COCOMPACT LATTICES OF MINIMAL COVOLUME IN RANK 2 KAC–MOODY GROUPS, PART II

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Abstract. Let G be a topological Kac–Moody group of rank 2 with symmetric Cartan matrix, defined over a finite field $\mathbb{F}_q$. An example is $G = SL(2, \mathbb{F}_q((t^{-1})))$. We determine a positive lower bound on the covolumes of cocompact lattices in G, and construct a cocompact lattice $\Gamma_0 < G$ which realises this minimum. This completes the work begun in Part I, which considered the cases when G admits an edge-transitive lattice.

Introduction

A classical theorem of Siegel [8] states that the minimum covolume among lattices in $G = SL_2(\mathbb{R})$ is $\frac{\pi}{21}$, and determines the lattice which realises this minimum. In the nonarchimedean setting, Lubotzky [6] and Lubotzky–Weigel [7] constructed the lattice of minimal covolume in $G = SL_2(K)$, where K is the field $\mathbb{F}_q((t^{-1}))$ of formal Laurent series over $\mathbb{F}_q$.

The group $G = SL_2(\mathbb{F}_q((t^{-1})))$ has, in recent developments, been viewed as the first example of a topological Kac–Moody group of rank 2 over the finite field $\mathbb{F}_q$. Such Kac–Moody groups are locally compact, totally disconnected topological groups, which may be thought of as infinite-dimensional analogues of semisimple algebraic groups. The so-called affine case $G = SL_2(\mathbb{F}_q((t^{-1})))$ is actually quite special among such Kac–Moody groups, since it is the only case in which there is a linear representation.

Our main result is Theorem 1 below. The groups G in this statement are *topological* Kac–Moody groups, meaning that each such G is the completion of a minimal Kac–Moody group Λ with respect to some topology. We use the completion in the 'building topology', discussed in [4]. The groups G in our result have Bruhat–Tits building a regular tree X, and the kernel of the G–action on X is the finite group $Z(G)$, the centre of G (see [4]). We recall in Section 1.3 below that if Γ is a cocompact lattice in G, then Γ acts on X cocompactly and with finite vertex stabilisers. Moreover, the Haar measure μ on G may be normalised so that the covolume $\mu(\Gamma\backslash G)$ is equal to $\sum |\Gamma_y|^{-1}$, where this sum is over the finitely many vertices y in a fundamental domain $Y \subset X$ for Γ. Using this normalisation, we obtain the following.

Theorem 1. *Let G be a topological Kac–Moody group of rank 2 defined over the finite field $\mathbb{F}_q$, with symmetric generalised Cartan matrix* $\begin{pmatrix} 2 & -m \\ -m & 2 \end{pmatrix}$, $m \geq 2$. *Then for $q \geq 514$*

$$\min\{\mu(\Gamma\backslash G) \mid \Gamma \text{ a cocompact lattice in } G\} = \frac{2}{(q+1)|Z(G)|\delta}$$

where $\delta \in \{1, 2, 4\}$ (depending upon the particular group G). Moreover, we construct a cocompact lattice $\Gamma_0 < G$ which realises this minimum.

The action of G on its Bruhat–Tits tree X is edge-transitive and induces the graph of groups

$\mathbb{G} =$

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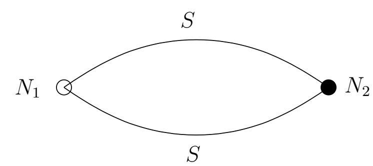
where P_1 and P_2 are the standard parabolic/parahoric subgroups of G, and $B = P_1 \cap P_2$ is the standard Borel/Iwahori subgroup. In Part I of this work, we classified the edge-transitive cocompact lattices in G (see Theorems 1 and 2 of [3]). We then proved for $q \geq 514$ that in the cases when G admits an edge-transitive lattice, the cocompact lattice of minimal covolume in G is edge-transitive (Theorem 3 of [3]). Hence in these cases, the lattice Γ_0 in Theorem 1 above appeared in our classification.

In order to prove Theorem 1 above in the cases where G does *not* admit any edge-transitive lattice, we first, in Section 3 below, construct a cocompact lattice $\Gamma_0 < G$ which acts on the tree X inducing a graph of groups of the form

$\mathbb{A} =$



The finite groups S, N_1 and N_2 will be defined in Section 3 below. Our construction generalises Example (6.2) of Lubotzky–Weigel [7].

In Section 4 below, we compute the covolume of the cocompact lattice Γ_0 to be $2/(q+1)|Z(G)|\delta$. We then complete the proof of Theorem 1 by showing that Γ_0 is the cocompact lattice in G of minimal covolume. A key ingredient here is Proposition 5 of our previous work [3], which concerns p–torsion in G (where $q = p^a$ with p prime), and which we restate in Section 4 below.

The lattice Γ_0 that we construct in Section 3 below is the only known cocompact lattice in such rank 2 complete Kac–Moody groups G, except for the free Schottky groups constructed by Carbone–Garland in [5].

Several methods can be used to show that the fundamental group Γ_0 of the graph of groups $\mathbb{A}$ above embeds as a cocompact lattice in G. One can achieve this by generalising the criterion of Lubotzky and Weigel in [7]. Alternatively this result can be obtained by explicitly constructing a covering of graphs of groups (see Bass [1]). After verifying that both of these methods worked, we realised that a third approach was possible. In Section 2 below we provide our own embedding criterion, which applies to many locally compact groups G acting on the edges of a regular tree X with fundamental domain an edge $f = [x_1, x_2]$. For $i = 1, 2$, denote by $E_X(x_i)$ the set of edges of X which are adjacent to the vertex x_i. We prove the following sufficient condition for the fundamental group of a graph of groups with two vertex groups A_1 and A_2 acting on respectively $E_X(x_1)$ and $E_X(x_2)$ with the same number of orbits to embed as a lattice in G.

Proposition 2. *Suppose that there are finite groups $A_1 \leq G_{x_1}$ and $A_2 \leq G_{x_2}$, and a positive integer n, such that:*

(1) *for $i = 1, 2$, the group A_i has n orbits of equal size on $E_X(x_i)$;*
(2) *there are representatives $f = [x_1, x_2] = f_1, f_2, \ldots, f_n$ of the orbits of A_1 on $E_X(x_1)$, and elements $1 = g_1, g_2, \ldots, g_n \in G_{x_1}$, and representatives $[x_2, x_1] = \hat{f}_1, \hat{f}_2, \ldots, \hat{f}_n$ of the orbits of A_2 on $E_X(x_2)$, and elements $1 = \hat{g}_1, \hat{g}_2, \ldots, \hat{g}_n \in G_{x_2}$, such that for $j = 1, \ldots, n$:*
   (a) *$g_j \cdot f_1 = f_j$ and $\hat{g}_j \cdot \hat{f}_1 = \hat{f}_j$;*
   (b) *$A_1 \cap G_f^{g_j} = A_1 \cap A_2 = A_2 \cap G_f^{\hat{g}_j}$; and*
   (c) *$(A_1 \cap A_2)^{g_j} = (A_1 \cap A_2)^{\hat{g}_j}$.*

Let A be the graph with two vertices a_1 and a_2 and n edges connecting a_1 to a_2. (The case $n = 2$ is sketched above.) Then there is a graph of groups $\mathbb{A}$ over A with vertex group A_i at a_i for $i = 1, 2$, and all edge groups $A_1 \cap A_2$, such that the fundamental group of the graph of groups $\mathbb{A}$ is a cocompact lattice Γ in G, with quotient $A = \Gamma \backslash X$.

Our criterion applies in particular to the Kac–Moody groups G of our main result, Theorem 1 above. The constructions of edge-transitive lattices in [6] and in [3] are a particular case. In [7], the covering theory developed for graphs of groups by Bass in [1] was employed. We are able to provide somewhat simpler proofs by using covering theory for complexes of groups, which was developed more recently by Bridson–Haefliger [2], and in which the notion of morphism is less complicated.

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1. Preliminaries

We recall necessary definitions and fix notation concerning graphs, in Section 1.1, and graphs of groups, in Section 1.2. Lattices in groups acting on trees are discussed briefly in Section 1.3 below, and then in Section 1.4 below we recall necessary definitions and a result from covering theory for graphs of groups. For all relevant terminology, notation and results for Kac–Moody groups, we refer the reader to our earlier work [3].

1.1. **Graphs.** Let A be a connected graph, with sets VA of vertices and EA of oriented edges. The initial and terminal vertices of $e \in EA$ are denoted by $\partial_0 e$ and $\partial_1 e$ respectively. The map $e \mapsto \overline{e}$ is orientation reversal, with $\overline{\overline{e}} = e$ and $\partial_{1-j}\overline{e} = \partial_j e$ for $j = 0, 1$ and all $e \in EA$. Given a vertex $a \in VA$, we denote by $E_A(a)$ the set of edges

$$E_A(a) := \{e \in EA \mid \partial_0 e = a\}$$

with initial vertex a.

Let A and B be graphs. A *morphism of graphs* is a function $\theta : A \to B$ taking vertices to vertices and edges to edges, such that for every edge $e \in EA$, $\theta(\overline{e}) = \overline{\theta(e)}$ and $\theta(\partial_i(e)) = \partial_i(\theta(e))$ for $i = 1, 2$.

1.2. **Bass–Serre theory.** A *graph of groups* $\mathbb{A} = (A, \mathcal{A})$ over a connected graph A consists of an assignment of vertex groups $\mathcal{A}_a$ for each $a \in VA$ and edge groups $\mathcal{A}_e = \mathcal{A}_{\overline{e}}$ for each $e \in EA$, together with monomorphisms $\alpha_e : \mathcal{A}_e \to \mathcal{A}_{\partial_0 e}$ for each $e \in EA$.

Let X be a tree. A group Γ is said to act on X *without inversions* if for all edges $e \in EX$ and all $g \in \Gamma$, $g \cdot e \neq \overline{e}$. Any action of a group Γ on a tree X without inversions induces a graph of groups over the quotient graph $A = \Gamma\backslash X$. See for example [1] for the definitions of the *fundamental group* $\pi_1(\mathbb{A}, a_0)$ and the *universal cover* $X = \widetilde{(A, a_0)}$ of a graph of groups $\mathbb{A} = (A, \mathcal{A})$, with respect to a basepoint $a_0 \in VA$. The universal cover X is a tree, on which $\pi_1(\mathbb{A}, a_0)$ acts by isometries inducing a graph of groups isomorphic to $\mathbb{A}$.

1.3. **Lattices in groups acting on trees.** Let X be a locally finite tree and let G be a cocompact group of automorphisms of X, which acts without inversions and with compact open vertex stabilisers. As recalled in our Part I [3], a subgroup $\Gamma < G$ is discrete if and only if it acts on X with finite vertex stabilisers, and the Haar measure μ on G may be normalised so that the covolume of a discrete $\Gamma < G$ is $\mu(\Gamma\backslash G) = \sum |\Gamma_y|^{-1}$ where the sum is over the vertices y of $Y \subset X$ a fundamental domain for Γ. Moreover, a discrete subgroup $\Gamma < G$ is a cocompact lattice in G if and only if the graph $\Gamma\backslash X$ is finite. Hence the graph of groups induced by a cocompact lattice $\Gamma < G$ will be a finite graph of finite groups.

1.4. **Definitions and a result from covering theory.** We adapt definitions from covering theory for complexes of groups (Chapter III.$\mathcal{C}$ of [2]) to graphs of groups, and recall a necessary result from covering theory. For the precise relationship between the category of graphs of groups and the category of complexes of groups over 1–dimensional spaces, see Proposition 2.1 of [9].

Definition 1 (Morphism of graphs of groups)**.** *Let $\mathbb{A} = (A, \mathcal{A})$ and $\mathbb{B} = (B, \mathcal{B})$ be graphs of groups, with monomorphisms from edge groups to vertex groups respectively $\alpha_e : \mathcal{A}_e \to \mathcal{A}_{\partial_0 e}$ for $e \in EA$ and $\beta_e : \mathcal{B}_f \to \mathcal{B}_{\partial_0 f}$ for $f \in EB$. Let $\theta : A \to B$ be a morphism of graphs.* A morphism of graphs of groups *$\Phi : \mathbb{A} \to \mathbb{B}$ over θ is given by:*

(1) *a homomorphism $\phi_x : \mathcal{A}_x \to \mathcal{B}_{\theta(x)}$ of groups, for every $x \in VA \cup EA$; and*

(2) *an element* $\phi(e) \in \mathcal{B}_{\partial_0(\theta(e))}$ *for each* $e \in EA$ *such that the following diagram commutes, where* $a = \partial_0 e$*:*

$$\begin{tikzcd} \mathcal{A}_e \arrow[r, "\phi_e"'] \arrow[d, "\alpha_e"] & \mathcal{B}_{\theta(e)} \arrow[d, "\mathrm{ad}(\phi(e))\circ\beta_{\theta(e)}"] \\ \mathcal{A}_a \arrow[r, "\phi_a"'] & \mathcal{B}_{\theta(a)} \end{tikzcd}$$

Definition 2 (Covering of graphs of groups)**.** *With notation as in Definition 1 above,* $\Phi : \mathbb{A} \to \mathbb{B}$ *is a* covering of graphs of groups *if in addition:*

(1) *for each* $x \in VA \cup EA$ *the homomorphism* $\phi_x : \mathcal{A}_x \to \mathcal{B}_{\theta(x)}$ *is injective; and*
(2) *for each edge* $f \in EB$ *and each vertex* $a \in VA$ *with* $\partial_0 f = b = \theta(a)$*, the map*

$$\Phi_{a/f} : \coprod_{e \in E_A(a) \cap \theta^{-1}(f)} \mathcal{A}_a/\alpha_e(\mathcal{A}_e) \to \mathcal{B}_b/\beta_f(\mathcal{B}_f)$$

induced by $g \mapsto \phi_a(g)\phi(e)$ *is bijective.*

The result from covering theory that we will need is:

Proposition 3 (Bass, Proposition 2.7 of [1])**.** *Let* $\mathbb{A} = (A, \mathcal{A})$ *and* $\mathbb{B} = (B, \mathcal{B})$ *be graphs of groups. Choose basepoints* $a_0 \in A$ *and* $b_0 \in B$*. If there is a covering of graphs of groups* $\Phi : \mathbb{A} \to \mathbb{B}$ *over* $\theta : A \to B$ *with* $\theta(a_0) = b_0$*, then* $\pi_1(\mathbb{A}, a_0)$ *injects into* $\pi_1(\mathbb{B}, b_0)$*.*

2. Embedding criterion

We now prove our embedding criterion, Proposition 4 below, which implies Proposition 2 of the introduction. We will apply Proposition 4 in Section 3 below to show that the fundamental group Γ_0 of the graph of groups $\mathbb{A}$ sketched in the introduction embeds as a cocompact lattice in the Kac–Moody group G of our main result, Theorem 1.

Our embedding criterion in fact applies to more general groups G. Let q be a positive integer and let X be the $(q+1)$–regular tree. Let G be any locally compact group of automorphisms of X, which acts on X without inversions, with compact open vertex stabilisers G_x for $x \in VX$, and with fundamental domain an edge $[x_1, x_2]$. Denote by P_i the stabiliser G_{x_i} for $i = 1, 2$, and let $B = P_1 \cap P_2$. For notational convenience, we denote by C the subgraph of X with vertex set $\{x_1, x_2\}$ and edge set $\{f, \overline{f}\}$, such that $\partial_0(f) = \partial_1(\overline{f}) = x_1$ and $\partial_1(f) = \partial_0(\overline{f}) = x_2$. Then G is the fundamental group of an edge of groups $\mathbb{G}$ over C, as sketched in the introduction.

For some integer $n \geq 1$ dividing $q + 1$, let $A = A_n$ be the graph with two vertices a_1 and a_2 and edge set $\{e_1, \ldots, e_n, \overline{e_1}, \ldots, \overline{e_n}\}$, so that $\partial_0(e_j) = \partial_1(\overline{e_j}) = a_1$ and $\partial_1(e_j) = \partial_0(\overline{e_j}) = a_2$. The case $n = 2$ is sketched in the introduction. We now state and prove a sufficient criterion for the fundamental group of a graph of groups over A to embed in G as a cocompact lattice.

Proposition 4. *Suppose that there are finite groups* $A_1 \leq P_1$ *and* $A_2 \leq P_2$ *such that:*

(1) *for* $i = 1, 2$*, the group* A_i *has* n *orbits of equal size on* $E_X(x_i)$*;*
(2) *there are representatives* $f = f_1, f_2, \ldots, f_n$ *of the orbits of* A_1 *on* $E_X(x_1)$*, and elements* $1 = g_1, g_2, \ldots, g_n \in P_1$*, and representatives* $\overline{f} = \hat{f}_1, \hat{f}_2, \ldots, \hat{f}_n$ *of the orbits of* A_2 *on* $E_X(x_2)$*, and elements* $1 = \hat{g}_1, \hat{g}_2, \ldots, \hat{g}_n \in P_2$*, such that for* $j = 1, \ldots, n$*:*
   (a) $g_j \cdot f_1 = f_j$ *and* $\hat{g}_j \cdot \hat{f}_1 = \hat{f}_j$*;*
   (b) $A_1 \cap B^{g_j} = A_1 \cap A_2 = A_2 \cap B^{\hat{g}_j}$*; and*
   (c) $(A_1 \cap A_2)^{g_j} = (A_1 \cap A_2)^{\hat{g}_j}$*.*

Let $\mathbb{A}$ *be the graph of groups over* A *with vertex groups* $\mathcal{A}_{a_i} = A_i$ *for* $i = 1, 2$*, and all edge groups* $\mathcal{A}_{e_j} = \mathcal{A}_{\overline{e_j}} = A_1 \cap A_2$*. Each monomorphism* α_{e_j} *from an edge group* $A_1 \cap A_2$ *into* A_1 *is inclusion composed with* $\mathrm{ad}(g_j \hat{g}_j^{-1})$*, and the monomorphisms* $\alpha_{\overline{e_j}}$ *from edge groups* $A_1 \cap A_2$ *into* A_2 *are inclusions.*

Then the fundamental group of the graph of groups $\mathbb{A}$ *is a cocompact lattice in* G*, with quotient* A*.*

Proof. We will construct a covering of graphs of groups $\Phi : \mathbb{A} \to \mathbb{G}$. Since A is a finite graph and the vertex groups A_1 and A_2 are finite, it follows from our discussion of lattices in Section 1.3 above and Proposition 3 above that the fundamental group of $\mathbb{A}$ is a cocompact lattice in G with quotient the graph A.

Let $\theta : A \to C$ be the graph morphism given by $\theta(a_i) = x_i$ for $i = 1, 2$, and $\theta(e_j) = f$ and $\theta(\overline{e_j}) = \overline{f}$ for $j = 1, \ldots, n$. We construct a morphism of graphs of groups $\Phi : \mathbb{A} \to \mathbb{G}$ over θ as follows. For $i = 1, 2$ let $\phi_{a_i} : \mathcal{A}_{a_i} \to P_i$ be the natural inclusion $A_i \hookrightarrow P_i$. For $j = 1, \ldots, n$ let $\phi_{e_j} : \mathcal{A}_{e_j} \to B$ be the composition of the natural inclusion $A_1 \cap A_2 \hookrightarrow B^{\hat{g}_j}$ with the map $\mathrm{ad}(\hat{g}_j^{-1}) : B^{\hat{g}_j} \to B$. Define $\phi(e_j) = g_j$ and $\phi(\overline{e_j}) = \hat{g}_j$. Then it may be checked that Φ is indeed a morphism of graphs of groups.

To show that Φ is a covering, we first show that the map

$$\Phi_{a_1/f} : \coprod_{j=1}^{n} \mathcal{A}_{a_1}/\alpha_{e_j}(\mathcal{A}_{e_j}) \to P_1/B$$

induced by $g \mapsto \phi_{a_1}(g)\phi(e_j) = gg_j$ for g representing a coset of $\alpha_{e_j}(\mathcal{A}_{e_j}) = (A_1 \cap A_2)^{g_j \hat{g}_j^{-1}} = A_1 \cap A_2$ in $\mathcal{A}_{a_1} = A_1$, is a bijection. For this, we note that since the edges $f_j = g_j \cdot f_1 = g_j \cdot f$ represent pairwise distinct A_1–orbits on $E_X(x_1)$, for all $g, h \in A_1$ and all $1 \leq j \neq j' \leq n$ the cosets gg_jB and $hg_{j'}B$ are pairwise distinct. The conclusion that $\Phi_{a_1/f}$ is a bijection then follows from the assumption that $A_1 \cap B^{g_j} = A_1 \cap A_2$.

The proof that the map

$$\Phi_{a_2/\overline{f}} : \coprod_{j=1}^{n} \mathcal{A}_{a_2}/\alpha_{\overline{e_j}}(\mathcal{A}_{\overline{e_j}}) \to P_2/B$$

is a bijection is similar. We conclude that $\Phi : \mathbb{A} \to \mathbb{G}$ is a covering of graphs of groups, as desired. □

3. Construction of the lattice Γ_0

Let G be as in Theorem 1 above, and assume that G does not admit any edge-transitive lattices. In this section we show that the fundamental group Γ_0 of the graph of groups $\mathbb{A}$ sketched in the introduction embeds as a cocompact lattice in G. We first in Section 3.1 define finite subgroups S, N_1 and N_2 of G and discuss their structure, then in Section 3.2 verify that our embedding criterion, Proposition 4 above, may be applied with $A_1 = N_1$ and $A_2 = N_2$.

3.1. **The groups S, N_1 and N_2.** As in our earlier work [3], for $i = 1, 2$ let P_i be a standard maximal parabolic/parahoric subgroup of G. Then P_i is the stabiliser in G of a vertex x_i of X, with $[x_1, x_2]$ an edge of X. Since G is rank 2 and has symmetric Cartan matrix, $P_1 \cong P_2$. Moreover, if L_i is a Levi complement of P_i, then $L_i = M_iT$ where $T \leq B \leq P_1 \cap P_2$ is a torus of G and $A_1(q) \cong M_i \triangleleft L_i$, where $A_1(q)$ is isomorphic to either $SL_2(q)$ or $PSL_2(q)$, depending upon G. Since by assumption G has no edge-transitive lattices, Theorem 1 of [3] implies that $q \equiv 1 \pmod 4$, and either $L_i/Z(L_i) \cong PSL_2(q)$, or if $L_i/Z(L_i) \cong PGL_2(q)$, $Q_i^0 \not\leq Z(G)$ where Q_i^0 is the unique subgroup of index 2 of the Sylow 2–subgroup of $Z(L_i)$.

For $i = 1, 2$ let H_i be a fixed non-split torus of M_i such that $N_T(H_i)$ is as big as possible. Then either $H_i \cong C_{\frac{q+1}{2}}$ or C_{q+1}, depending on whether $M_i \cong PSL_2(q)$ or $SL_2(q)$ respectively. Also, $N_T(H_i)/C_T(H_i) \cong C_2$ and $H_i \cap N_T(H_i) = Z(M_i)$. Define

$$S := N_T(H_1) \cap N_T(H_2).$$

Let us try to describe S in more definite terms. Let Q be the Sylow 2–subgroup of T (it is unique since T is abelian). First of all, let us notice that if $z \in N_T(H_i)$, $i = 1, 2$, is of odd order, then $[z, H_i] = 1$ and $[z, M_i] = 1$. Hence, if $z \in S$ and z is of odd order, $z \in C_G(\langle M_1, M_2 \rangle)$ thus $z \in Z(G)$. It follows immediately that $Z(G) \leq S \leq Z(G)Q$. Let us now investigate what happens when $z \in N_T(H_i) \cap Q$, $i = 1, 2$.

Take $x \in Q$ such that x normalises but not centralises H_i for some $i \in \{1, 2\}$. Then x acts on H_i as an element of order 2, and so x^2 centralises H_i. It follows that x^2 centralises M_i. Now consider $R_i := \{x \in Q \mid x^2 \in C_T(M_i)\}$. Then $R_i \leq Q$ and $R_i \leq N_T(H_i)$. Define

$$Q_0 := R_1 \cap R_2 = \{x \in Q \mid x^2 \in C_T(M_1) \cap C_T(M_2)\} = \{x \in Q \mid x^2 \in Z(G)\}$$

Clearly, $Q_0 \leq S$. On the other hand, take $s \in S \cap Q$. If $[s, H_i] = 1$ for both $i = 1, 2$, then $[s, M_i] = 1$ for $i = 1, 2$ implying $s \in Z(G) \cap Q \leq Q_0$. Let $s \in S \cap Q$ be such that $[s, H_i] \neq 1$ for some $i \in \{1, 2\}$. As noticed above, $s^2 \in C_T(H_i) \leq C_T(M_i)$. Hence, $s^2 \in C_T(M_j)$, $\{i, j\} = \{1, 2\}$. Therefore, $s^2 \in Z(G)$. Thus $S \cap Q \leq Q_0$. It follows that:

Lemma 5. $S = Z(G)Q_0$.

Notice that $|N_S(H_i) : C_S(H_i)| = 2$. We also define

$$N_1 := SH_1 \quad \text{and} \quad N_2 := SH_2.$$

3.2. **Application of embedding criterion.** By construction, for $i = 1, 2$, N_i is a finite subgroup of P_i, and $S = N_1 \cap N_2$. We now verify that our embedding criterion, Proposition 4 above, may be applied with $A_1 = N_1$ and $A_2 = N_2$.

Notice first that the intersection of N_i with an edge stabiliser in L_i is of index $\frac{q+1}{2}$. The Orbit-Stabiliser Theorem yields immediately that N_i has 2 orbits of equal size $\frac{1}{2}(q + 1)$ on $E_X(x_i)$. That is, with $n = 2$, condition (1) in the statement of Proposition 4 above holds.

Denote by f_1 the edge $[x_1, x_2]$ of X and by $\hat{f}_1$ the edge $[x_2, x_1]$. Choose an edge $f_2 \in E_X(x_1)$ so that the edges f_1 and f_2 represent the two N_1–orbits on $E_X(x_1)$, and choose an edge $\hat{f}_2 \in E_X(x_2)$ so that the edges $\hat{f}_1$ and $\hat{f}_2$ represent the two N_2–orbits on $E_X(x_2)$.

The edges f_1 and $\hat{f}_1$ are fixed by S, since $S \leq T \leq B = P_1 \cap P_2$. We claim that the edges f_2 and $\hat{f}_2$ may be chosen so that S fixes both f_2 and $\hat{f}_2$. To see this, consider first the action of N_1 on the edges $E_X(x_1)$. Now $N_1 \leq L_1$, and L_1 acts on the set $E_X(x_1)$ as on the points of projective line, i.e., we observe this action via a homomorphism $\phi : L_1 \to PGL_2(q)$. The kernel of this action is $\ker(\phi) = Z(L_1) = C_T(M_1)$. We know that N_1 has 2 orbits, say θ_1 and θ_2 in this action, each of length $\frac{q+1}{2}$, which is odd. Assume that the fixed points of S all lie inside the same orbit of N_1, say θ_1. Then S would act fixed points free on θ_2. Now, $S\ker(\phi)/\ker(\phi) \cong S/S \cap \ker(\phi)$ and as $|S\ker(\phi)/\ker(\phi)| = 2$, S would have a fixed point on θ_2, a contradiction. Hence we may choose the edge $f_2 \in E_X(x_1)$ so that f_2 is fixed by S. Similarly, we may choose $\hat{f}_2 \in E_X(x_2)$ to be fixed by S.

Now let $g_1 = \hat{g}_1 = 1_G$. Consider the fixed points of S on $E_X(x_i)$, $i = 1, 2$. Since $|S \cap M_i : Z(M_i)| = 2$, they are the two points fixed by the whole of T. Choose $g_2 \in N_{P_1}(T)$ that represents $w_1 \in W$. Then $g_2 \cdot f_1 = f_2$. Similarly, we may choose $\hat{g}_2 \in N_{P_2}(T)$ that represents w_2 and such that $\hat{g}_2 \cdot \hat{f}_1 = \hat{f}_2$. Then (2a) in Proposition 4 above holds. Let $\tau := g_2\hat{g}_2^{-1}$. We observe that $S^\tau = S$, since by Lemma 5 above $S = Z(G)Q_0$, a characteristic subgroup of T which is therefore $N_G(T)$–invariant. Hence $(N_1 \cap N_2)^{g_2} = (N_1 \cap N_2)^{\hat{g}_2}$, and so (2c) in Proposition 4 above is satisfied.

To show that (2b) in Proposition 4 above holds, we must show that $N_1 \cap B = N_1 \cap B^{g_2} = N_2 \cap B = N_2 \cap B^{\hat{g}_2} = S$. Since $N_1 \leq L_1 \leq P_1$, we have that $N_1 \cap B = N_1 \cap (B \cap L_1)$. Now, $B \cap L_1$ is isomorphic to a Borel subgroup TU_0 of L_1, where $U_0 \cong E_{p^a}$, the elementary abelian group of exponent p and order $q = p^a$, is normalised by T. On the other hand N_1 is a finite subgroup of L_1. The order of N_1 is $|S|\frac{q+1}{2}$ and it divides $|T|\frac{q+1}{2}$. Moreover, $(|S|, \frac{q+1}{2}) = 1$. Therefore, numerical reasons imply that $N_1 \cap B$ is a finite group whose order divides $|T|$ and is actually at most $|S|$. But $S \leq N_1$ and $S \leq T \leq B$. Hence $N_1 \cap B = S$ as required. The argument that $N_2 \cap B = S$ is similar.

Since S fixes the edge $f_2 = g_2 \cdot f_1$, we have $S \leq B^{g_2}$. The argument that $N_1 \cap B^{g_2} = S$ is then similar to the previous paragraph. Finally, S also fixes the edge $\hat{f}_2 = \hat{g}_2 \cdot \hat{f}_1$, and again by similar arguments we conclude that $N_2 \cap B^{\hat{g}_2} = S$. Therefore all hypotheses of Proposition 4 above are satisfied with $A_1 = N_1$ and $A_2 = N_2$, and so the fundamental group Γ_0 of the graph of groups $\mathbb{A}$ as sketched in the introduction is a cocompact lattice in G with quotient the graph A.

4. Minimality of covolume

Let G be as in Theorem 1 above. In this section we compute the covolume of the lattice Γ_0 constructed in Section 3 above, and prove that for $q \geq 514$, the lattice Γ_0 is the cocompact lattice of minimal covolume in G. We will make repeated use of the following result from our earlier work [3]:

Proposition 6. *Let G be as in Theorem 1 above, with $q = p^a$ where p is prime. If Γ is a cocompact lattice in G, then Γ does not contain p–elements.*

From the construction of Γ_0 in Section 3 above and the discussion of covolumes in Section 1.3 above, it follows that the covolume of Γ_0 is given by $\mu(\Gamma_0\backslash G) = \frac{1}{|\mathcal{A}_{x_1}|} + \frac{1}{|\mathcal{A}_{x_2}|} = \frac{1}{|SH_1|} + \frac{1}{|SH_2|}$. Recall that $S \cap H_i = Z(M_i) \le Q_0$ and $|H_i : Z(M_i)| = \frac{q+1}{2}$. Hence

$$|SH_i| = \frac{|S||H_i|}{|S \cap H_i|} = |S|\frac{q+1}{2} = |Z(G)||Q_0 : (Q_0 \cap Z(G))|\frac{q+1}{2}.$$

Since $|Q_0 : (Q_0 \cap Z(G))| = 2\delta$ where $\delta \in \{1, 2\}$ and its precise value depends on G, we obtain that

$$\mu(\Gamma_0\backslash G) = \frac{2}{\delta|Z(G)|(q+1)} \quad \text{with } \delta \in \{1,2\} \text{ depending on } G. \tag{1}$$

Let us now discuss the issue of minimality of covolume of Γ_0. As in Part I, in order to simplify arguments we assume that G has trivial centre, that is, the finite group $Z(G)$ satisfies $|Z(G)| = 1$. To avoid tedious technical calculations we suppose that $q \ge 514$ (the case when $q < 514$ can be done in a similar manner, but requires more patience).

Assume now that there is a cocompact lattice Γ of G whose covolume $\mu(\Gamma\backslash G)$ is strictly smaller than the covolume Γ_0 given above. Let $Y \subset X$ be a connected fundamental domain for Γ and let A be the graph $A = \Gamma\backslash X$. Since Γ has at least two orbits of vertices, Y contains at least two vertices x_1 and x_2 (connected by at least one edge), such that without loss of generality $G_{x_i} \le P_i$ for $i = 1, 2$. By the discussion in Section 1.3 above,

$$\mu(\Gamma\backslash G) = \sum_{y \in VY} \frac{1}{|\Gamma_y|} \ge \frac{1}{|\Gamma_{x_1}|} + \frac{1}{|\Gamma_{x_2}|}$$

Since Γ is discrete, $|\Gamma_{x_i}|$ is finite. But Γ is cocompact, and so Proposition 6 above implies that, in fact, we may suppose that Γ_{x_i} is a subgroup of L_i of order co-prime to p (where L_i is a Levi complement of the parabolic P_i, $i = 1, 2$).

Remark 7. *Notice that $T \le P_1 \cap P_2$ together with $\Gamma \cap P_i = \Gamma_{x_i}$ yields $\Gamma_{x_1} \cap T = \Gamma_{x_2} \cap T$.*

As in the proof of Theorem 3 of Part I, we organise our remaining discussion based on the following cases:

Case 1 : *For $i = 1, 2$, $L_i/Z(L_i) \cong PSL_2(q)$,* and **Case 2** : *For $i = 1, 2$, $L_i/Z(L_i) \cong PGL_2(q)$.*

4.1. **Case 1.** In this case $L_i = M_i \circ T_i$, that is, L_i is a central (commuting) product of M_i and $T_i = C_T(M_i)$. Moreover, if an element of T centralises a non-split torus of M_i, then from the structure of M_i and L_i, it follows immediately that it centralises M_i. Now, $Z(G) = 1$ implies that $T_i \cap T_j = 1$ and T_i acts faithfully on M_j with $\{i, j\} = \{1, 2\}$. Let us make a few more comments about the structure of the L_i's. Recall that for a finite group F, $O_2(F)$ denotes the largest normal 2-subgroup of F.

Suppose first that $L_i = M_i \times T_i$. Assume that $Z(M_i) \ne 1$, i.e., $M_i \cong SL_2(q)$. Then $1 \ne Q_0 \le C_G(M_i)$ for $i = 1, 2$, and so $Q_0 \le C_G(\langle M_1, M_2\rangle) \le Z(G) = 1$, a contradiction. Thus if $L_i = M_i \times T_i$, $M_i \cong PSL_2(q)$. Moreover, as far as the value of our parameter δ is concerned, it follows immediately that $|T_i|$ is odd whenever $\delta = 1$, and $|T_i|$ is even whenever $\delta = 2$. (In particular, in the key example $G = PSL_2(\mathbb{F}_q((t)))$, $|T_i| = 1$ and $\delta = 1$.) Furthermore, T_i must act faithfully on M_j and so T_i must be isomorphic to a subgroup of M_j. It follows that $|T_i|$ divides $\frac{q-1}{2}$. Suppose now that $Z(M_i) \ne 1$, $T_i \cap M_i \ne 1$, $4 \mid |T_i|$ and $L_i = M_i \circ_{\langle -I\rangle} T_i$. In particular, $M_i \cong M_j \cong SL_2(q)$. Choose an element $g_i \in T \cap M_i$ of order $(q-1)$. Then $\langle g_i^{\frac{q-1}{2}}\rangle = Z(M_i)$. Since $g_i \in T$, it follows that $g_i \in L_j$ and $g_i^{\frac{q-1}{2}}$ must act faithfully on M_j. Thus $O_2(\langle g_i\rangle)$ acts faithfully on M_j via inner automorphisms, which is a contradiction since $O_2(\langle g_i\rangle) \cong C_{2-part\ of\ (q-1)}$ while $Inn(SL_2(q)) = PSL_2(q)$ does not contain such a subgroup. Therefore, this case does not happen. Hence $M_i \cong PSL_2(q)$ and $L_i \cong T_i \times M_i$.

If $|\Gamma_{x_i} \cap T_i| \le \delta$, then

$$\Gamma_{x_i}/\Gamma_{x_i} \cap T_i \cong \Gamma_{x_i}T_i/T_i \le M_iT_i/T_i \cong M_i \cong PSL_2(q). \tag{2}$$

Notice, that $|\Gamma_{x_i} \cap T_i| = 2$ implies that $\delta = 2$. By Dickson's Theorem, it follows that $|\Gamma_{x_i}| \leq \delta(q+1)$. Therefore if $|\Gamma_{x_i} \cap T_i| \leq \delta$ for both $i = 1, 2$, it follows that $\mu(\Gamma\backslash G) \geq \mu(\Gamma_0\backslash G)$, a contradiction (this is precisely the case in [7] implying the minimality of the lattice constructed there). Hence, without loss of generality we may assume that there exists $1 \neq y_1 \in \Gamma_{x_1} \cap T_1$ such that $\langle y_1 \rangle = \Gamma_{x_1} \cap T_1$ with $o(y_1) > \delta$. Then $o(y_1) \mid \frac{q-1}{2}$ and $\langle y_1 \rangle$ acts faithfully on M_2 via inner automorphisms. Notice that if $\delta = 1$, $o(y_1) \neq 1$ is odd, and so for $\delta \in \{1, 2\}$, $o(y_1) \geq 3$. As noticed in Remark 7, since $y_1 \in \Gamma_{x_1} \cap T$, $y_1 \in \Gamma_{x_2}$. Thus Γ_{x_2} acts non-trivially on M_2. Now Dickson's Theorem asserts that Γ_{x_2} must act on M_2 either as a subgroup of a normaliser of a split torus of M_2, or as a subgroup of K_2 with $K_2 \in \{S_4, A_5\}$ (notice that in this case $|o(y_1)| \leq 5$).

Assume first that $|\Gamma_{x_2} \cap T_2| \leq \delta$. Then (2) implies that Γ_{x_2} is actually isomorphic to a subgroup of $PSL_2(q) \times C_\delta$. If Γ_{x_2} acts on M_2 as a subgroup of K_2, then using the previous paragraph we obtain that $\mu(\Gamma\backslash G) \geq \frac{1}{5\cdot(q+1)} + \frac{1}{60\delta} > \frac{2}{\delta(q+1)} = \mu(\Gamma_0\backslash G)$ for $q > 107$. Since this obviously contradicts the minimality of covolume of Γ, Γ_{x_2} must be acting on M_2 as a subgroup of a normaliser of a split torus of M_2. It follows that $\langle y_1 \rangle$ is normal in Γ_{x_2}.

We are now interested in the action of Γ_{x_1} on M_1. By abuse of notation, we identify x_i with its image in the quotient graph $A = \Gamma\backslash X$ for $i = 1, 2$. Then in the graph A, x_1 is a neighbour of x_2. If $|A| = 2$, it follows immediately that $\langle y_1 \rangle \triangleleft \Gamma$, a contradiction. And so $|A| > 2$. Let $z_1, ..., z_k$ be representatives of the other neighbouring vertices of x_2 in A. If for some i, $|C_{\Gamma_{z_i}}(M_{z_i})| \leq \delta$, $|\Gamma_{z_i}| \leq (q+1)\delta$. Hence, $\mu(\Gamma\backslash G) \geq \mu(\Gamma_0\backslash G)$, a contradiction. Therefore for all i, $|C_{\Gamma_{z_i}}(M_{z_i})| > \delta$. Denote by y_{z_i} an element of Γ_{z_i} such that $\langle y_{z_i} \rangle = C_{\Gamma_{z_i}}(M_{z_i})$. We may use exactly the same arguments for y_{z_i} as we did for y_1. Then $o(y_{z_i}) \geq 3$, just like y_1, y_{z_i} acts faithfully on M_2 and $y_{z_i} \in \Gamma_{x_2}$. But Γ_{x_2} acts on M_2 as a subgroup of the normaliser of the split torus. Hence, $[y_1, y_{z_i}] = 1$. Using the fact that we know how L_2 acts on $E_X(x_2)$, we observe that y_1 then must fix the edge (x_2, z_i). It follows that y_1 fixes $x_1, x_2, z_1, ..., z_k$ and that $y_1 \in \Gamma_{z_i}$.

Assume that y_1 acts on M_{z_i} as a non-trivial element of $K_{z_i} \in \{S_4, A_5\}$. Suppose first that $o(y_1) = 2^{\beta_1} \cdot 3^{\beta_2} \cdot 5^{\beta_3}$ where either $\beta_1 \leq 2$ and $\beta_i \leq 1$ for $i = 2, 3$ (this corresponds to $K_{z_i} \cong A_5$), or $\beta_1 \leq 3$, $\beta_2 \leq 1$ while $\beta_3 = 0$ (this is when $K_{z_i} \cong S_4$). Let us evaluate the covolume of Γ. Let $a_2 \in \Gamma_{x_2}$ be such that $\langle a_2 \rangle$ acts faithfully on M_2 and $\langle a_2 \rangle \leq \Gamma_{x_2} \leq (\langle a_2 \rangle \rtimes \langle s_2 \rangle)(\Gamma_{x_2} \cap T_2)$ where $\langle a_2 \rangle \rtimes \langle s_2 \rangle \cong D_{2\cdot o(a_2)}$, $|\Gamma_{x_2} \cap T_2| \leq \delta$ and $\langle a_2 \rangle \times (\Gamma_{x_2} \cap T_2) = \Gamma_2 \cap T$. Then $a_2 \in \Gamma_{x_1}$ and without loss of generality we may assume that $y_1 \in \langle a_2 \rangle$. If $o(y_1) = o(a_2)$, then $|\Gamma_{x_2}| \leq (2^2 \cdot 3 \cdot 5) \cdot 2\delta$ which immediately contradicts the minimality of covolume of Γ ($\frac{1}{120\delta} \geq \frac{2}{(q+1)\delta}$ for $q \geq 240$). Hence, $o(a_2) > o(y_1)$ and so Γ_{x_1} acts on M_1 as either a subgroup of $K_1 \in \{S_4, A_5\}$, or as a subgroup of $N_{M_1}(M_1 \cap T)$.

In the former case $|\Gamma_{x_1}| \leq 2^{\beta_1} \cdot 3^{\beta_2} \cdot 5^{\beta_3} \cdot 60$. In particular, $|\Gamma_{x_1} \cap T| \leq 2^{\beta_1} \cdot 3^{\beta_2} \cdot 5^{\beta_3} \cdot 5$. As we noticed earlier, $\Gamma_{x_1} \cap T = \Gamma_{x_2} \cap T$, and so $|\Gamma_{x_2} \cap T| \leq 2^{\beta_1} \cdot 3^{\beta_2} \cdot 5^{\beta_3+1}$. Since Γ_{x_2} acts on M_2 as a subgroup of a normaliser of split torus of M_2, $|\Gamma_{x_2}| \leq (2^{\beta_1} \cdot 3^{\beta_2} \cdot 5^{\beta_3+1}) \cdot 2$. Note that if $\beta_1 = 1$, $\delta = 2$. If follows that $\mu(\Gamma\backslash G) > \frac{1}{2^{\beta_1+2} \cdot 3^{\beta_2+1} \cdot 5^{\beta_3+1}} + \frac{1}{2^{\beta_1+1} \cdot 3^{\beta_2} \cdot 5^{\beta_3+1}} \geq \frac{2}{(q+1)\delta}$ for $q \geq 514$, again a clear contradiction.

In the latter case, $|\Gamma_{x_1}| \leq o(y_1) 2 \frac{o(a_2)\delta}{o(y_1)} \leq (q-1)\delta$. Since $|\Gamma_{x_2}| \leq (q-1)\delta$, we again get a contradiction with the minimality of covolume of Γ. Therefore either $o(y_1)$ is divisible by $2^{\beta_1} \cdot 3^{\beta_2} \cdot 5^{\beta_3}$ with either $\beta_1 \geq 3$ or $\beta_i \geq 2$ for some $i = 2, 3$, or by $2^{\beta_1} \cdot 3^{\beta_2}$ with $\beta_1 \geq 4$ or $\beta_2 \geq 2$, or $o(y_1)$ is divisible by $\alpha_1 \neq 1$ with $(\alpha_1, 30) = 1$. In all the cases there exists $1 \neq y_1' \in \langle y_1 \rangle$ such that $[y_1', M_{z_i}] = 1$ for all i's. Moreover, $o(y_1') \geq 3$. We may now replace y_1 by y_1' if necessary in all the previous subgroups to obtain the following conclusion: either y_1 centralises M_{z_i} or acts on it as a subgroup of a normaliser of a split torus of M_{z_i} where $i = 1, ..., k$. In both situations, $\langle y_1 \rangle$ is normal in Γ_{z_i} for $i = 1, ..., k$. If x_1 has no other neighbouring vertices than x_2 in A, we continue with the argument (i.e., next look at y_1 in the Γ–stabilisers of the neighbouring vertices of the y_{z_i}'s in A and so on) only to conclude that $\langle y_1 \rangle \triangleleft \Gamma$, an obvious contradiction.

Therefore, it is possible that x_1 has more than one neighbouring vertex in A. One is x_2 and let z be among the other neighbouring vertices of x_1. If $|\Gamma_z \cap T_z| \leq \delta$, then using $|\Gamma_{x_2}|$ and $|\Gamma_z|$, we obtain a contradiction with the minimality of covolume of Γ. Therefore $|\Gamma_z \cap T_z| > \delta$ and we may take $x_2 = z$. Thus whether x_1 has one or more neighbouring vertices in A, we may assume that $|\Gamma_{x_2} \cap T_2| > \delta$. Hence, there exists $y_2 \in \Gamma_{x_2} \cap T_2$ with $o(y_2) > \delta$ and $\langle y_2 \rangle = \Gamma_{x_2} \cap T_2$. As for y_1, we notice that $o(y_2) \mid \frac{q-1}{2}$, $o(y_2) \geq 3$, $y_2 \in \Gamma_{x_1}$ and $\langle y_2 \rangle$ acts faithfully on M_1 via inner automorphisms. Now Dickson's Theorem allows us to conclude that either Γ_{x_1} acts on M_1 as a subgroup of K_1 where $K_1 \in \{S_4, A_5\}$ (in which case $o(y_2) \le 5$), or Γ_{x_1} acts on M_1 as a subgroup of $N_{M_1}(M_1 \cap T)$.

Let us begin with the case when Γ_{x_1} acts on M_1 as a subgroup of K_1. If Γ_{x_2} acts on M_2 as a subgroup of K_2, then $\mu(\Gamma\backslash G) \ge \frac{2}{60\cdot 5} > \frac{2}{(q+1)\delta} = \mu(\Gamma'\backslash G)$ for $q \ge 514$, a contradiction. Hence, Γ_{x_2} acts on M_2 as a subgroup $N_{M_2}(M_2 \cap T)$ and in particular, $\langle y_1\rangle \lhd \Gamma_{x_2}$. Again, if $|VA| = 2$, $\langle y_1\rangle \lhd \Gamma$, a clear contradiction. Thus $|VA| > 2$ and let $v_1, ..., v_k$ be the neighbours of x_1 in $VA - \{x_2\}$. Since y_1 fixes every edge in $E_X(x_1)$, it follows that y_1 acts faithfully on M_{v_i} and holding a discussion similar to the above one with v_i in place of x_2, we may assume that Γ_{v_i} acts on M_{v_i} as a subgroup of a normaliser of a split torus of M_{v_i}. It follows that $\langle y_1\rangle$ is normal in each Γ_{v_i}. Now let $z_1, ..., z_m$ be the neighbours of x_2 in $VA - \{x_1\}$. Let us consider $\Gamma_{z_i} = \Gamma \cap P_{z_i}$. If $|C_{\Gamma_{z_i}}(M_{z_i})| \le \delta$, then there is at most one such vertex, otherwise we would contradict the minimality of covolume of Γ. Hence, we may assume that if it happens, $i = 1$, i.e., $|C_{\Gamma_{z_1}}(M_{z_1})| \le \delta$. Then we may further assume that $T \le P_{z_1}$. Thus $y_1, y_2 \in \Gamma_{z_1}$. If Γ_{z_1} acts on M_{z_1} as a subgroup of $K_{z_1} \in \{S_4, A_5\}$, then $|\Gamma_{z_1}| \le 60\delta$, which is a contradiction, as always ($\frac{1}{60\delta} \ge \frac{2}{(q+1)\delta}$ for $q > 120$). Hence, Γ_{z_1} acts on M_{z_1} as a subgroup of a normaliser of a split torus of M_{z_1}. It follows that $\langle y_1\rangle$ is a normal subgroup of Γ_{z_1}. Now for $i > 1$, there exists $y_{z_i} \in C_G(M_{z_i})$ whose order $o(y_{z_i}) > \delta$ (and thus is at least 3) and does divide $\frac{q-1}{2}$. But this element sits in the kernel of action of L_{z_i} on $E_X(z_i)$ and therefore, $y_{z_i} \in \Gamma_{x_2}$. On the other hand by the usual argument, y_2 acts faithfully on M_{z_i} and so $[y_2, y_{z_i}] = 1$. It follows that $\langle y_{z_i}\rangle$ is normal in Γ_{x_2}. Finally, as $C_{\Gamma_{x_2}}(M_2)$ stabilises (x_2, z_i), it follows that $C_{\Gamma_{x_2}}(y_{z_i}) \le \Gamma_{z_i}$. It follows that $y_1 \in \Gamma_{z_i}$ and so $\langle y_1, y_2\rangle \le \Gamma_{z_i}$. Assume that y_1 acts on M_{z_i} as a subgroup of $K_{z_i} \in \{S_4, A_5\}$. Using the same argument as before we obtain that there exists $y_1' \in \langle y_1\rangle$ with $[y_1', M_{z_i}] = 1$ for all $i > 1$ and with $o(y_1') \ge 3$. In this case we will replace y_1 by y_1' if necessary in all the previous subgroups to obtain the following conclusion: $\langle y_1\rangle$ is normal in Γ_v for all the vertices mentioned so far, i.e., $x_1, x_2, z_1, ..., z_m, v_1, ..., v_k$. By iterating this argument we may show that $\langle y_1\rangle \lhd \Gamma$ which is a contradiction.

We are now reduced to the last possible situation: Γ_{x_1} acts on M_1 as a subgroup of $N_{M_1}(M_1 \cap T)$. Notice, that because of the symmetry between x_1 and x_2 to finish the analysis it remains to consider the case when Γ_{x_2} acts on M_2 as a subgroup of $N_{M_2}(M_2 \cap T)$. But in this case $\langle \Gamma_{x_1}, \Gamma_{x_2}\rangle \le N$. Hence, we may move to the next vertex y on our graph. Using the previous argument we obtain that again that the only possible case will be $\Gamma_y \le N$, and so on and so forth. Therefore, in the end of this case, the only possible conclusion will be $\Gamma \le N$, which is a contradiction as N is not a uniform lattice of G, not does it contain any uniform lattice.

4.2. **Case 2.** We are now in the situation when T induces some non-trivial outer-diagonal automorphisms on M_i, that is $L_i/Z(L_i) \cong PGL_2(q)$. Consider $L_i = M_iT$. As before $M_i \lhd L_i$ and $T_i = C_T(M_i)$. Then there exists an element $t_i \in T - T_iM_i$ such that $t_i^2 \in T_iM_i$ and t_i induces an outer diagonal automorphism on M_i. Since $q \equiv 1 \pmod 4$, if x is an involution in $L_i \cap T$, $x \in M_iT_i$.

Recall that G does not admit any edge-transitive lattice. Therefore, if $Q_i \in \mathcal{S}yl_2(L_i)$ and Q_i^0 is its unique subgroup of index 2, then $Q_i^0 \not\le Z(G)$. It follows that $|Q_i/Q_i \cap Z(G)| \ge 4$ and so, $\delta = 2$.

The minimality of covolume of Γ_0 can be now shown by repeating exactly the same sequence of arguments as in Case 1 applied to subgroups of L_i, $i = 1, 2$. It turns out that the difference in the structure of L_i (which is now a quotient of $GL_2(q)$) does not significantly affect the argument and so we omit it here in order to avoid a fairly routine repetition.

References

[1] H. Bass, *Covering theory for graphs of groups*, J. Pure Appl. Algebra **89** (1993), 3–47.

[2] M. R. Bridson and A. Haefliger, *Metric Spaces of Non-Positive Curvature*, Springer-Verlag, Berlin, 1999.

[3] I. (Korchagina) Capdeboscq and A. Thomas, *Cocompact lattices of minimal covolume in rank* 2 *Kac–Moody groups, Part I: Edge-transitive lattices*, submitted, `arXiv:0907.1350`.

[4] P.-E. Caprace and B. Rémy, *Simplicity and superrigidity of twin building lattices*, Invent. Math. **176** (2009) 169–221.

[5] L. Carbone and H. Garland, *Existence of lattices in Kac–Moody groups over finite fields*, Commun. Contemp. Math. **5** (2003), 813–867.

[6] A. Lubotzky, *Lattices of minimal covolume in SL_2: a nonarchimedean analogue of Siegel's theorem $\mu \ge \pi/21$*, J. Amer. Math. Soc. **3** (1990), 961–975.

[7] A. Lubotzky and Th. Weigel, *Lattices of minimal covolume in SL_2 over local fields*, Proc. London Math. Soc. **78** (1999), 283–333.
[8] C. L. Siegel, *Some remarks on discontinuous groups*, Ann. of Math. **46** (1945), 708–718.
[9] A. Thomas, *Lattices acting on right-angled buildings*, Alg. Geom. Topol. **6** (2006), 1215–1238.

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