{H}_{n+1}$. Therefore,

$$\operatorname{Ind} M(\boldsymbol{\mu}) = m_{\boldsymbol{\mu}}\mathscr{H}_n \otimes_{\mathscr{H}_n} \mathscr{H}_{n+1} = m_{\boldsymbol{\mu}}\mathscr{H}_{n+1} = m_{\boldsymbol{\mu}\cup\omega}\mathscr{H}_{n+1} = M(\boldsymbol{\mu} \cup \omega),$$

proving (a). As induction is exact, part (b) follows from part (a) and **(2.3)**(b). □

If $\boldsymbol{\mu} = ((n), (0), \dots, (0))$ then $S(\boldsymbol{\mu}) = M(\boldsymbol{\mu})$. The Main Theorem in this special case is just part (b) of the Lemma. To prove the theorem when $\boldsymbol{\mu} \ne ((n), (0), \dots, (0))$ we explicitly describe the filtration of $\operatorname{Ind} M(\boldsymbol{\mu})$ given by the Lemma in terms of the basis of $M(\boldsymbol{\mu} \cup \omega)$ from **(2.3)**.

Let ω be the lowest addable node of $\boldsymbol{\mu}$. Then $\omega=(z,1,\ell)$, where $z\ge 1$ is minimal such that $(z,1,\ell)\notin[\boldsymbol{\mu}]$. Suppose that $\mathsf{S}\in\mathcal{T}^{\text{SStd}}_{\boldsymbol{\mu}}(\boldsymbol{\lambda})$, for some $\boldsymbol{\lambda}\in\Lambda^+_{\ell,n}$, and that β is an addable node of $\boldsymbol{\lambda}$. Let $\mathsf{S}\cup\beta$ be the semistandard $(\boldsymbol{\lambda}\cup\beta)$-tableau given by

$$(\mathsf{S}\cup\beta)(\eta)=\begin{cases}\mathsf{S}(\eta), & \text{if } \eta\in[\boldsymbol{\lambda}],\\ (z,\ell), & \text{if } \eta=\beta.\end{cases}$$

Thus $\mathsf{S}\cup\beta$ is the semistandard $(\boldsymbol{\lambda}\cup\beta)$-tableau of type $\boldsymbol{\mu}\cup\omega$ obtained by adding the node β to S with label (z,ℓ). Let $\mathcal{T}^{\text{SStd}}_{\boldsymbol{\mu}\cup\omega}(\mathsf{S})$ be the set of semistandard tableau of type $\boldsymbol{\mu}\cup\omega$ obtained in this way from S as β runs over the addable nodes of $\boldsymbol{\lambda}$. It is easy to see that every semistandard tableau of type $\boldsymbol{\mu}\cup\omega$ arises uniquely in this way, so

$$\mathcal{T}^{\text{SStd}}_{\boldsymbol{\mu}\cup\omega}(\Lambda^+_{\ell,n+1})=\coprod_{\mathsf{S}\in\mathcal{T}^{\text{SStd}}_{\boldsymbol{\mu}}(\Lambda^+_{\ell,n})}\mathcal{T}^{\text{SStd}}_{\boldsymbol{\mu}\cup\omega}(\mathsf{S}). \tag{3.2}$$

Armed with this notation, observe that if $\mathsf{S}\in\mathcal{T}^{\text{SStd}}_{\mu}(\boldsymbol{\lambda})$ then $m_{\mathsf{S}\mathfrak{t}^{\boldsymbol{\lambda}}}=m_{(\mathsf{S}\cup\beta)\mathfrak{t}^{\boldsymbol{\lambda}\cup\beta}}$, as an element of $\mathscr{H}_{n+1}$, where β is the lowest addable node of $\boldsymbol{\lambda}$.

Suppose that $1\le a\le b<n$. Let $\mathfrak{S}_{a,b}$ be the symmetric group on $\{a,a+1,\dots,b\}$ and set $s_{b,a}=(b,b+1)\dots(a,a+1)\in\mathfrak{S}_n$ and $T_{b,a}=T_{s_{b,a}}=T_b\dots T_a$. For convenience, we set $T_{b,a}=1$ if $b<a$. The following useful identity is surely known.

(3.3). Lemma. *Suppose that* $1\le a<b\le n$. *Then*

$$\Big(\sum_{w\in\mathfrak{S}_{a,b}}T_w\Big)T_{b,a}=T_{b,a}\Big(\sum_{v\in\mathfrak{S}_{a+1,b+1}}T_v\Big).$$

Proof. It is easy to check that $\mathfrak{S}_{a,b}s_{b,a}=s_{b,a}\mathfrak{S}_{a+1,b+1}$ and that $s_{b,a}$ is a distinguished $(\mathfrak{S}_{a,b},\mathfrak{S}_{a+1,b+1})$-double coset representative (in the sense of [**16**, Prop. 4.4], for example). Therefore, if $w\in\mathfrak{S}_{a,b}$ and $v=s_{b,a}ws_{b,a}\in\mathfrak{S}_{a+1,b+1}$ then $T_wT_{b,a}=T_{ws_{b,a}}=T_{s_{b,a}v}=T_{b,a}T_v$ by [**16**, Prop. 3.3]. This implies the Lemma. □

(3.4). Lemma. *Suppose that* $\boldsymbol{\lambda}\in\Lambda^+_{\ell,n}$ *and* $\boldsymbol{\nu}=\boldsymbol{\lambda}\cup\beta$, *where* $\beta=(r,c,e)$ *is an addable node of* $\boldsymbol{\lambda}$. *Then* $T_{n-1,a+1}m_{\boldsymbol{\nu}}\in m_{\boldsymbol{\lambda}}\mathscr{H}_{n+1}$, *where* $a=a^{\boldsymbol{\lambda}}_1+\dots+a^{\boldsymbol{\lambda}}_e+\lambda^{(e)}_1+\dots+\lambda^{(e)}_r$.

Proof. Let $D_{d,a}=1+T_a+T_{a,a-1}+\dots+T_{a,d}$, where $d=a-\lambda^{(e)}_r+1$. Then $D_{d,a}$ is the sum of distinguished right coset representatives for $\mathfrak{S}_{d,a}$ in $\mathfrak{S}_{d,a+1}$. Therefore, $x_{\boldsymbol{\lambda}}T_{n-1,a+1}D_{d,a}=T_{n-1,a+1}x_{\boldsymbol{\nu}}$ by Lemma **(3.3)**. On the other hand, it follows directly from the definitions that $u^+_{\boldsymbol{\nu}}=u^+_{\boldsymbol{\lambda}}(L_{a^{\boldsymbol{\lambda}}_\ell+1}-Q_\ell)\dots(L_{a^{\boldsymbol{\lambda}}_{e+1}+1}-Q_{e+1})$. Therefore, writing $m_{\boldsymbol{\lambda}}=x_{\boldsymbol{\lambda}}u^+_{\boldsymbol{\lambda}}$ and using **(2.1)** we see that

$$m_{\boldsymbol{\lambda}}\Big(\prod_{s=\ell,\dots,e+1}T_{a^{\boldsymbol{\lambda}}_{s+1},a^{\boldsymbol{\lambda}}_s+1}(L_{a^{\boldsymbol{\lambda}}_s+1}-Q_s)\Big)T_{a^{\boldsymbol{\lambda}}_{e+1},a+1}D_{d,a}=T_{n-1,a+1}m_{\boldsymbol{\nu}},$$

where the product on the left-hand side is read in order, from left to right, with decreasing values of s. (Recall that, for convenience, $a^{\boldsymbol{\lambda}}_{\ell+1}=n-1$ and $T_{n-1,n}=1$.) □

Let $\le$ be the Bruhat order on $\mathfrak{S}_n$; see, for example, [**16**, p.30]. If S is a semistandard $\boldsymbol{\lambda}$-tableau of type $\boldsymbol{\mu}$ let $\dot{\mathsf{S}}$ be the unique standard $\boldsymbol{\lambda}$-tableau such that $\boldsymbol{\mu}(\dot{\mathsf{S}})=\mathsf{S}$ and $d(\dot{\mathsf{S}})\le d(\mathfrak{s})$ whenever $\mathfrak{s}\in\mathcal{T}^{\text{Std}}(\boldsymbol{\lambda})$ and $\boldsymbol{\mu}(\mathfrak{s})=\mathsf{S}$. Such a tableau $\dot{\mathsf{S}}$ exists by [**13**, Lemma 3.9].

(3.5). Lemma. *Suppose that* $\mathsf{S}\in\mathcal{T}^{\text{SStd}}_{\boldsymbol{\mu}}(\boldsymbol{\lambda})$ *and that* $\mathsf{U}\in\mathcal{T}^{\text{SStd}}_{\boldsymbol{\mu}\cup\omega}(\mathsf{S})$. *Let* $\boldsymbol{\nu}=\operatorname{Shape}(\mathsf{U})$. *Then* $m_{\mathsf{U}\mathfrak{t}^{\boldsymbol{\nu}}}\in m_{\mathsf{S}\mathfrak{t}^{\boldsymbol{\lambda}}}\mathscr{H}_{n+1}$.

Proof. Definition, $m_{\mathsf{S}\mathfrak{t}^{\boldsymbol{\lambda}}} = \sum_{\mathfrak{s}} m_{\mathfrak{s}\mathfrak{t}^{\boldsymbol{\lambda}}}$ where $d(\mathfrak{s})$ runs over a set of right $\mathfrak{S}_{\boldsymbol{\mu}}$-coset representatives in the double coset $\mathfrak{S}_{\boldsymbol{\lambda}} d(\dot{\mathsf{S}})\mathfrak{S}_{\boldsymbol{\mu}}$. Therefore, $m_{\mathsf{S}\mathfrak{t}^{\boldsymbol{\lambda}}} = h_{\mathsf{S}} T^*_{d(\dot{\mathsf{S}})} m_{\boldsymbol{\lambda}}$ for some $h_{\mathsf{S}} \in \mathscr{H}_q(\mathfrak{S}_{\boldsymbol{\mu}})$. (Explicitly, $h_{\mathsf{S}} = \sum_d T_d$ where d runs over the set of distinguished left coset representatives of $\mathfrak{S}_{\boldsymbol{\mu}} \cap d(\dot{\mathsf{S}})\mathfrak{S}_{\boldsymbol{\lambda}} d(\dot{\mathsf{S}})^{-1}$ in $\mathfrak{S}_{\boldsymbol{\mu}}$.)

As in Lemma **(3.4)**, write $\boldsymbol{\nu} = \boldsymbol{\lambda} \cup \beta$, where $\beta = (r, c, e)$ and set $a = a^{\boldsymbol{\lambda}}_1 + \cdots + a^{\boldsymbol{\lambda}}_e + \lambda^{(e)}_1 + \cdots + \lambda^{(e)}_r$. Then $\mathsf{U} = \mathsf{S} \cup \beta$. Therefore, $d(\dot{\mathsf{U}}) = s^{-1}_{n-1,a+1} d(\dot{\mathsf{S}})$, so that $m_{\mathsf{U}\mathfrak{t}^{\boldsymbol{\nu}}} = h T^*_{d(\dot{\mathsf{S}})} T_{n-1,a+1} m_{\boldsymbol{\nu}}$.

Finally, $T_{n-1,a+1} m_{\boldsymbol{\nu}} = m_{\boldsymbol{\lambda}} h_{\boldsymbol{\nu},a}$, for some $h_{\boldsymbol{\nu},a} \in \mathscr{H}_{n+1}$, by Lemma **(3.4)**. Therefore,

$$m_{\mathsf{U}\mathfrak{t}^{\boldsymbol{\nu}}} = h_{\mathsf{S}} T^*_{d(\dot{\mathsf{S}})} T_{n-1,a+1} m_{\boldsymbol{\nu}} = h_{\mathsf{S}} T^*_{d(\dot{\mathsf{S}})} m_{\boldsymbol{\lambda}} h_{\boldsymbol{\nu},a} = m_{\mathsf{S}\mathfrak{t}^{\boldsymbol{\lambda}}} h_{\boldsymbol{\nu},a} \in m_{\mathsf{S}\mathfrak{t}^{\boldsymbol{\lambda}}} \mathscr{H}_{n+1},$$

as required. □

We can now make the filtration of Lemma **(3.1)**(b) explicit. As a result we will show that we can obtain a basis for the induced module by adding a node labeled (z, ℓ) to the basis elements of $M(\boldsymbol{\mu})$ in all possible ways.

(3.6). **Theorem.** *Suppose that $\boldsymbol{\mu} \in \Lambda^+_{\ell,n}$ and order $\mathcal{T}^{SStd}_{\boldsymbol{\mu}}(\Lambda^+_{\ell,n}) = \{\mathsf{S}_1, \dots, \mathsf{S}_m\}$ as above, with $\boldsymbol{\lambda}_i = \operatorname{Shape}(\mathsf{S}_i)$. Let N_i be the R-submodule of $M(\boldsymbol{\mu} \cup \omega)$ spanned by the elements*

$$\{\, m_{\mathsf{U}\mathfrak{v}} \mid \mathsf{U} \in \mathcal{T}^{SStd}_{\boldsymbol{\mu}\cup\omega}(\mathsf{S}_j), \mathfrak{v} \in \mathcal{T}^{Std}(\operatorname{Shape}(\mathsf{U})) \text{ for } 1 \le j \le i \,\},$$

for $i = 0, 1, \dots, m$. Then N_i is an $\mathscr{H}_{n+1}$-submodule of $\operatorname{Ind} M(\boldsymbol{\lambda})$ and

$$\operatorname{Ind} S(\boldsymbol{\lambda}_i) \cong N_i / N_{i-1},$$

for $1 \le i \le m$.

Proof. By Lemma **(3.1)**(a), $\operatorname{Ind} M(\boldsymbol{\mu}) = M(\boldsymbol{\mu} \cup \omega)$ and by **(3.2)** the set of elements

$$\{\, m_{\mathsf{U}\mathfrak{v}} \mid \mathsf{U} \in \mathcal{T}^{\mathrm{SStd}}_{\boldsymbol{\mu}\cup\omega}(\mathsf{S}_j), \mathfrak{v} \in \mathcal{T}^{\mathrm{Std}}(\operatorname{Shape}(\mathsf{U})) \text{ for } 1 \le j \le m \,\}$$

is precisely the basis of $M(\boldsymbol{\mu} \cup \omega)$ given by **(2.3)**, so $M(\boldsymbol{\mu} \cup \omega) = N_m$. Moreover, since $\mathscr{H}_{n+1}(\boldsymbol{\nu})$ is a two-sided ideal of $\mathscr{H}_{n+1}$ for all $\boldsymbol{\nu} \in \Lambda^+_{\ell,n+1}$, the action of $\mathscr{H}_{n+1}$ on the basis $\{m_{\mathsf{U}\mathfrak{v}}\}$ respects dominance, so N_i is a submodule of $M(\boldsymbol{\mu} \cup \omega)$, for $0 \le i \le m$.

Recall the filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_m = M(\boldsymbol{\lambda})$ of $M(\boldsymbol{\lambda})$ given in **(2.3)**. By Lemma **(3.1)**(b), to prove the Theorem it is enough to show by induction on i that $\operatorname{Ind} M_i = N_i$, for $0 \le i < m$. This is trivially true when $i = 0$ so we may assume that $i > 0$.

To show that $\operatorname{Ind} M_i \subseteq N_i$ note that $m_{\mathsf{S}_i\mathfrak{t}^{\boldsymbol{\lambda}_i}} \in N_i$ and $m_{\mathsf{S}_i\mathfrak{t}^{\boldsymbol{\lambda}_i}} + M_{i-1}$ generates M_i / M_{i-1} as an $\mathscr{H}_n$-module. Therefore, $\operatorname{Ind} M_i \subseteq N_i$ by induction on i.

To prove the reverse inclusion, suppose that $\mathsf{U} \in \mathcal{T}^{\mathrm{SStd}}_{\boldsymbol{\mu}\cup\omega}(\mathsf{S}_i)$ and let $\boldsymbol{\nu} = \operatorname{Shape}(\mathsf{U})$. Then $m_{\mathsf{U}\mathfrak{t}^{\boldsymbol{\nu}}} \in m_{\mathsf{S}_i\mathfrak{t}^{\boldsymbol{\lambda}_i}} \mathscr{H}_{n+1} \subseteq \operatorname{Ind} M_i$ by Lemma **(3.5)**. Therefore, $m_{\mathsf{U}\mathfrak{v}} \in \operatorname{Ind} M_i$, for any $\mathfrak{v} \in \mathcal{T}^{\mathrm{Std}}(\boldsymbol{\nu})$. It follows by induction that $N_i \subseteq \operatorname{Ind} M_i$ as required. □

For each addable node β of $\boldsymbol{\mu}$ let N^β be the submodule of $M(\boldsymbol{\mu} \cup \omega)$ spanned by

$$\{\, m_{\mathsf{U}\mathfrak{v}} \mid \mathsf{U} \in \mathcal{T}^{\mathrm{SStd}}_{\boldsymbol{\mu}\cup\omega}(\boldsymbol{\lambda}), \mathfrak{v} \in \mathcal{T}^{\mathrm{Std}}(\boldsymbol{\lambda}) \text{ where } \boldsymbol{\lambda} \in \Lambda^+_{\ell,n+1} \text{ and } \boldsymbol{\lambda} \rhd \boldsymbol{\mu} \cup \beta \,\} + N_{m-1},$$

where N_{m-1} is the submodule of $M(\boldsymbol{\mu} \cup \omega)$ defined in Theorem **(3.6)**. Note, in particular, that $N^\alpha = N_{m-1}$.

We can now prove a more explicit version of the Main Theorem of this paper.

(3.7). **Corollary.** *Suppose that $\boldsymbol{\mu}$ is a multipartition of n and let $\alpha_1=\alpha>\dots>\alpha_a=\omega$ be the addable nodes of $\boldsymbol{\mu}$. Then $\operatorname{Ind} S(\boldsymbol{\mu})\cong M(\boldsymbol{\mu}\cup\omega)/N^\alpha$ is a free R-module with basis*

$$\{\, m_{\mathsf{U}\mathfrak{v}}+N^\alpha \mid \mathsf{U}\in\mathcal{T}^{\mathit{SStd}}_{\boldsymbol{\mu}\cup\omega}(\boldsymbol{\mu}\cup\alpha_j), \mathfrak{v}\in\mathcal{T}^{\mathit{Std}}(\boldsymbol{\mu}\cup\alpha_j),\ \textit{for } 1\le j\le a\,\}\,.$$

In particular, $\operatorname{Ind} S(\boldsymbol{\mu})$ has a filtration $0=I_0\subset I_1\subset\dots\subset I_a=\operatorname{Ind} S(\boldsymbol{\mu})$ such that $I_j/I_{j-1}\cong S(\boldsymbol{\mu}\cup\alpha_j)$, for $j=1,\dots,a$.

Proof. That $\operatorname{Ind} S(\boldsymbol{\mu})\cong M(\boldsymbol{\mu}\cup\omega)/N^\alpha$ is a special case of Theorem **(3.6)**. The second claim follows from **(2.3)** by setting $I_j=N^{\alpha_{j+1}}/N^\alpha$, for $0\le j<a$. To prove that $S(\boldsymbol{\mu}\cup\alpha_j)\cong I_j/I_{j-1}$, for $1\le j\le a$, observe that the bijective map

$$S(\boldsymbol{\mu}\cup\alpha_j)\longrightarrow I_j/I_{j-1}; m_{\mathfrak{s}}\mapsto m_{(\mathsf{T}^{\boldsymbol{\mu}}\cup\alpha_j)\mathfrak{s}}+I_{j-1}, \qquad \text{for } \mathfrak{s}\in\mathcal{T}^{\text{Std}}(\boldsymbol{\mu}\cup\alpha_j),$$

commutes with the action of $\mathscr{H}_{n+1}$. (Here, $\mathsf{T}^{\boldsymbol{\mu}}=\boldsymbol{\mu}(\mathfrak{t}^{\boldsymbol{\mu}})$ is the unique semistandard $\boldsymbol{\mu}$-tableau of type $\boldsymbol{\mu}$.) □

(3.8). **Remark.** Maintain the notation of Theorem **(3.6)** and define integers a_i and multipartitions $\boldsymbol{\lambda}_{i,j}$ by writing $\{\boldsymbol{\lambda}_{i,1}\rhd\dots\rhd\boldsymbol{\lambda}_{i,a_i}\}=\{\,\operatorname{Shape}(\mathsf{U})\mid\mathsf{U}\in\mathcal{T}^{\text{SStd}}_{\boldsymbol{\mu}\cup\omega}(\mathsf{S}_i)\,\}$, for $i=1,\dots,m$. Theorem **(3.6)** then implies, just as in the proof of Corollary **(3.7)**, that $M(\boldsymbol{\mu}\cup\omega)$ has a Specht filtration

$$0\subset I_{1,1}\subset\dots\subset I_{1,a_1}\subset I_{2,1}\subset\dots\subset I_{m,a_m}=M(\boldsymbol{\mu}\cup\omega),$$

with $I_{i,a}/I^<_{i,a}\cong S(\boldsymbol{\lambda}_{i,a})$, where $I_{i,a}$ is the submodule of $M(\boldsymbol{\mu}\cup\omega)$ with basis

$$\{\, m_{\mathsf{U}\mathfrak{v}}\mid\mathsf{U}\in\mathcal{T}^{\text{SStd}}_{\boldsymbol{\mu}\cup\omega}(\boldsymbol{\lambda}_{j,b}), \mathfrak{v}\in\mathcal{T}^{\text{Std}}(\boldsymbol{\lambda}_{j,b}) \text{ where } j<i, \text{ or } j=i \text{ and } b\le a\,\}$$

and where $I^<_{i,a}=I_{i,a-1}$ if $a>1$, $I^<_{i,1}=I_{i-1,a_{i-1}}$ if $i>1$ and $I^<_{1,1}=0$.

Fred Goodman has pointed out that this filtration of $M(\boldsymbol{\mu}\cup\omega)$ is, in general, different to that given by **(2.3)** because the order in which the Specht modules appear does not have to be compatible with the dominance ordering–note, however, that the Specht modules in each 'layer' N_i/N_{i-1} are totally ordered by dominance. For example, suppose that $\ell=1$ and let $\mu=(3^2,1)$ so that $\mu\cup\alpha=(4,3,1)$ and $\mu\cup\omega=(3^2,1^2)$. Then

$$\mathsf{U}=\begin{array}{|c|c|c|c|}\hline 1&1&1&2\\\hline 2&2\\\cline{1-2} 3&4\\\cline{1-2}\end{array}$$

is a semistandard ν-tableau of type $\mu\cup\omega$, where $\nu=(4,2^2)$. (As $\ell=1$ we can label semistandard tableaux with the integers $1,\dots,n$.) However, $\mu\cup\alpha\rhd\nu$ even though $\nu\ne\mu\cup\beta$ for any addable node β of μ.

As induction and restriction are both exact functors the main result of this note, together with [**2**, Prop. 1.9] (and the corresponding argument for the degenerate case), shows that the full subcategory of $\mathscr{H}_n$-**mod** which consists of modules which have a Specht filtration is closed under induction and restriction.

(3.9). **Corollary.** *Suppose that M has a Specht filtration. Then the modules $\operatorname{Res} M$ and $\operatorname{Ind} M$ both have Specht filtrations.*

In [**17**, Theorem 3.6] and [**6**, Theorem 4.6] it is shown that for each multipartition $\boldsymbol{\mu}\in\Lambda^+_{\ell,n}$ there exists an indecomposable $\mathscr{H}_n$-module $Y(\boldsymbol{\mu})$, a **Young module**, such that

$$M(\boldsymbol{\mu})\cong Y(\boldsymbol{\mu})\oplus\bigoplus_{\boldsymbol{\lambda}\rhd\boldsymbol{\mu}}Y(\boldsymbol{\lambda})^{\oplus c_{\boldsymbol{\lambda}\boldsymbol{\mu}}}$$

for some non-negative integers $c_{\boldsymbol{\lambda}\boldsymbol{\mu}}$. Each Young module $Y(\boldsymbol{\mu})$ has a Specht filtration. Therefore, by Corollary **(3.9)**, $\operatorname{Res} Y(\boldsymbol{\mu})$ and $\operatorname{Ind} Y(\boldsymbol{\mu})$ both have Specht filtrations.

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REFERENCES

[1] S. ARIKI AND K. KOIKE, *A Hecke algebra of* $(\mathbf{Z}/r\mathbf{Z}) \wr \mathfrak{S}_n$ *and construction of its irreducible representations*, Adv. Math., **106** (1994), 216–243.

[2] S. ARIKI AND A. MATHAS, *The number of simple modules of the Hecke algebras of type* $G(r, 1, n)$, Math. Zeit., **233** (2000), 601–623.

[3] S. ARIKI, A. MATHAS, AND H. RUI, *Cyclotomic Nazarov–Wenzl algebras*, Nagoya J. Math., **182** (2006), 47–134. (Special issue in honour of George Lusztig).

[4] J. BRUNDAN, *Centers of degenerate cyclotomic Hecke algebras and parabolic category* $\mathcal{O}$, Represent. Theory, **12** (2008), 236–259.

[5] J. BRUNDAN AND A. KLESHCHEV, *Blocks of cyclotomic Hecke algebras and Khovanov-Lauda algebras*, preprint, 2008. arXiv:math.RT/0808.2032.

[6] ———, *The degenerate analogue of Ariki's categorification theorem*, preprint, 2008. arXiv:math.RT/0901.0057.

[7] ———, *Schur-Weyl duality for higher levels*, Selecta Math., **14** (2008), 1–57.

[8] J. BRUNDAN, A. KLESHCHEV, AND W. WANG, *Graded Specht modules*, preprint, 2009. arXiv:math.RT/0901.0218.

[9] R. DIPPER AND G. JAMES, *Representations of Hecke algebras of general linear groups*, Proc. Lond. Math. Soc. (3), **52** (1986), 20–52.

[10] R. DIPPER, G. JAMES, AND A. MATHAS, *Cyclotomic q–Schur algebras*, Math. Z., **229** (1999), 385–416.

[11] R. DIPPER AND A. MATHAS, *Morita equivalences of Ariki–Koike algebras*, Math. Zeit., **240** (2002), 579–610.

[12] G. D. JAMES, *The representation theory of the symmetric groups*, SLN, **682**, Springer–Verlag, New York, 1978.

[13] G. D. JAMES AND A. MATHAS, *The Jantzen sum formula for cyclotomic q–Schur algebras*, Trans. Amer. Math. Soc., **352** (2000), 5381–5404.

[14] A. S. KLESHCHEV, *Linear and projective representations of symmetric groups*, CUP, 2005.

[15] S. LYLE AND A. MATHAS, *Blocks of cyclotomic Hecke algebras*, Adv. Math., **216** (2007), 854–878.

[16] A. MATHAS, *Hecke algebras and Schur algebras of the symmetric group*, Univ. Lecture Notes, **15**, Amer. Math. Soc., 1999.

[17] ———, *Tilting modules for cyclotomic Schur algebras*, J. Reine Angew. Math., **562** (2003), 137–169.

[18] ———, *The representation theory of the Ariki-Koike and cyclotomic q-Schur algebras*, in Representation theory of algebraic groups and quantum groups, Adv. Stud. Pure Math., **40**, Math. Soc. Japan, Tokyo, 2004, 261–320.

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