S^2-BUNDLES OVER 2-ORBIFOLDS

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ABSTRACT. Let M be a closed 4-manifold with $\pi = \pi_1(M) \neq 1$ and $\pi_2(M) \cong Z$, and let $u : \pi \to Aut(\pi_2(M))$ be the natural action. If $\pi \cong \mathrm{Ker}(u) \times Z/2Z$ then M is homotopy equivalent to the total space of an RP^2-bundle over an aspherical surface. We show here that if π is not such a product then M is homotopy equivalent to the total space of an S^2-orbifold bundle over a 2-orbifold B. There are at most two such orbifold bundles for each pair (π, u). If B is the orbifold quotient of the orientable surface of genus g by the hyperelliptic involution there are two homotopy types of such orbifold bundles and only one of these is geometric.

Every closed 4-manifold with geometry $\mathbb{S}^2 \times \mathbb{E}^2$ or $\mathbb{S}^2 \times \mathbb{H}^2$ has a foliation with regular leaves S^2 or RP^2. The leaf space of such a foliation may be regarded as a compact 2-orbifold. If the regular leaves are S^2 the singularities of this orbifold are cone points of order 2 or reflector curves, and the projection to the leaf space is an orbifold bundle projection, with general fibre S^2. If there are no exceptional leaves the projection is a bundle projection, and the total space is geometric. (See Theorem 10.9 of [2].)

Each pair (π, u) where $\pi = \pi^{orb}(B)$ is a 2-orbifold group and $u : \pi \to Z/2Z$ is an epimorphism with torsion-free kernel is realized by a standard geometric manifold M_{st}. In §1 we review the key invariants that we shall use, and consider aspects of the cup-product in $H^*(M_{st}; \mathbb{F}_2)$. In §2 we show that if M is any 4-manifold realizing (π, u) then $k_1(M) = k_1(M_{st}$, and that if $\chi(M_{st})$ is even and B has cone points then $v_2(M_{st}) = U^2$, where $U \in H^1(\pi; \mathbb{F}_2) = Hom(\pi, Z/2Z)$ corresponds to the action u.

In §3 we consider local models for orbifold bundle projections, and in §4 we show that there are at most two 4-manifolds M which are total spaces of orbifold bundles over B with regular fibre S^2 and action u on $\pi_2(M) \cong Z$. The base orbifold B must have a nonsingular double cover. In particular, its singular locus consists of cone points of order 2 and reflector curves. If B has an "untwisted" reflector curve, the

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bundle is unique. We show also that if B is an $\mathbb{H}^2$-orbifold then every such bundle space is either geometric or has a decomposition into two geometric pieces. In §5 we review briefly the cases with spherical base orbifold.

We return to the homotopy classification in §6. Here we show that if u is nontrivial and $\pi \not\cong \mathrm{Ker}(u) \times Z/2Z$ then M is homotopy equivalent to an S^2-orbifold bundle space. If $B = S(2_{2g+2})$ is the orbifold quotient of the orientable surface of genus g by the hyperelliptic involution then there are two such orbifold bundles, representing the two homotopy types of 4-manifolds with fundamental group $\pi^{orb}(B)$ and $\pi_2 \cong Z$. Only one of these is geometric.

In the final three sections we show first that the 22 $\mathbb{S}^2 \times \mathbb{E}^2$-manifolds have distinct homotopy types, and there is one more homotopy type represented by a non-geometric S^2-orbifold bundle over $S(2,2,2,2)$. The TOP structure sets of such manifolds are infinite if π has torsion but is not a product with $Z/2Z$. If moreover π/π' is finite then there are infinitely many homeomorphism types within each such homotopy type. Finally we apply the main result to a characterization of the homotopy types of orientable 4-manifolds which are total spaces of bundles over RP^2 with aspherical fibre and a section.

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1. THE STANDARD EXAMPLE

Although we shall consider quotients of $S^2 \times S^2$ briefly in §3, our main concern is with 4-manifolds M covered by $S^2 \times R^2$. We shall identify S^2 with $CP^1 = \mathbb{C} \cup \{\infty\}$, via stereographic projection from $(0, 1) \in \mathbb{C} \times \mathbb{R}$. Under this identification the antipodal map a is given by $a(z) = -z/|z|^2$ (i.e, $a([z_0 : z_1]) = [-\overline{z_1} : \overline{z_0}]$), and rotation through an angle θ about the axis through 0 and ∞ is given by $R_\theta(z) = e^{i\theta}z$. (Care! Multiplication by -1 in CP^1 is R_π, not a!)

Let M be a closed 4-manifold with $\pi_2(M) \cong Z$ and $\pi = \pi_1(M) \neq 1$, and let $u : \pi \to Aut(\pi_2(M)) = \{\pm 1\}$ be the natural action. Then M has universal cover $\widetilde{M} \cong S^2 \times R^2$ and $\kappa = \mathrm{Ker}(u)$ is a PD_2-group, and $w = w_1(M)$ is determined by the pair (π, u). (See Chapter 10 of [2]. Note that if u is nontrivial π may have automorphisms that do not preserve u.) Let $[M] \in H_4(M; Z^w) \cong Z$ be a fundamental class.

If π is torsion-free then M is TOP s-cobordant to the total space of an S^2-bundle over an aspherical surface. If $\pi \cong \kappa \times Z/2Z$ then any 4-manifold M with $\pi_1(M) \cong \pi$ and $\pi_2(M) \cong Z^u$ is simple homotopy equivalent to the total space of an RP^2-bundle over $K(\kappa,1)$. For each PD_2-group κ there are two such bundles, distinguished by whether $v_2(M)=0$ or not. As these cases are well-understood, we shall usually assume that M is not homotopy equivalent to a bundle space.

If π has torsion but is not a direct product then u is nontrivial and $\pi\cong\kappa\rtimes Z/2Z$. Moreover π is the orbifold fundamental group of a $\mathbb{E}^2$- or $\mathbb{H}^2$-orbifold B. Since κ is torsion free the singular locus ΣB consists of cone points of order 2 and reflector curves.

The surface $K(\kappa,1)$ has an involution ζ corresponding to the action of $\pi/\kappa\cong Z/2Z$. The "standard" example of a closed 4-manifold realizing (π,u) is

$$M_{st}=S^2\times K(\kappa,1)/(s,k)\sim(-s,\zeta(k)).$$

This is a $\mathbb{S}^2\times\mathbb{E}^2$-manifold if $\chi(\pi)=0$, and is a $\mathbb{S}^2\times\mathbb{H}^2$-manifold otherwise. Projection onto the first factor induces a bundle projection from M_{st} to RP^2, with fibre $F=K(\kappa,1)$. Projection onto the second factor induces an orbifold bundle projection $p_{st}:M_{st}\to B$ with regular fibre $F\cong S^2$.

As involutions have fixed points in R^2, they must act without fixed points on S^2. Therefore if π is generated by involutions every geometric 4-manifold with group π is diffeomorphic to M_{st}.

The algebraic 2-type $[\pi,\pi_2(M),k_1(M)]$ determines $P_2(M)$, the second stage of the Postnikov tower for M, and the homotopy type of M is determined by the image of $[M]$ in $H_4(P_2(M);Z^w)$, modulo the action of $Aut(P_2(M))$. There are at most two possible values for this image, up to sign and automorphisms of the algebraic 2-type, by Theorem 10.6 of [2]. It is clear from this Theorem that the homotopy type of M is in fact detected by the image of $[M]$ in $H_4(P;\mathbb{F}_2)$. We shall construct a model for $P_2(M_{st})$ in §6.

In the remainder of this section all cohomology shall be with coefficients $\mathbb{F}_2$, and we shall drop the coefficients from the notation. Let $M=M_{st}$, and let $x\in\pi=\pi_1(M)$ be an element of order 2. The generator of $\pi_2(M)$ factors through the double cover $\widetilde{M}\simeq S^2\to\widetilde{M}/\langle x\rangle\simeq RP^2$, and so the *mod*-(2) Hurewicz homomorphism is trivial. Hence $H^i(\pi)\cong H^i(M)$ for $i\leq 2$.

We shall identify $H^1(\pi)$ with $Hom(\pi,Z/2Z)$. Let $U\in H^1(\pi)$ be the cohomology class corresponding to the epimorphism u. Thus $U(x)=1$ and $U(k)=0$ for all $k\in\kappa$, while $U^3=0$ since U is in the image of $H^*(RP^2)$.

Theorem 1. *The restriction $Res^\kappa_\pi:H^2(\pi)\to H^2(\kappa)=\mathbb{F}_2$ is surjective, and cup-product with U maps $H^1(\pi)$ onto* $\mathrm{Ker}(Res^\kappa_\pi)$.

Proof. Let θ be the automorphism of $H_1(\kappa)$ given by $\theta(A)(k) = A(xkx)$ for all $A \in H^1(\pi)$ and $k \in \kappa$. Let $r = \dim_{\mathbb{F}_2}\mathrm{Ker}(\theta+1)$ and $s = \dim_{\mathbb{F}_2}\mathrm{Im}(\theta+1)$. Then $\beta_1(\kappa;\mathbb{F}_2) = r+s$ and $\dim_{\mathbb{F}_2}H^1(Z/2Z;H^1(\kappa)) = r-s$. It follows from the LHS spectral sequence that $\beta_1(\pi;\mathbb{F}_2) = 1+r$ and $\beta_2(\pi;\mathbb{F}_2) = 1+r-s+\delta$, where $\delta = \dim_{\mathbb{F}_2}\mathrm{Im}(Res^\kappa_\pi) \leq 1$. Since $\chi(M) = 2-2\beta_1(\pi;\mathbb{F}_2)+\beta_2(\pi;\mathbb{F}_2)$ and also $\chi(M) = \chi(\kappa) = 2-\beta_1(\kappa;\mathbb{F}_2)$, we see that in fact $\delta = 1$. Therefore Res^κ_π is surjective.

Certainly $Res^\kappa_\pi(U\cup A) = 0$ for all $A \in H^1(\pi)$, and $U^2 \neq 0$. Suppose that $A \in H^1(\pi)$ is such that $A(x) = 0$. If $U \cup A = 0$ there is a function $f : \pi \to \mathbb{F}_2$ such that $U(g)A(h) = f(g)+f(h)+f(gh)$ for all $g,h \in \pi$. If $g \in \kappa$ then $U(g) = 0$ and so $f|_\kappa$ is a homomorphism. Taking $g = x$ we have $A(h) = f(x)+f(h)+f(xh)$, for all $h \in \pi$, while taking $h = x$ we have $f(gx) = f(g)+f(x)$ for all $g \in \pi$. In particular, $f(xhx) = f(xh)+f(x)$, for all $h \in \pi$. Therefore $A(h) = f(h)+f(xhx)$, for all $h \in \pi$, and so $A \in \mathrm{Im}(\theta+1)$. Thus $\dim_{\mathbb{F}_2}\mathrm{Ker}(U\cup -) \leq s$, and so the image of cup-product with U has rank at least $r-s+1 = \dim_{\mathbb{F}_2}\mathrm{Ker}(Res^\kappa_\pi)$. □

If $s : RP^2 \to M_{st}$ is a section then $U^2(s_*[RP^2]) = 1$, and so the Poincaré dual of $s_*[RP^2]$ has nonzero restriction to $H^2(\kappa)$. (If $r > s$ then there are classes $A, B \in H^1(\pi)$ such that $A(x) = B(x) = 0$ and $Res^\kappa_\pi(A\cup B) \neq 0$. However if $r = s$ then $U \cup H^1(\pi) = \langle U^2\rangle$.)

2. THE k-INVARIANT

Let $M = M_{st}$ and $P = P_2(M)$. The image of $H_4(CP^\infty;\mathbb{F}_2)$ in $H_4(P;\mathbb{F}_2)$ is fixed under the action of $Aut(P)$, and so $Aut(P)$ acts on this homology group through a quotient of order at most 2. Since M is geometric $Aut(\pi)$ acts isometrically. More generally, if M is the total space of an orbifold bundle then $Aut(\pi)$ acts by orbifold automorphisms of the base. The antipodal map on the fibres defines a self-homeomorphism which induces -1 on $\pi_2(M)$. These automorphisms clearly fix $H_4(P;\mathbb{F}_2)$. Thus it shall be enough to consider the action of the subgroup of $Aut(P)$ which acts trivially on π_1 and π_2. Since P is a connected cell-complex with $\pi_i(P) = 0$ for $i > 2$ this subgroup is isomorphic to $H^2(\pi;Z^u)$ [7].

In Lemma 10.4 of [2] it is shown that the u-twisted Bockstein β^u maps $H^2(\pi;\mathbb{F}_2)$ onto $H^3(\pi;Z^u)$, and that the restriction of $k_1(M)$ to each subgroup of order 2 in π is nontrivial. On looking more closely at the structure of such groups and applying Mayer-Vietoris arguments to compute these cohomology groups, we can show that there is only one possible k-invariant.

Lemma 2. *Let* $\alpha = *^k Z/2Z = \langle x_i, 1 \le i \le k \mid x_i^2 \ \forall\ i \rangle$ *and let* $u(x_i) = -1$ *for all* i. *Then restriction from* α *to* $\phi = \mathrm{Ker}(u)$ *induces an epimorphism from* $H^1(\alpha; Z^u)$ *to* $H^1(\phi; Z)$.

Proof. Let $x = x_1$ and $y_i = x_1 x_i$ for all $i > 1$. Then $\phi = \mathrm{Ker}(u)$ is free with basis $\{y_2, \dots, y_k\}$ and so $\alpha \cong F(k-1) \rtimes Z/2Z$.

If $k = 2$ then α is the infinite dihedral group D and the lemma follows by direct calculation with resolutions. In general, the subgroup D_i generated by x and y_i is an infinite dihedral group, and is a retract of α. The retraction is compatible with u, and so restriction maps $H^1(\alpha; Z^u)$ onto $H^1(D_i; Z^u)$. Hence restriction maps $H^1(\alpha; Z^u)$ onto each summand $H^1(\langle y_i \rangle; Z)$ of $H^1(\phi; Z)$, and the result follows. □

In particular, if k is even then $z = \Pi x_i$ generates a free factor of ϕ, and restriction maps $H^1(\alpha; Z^u)$ onto $H^1(\langle z \rangle; Z)$.

Let $S(2_k)$ be the sphere with k cone points of order 2.

Theorem 3. *Let* B *be an aspherical* 2*-orbifold, and let* $u : \pi = \pi_1^{orb}(B) \to \{\pm 1\}$ *be an epimorphism with torsion-free kernel* κ. *Suppose that* $\Sigma B \neq \emptyset$, *and that* B *has* r *reflector curves and* k *cone points. Then* $H^2(\pi; Z^u) \cong (Z/2Z)^r$ *if* $k > 0$ *and* $H^2(\pi; Z^u) \cong Z \oplus (Z/2Z)^{r-1}$ *if* $k = 0$. *In all cases* $\beta^u(U^2)$ *is the unique element of* $H^3(\pi; Z^u)$ *which restricts non-trivially to each subgroup of order* 2.

Proof. Suppose first that B has no reflector curves. Then B is the connected sum of a closed surface G with $S(2_k)$, and k is even, by Lemma 2. If $B = S(2_k)$ then $k \geq 4$, since B is aspherical. Hence $\pi \cong \mu *_Z \nu$, where $\mu = *^{k-2} Z/2Z$ and $\nu = Z/2Z * Z/2Z$ are generated by cone-point involutions. Otherwise $\pi \cong \mu *_Z \nu$, where $\mu = *^k Z/2Z$ and $\nu = \pi_1(G \setminus D^2)$ is a non-trivial free group. Every non-trivial element of finite order in such a generalized free product must be conjugate to one of the involutions. In each case a generator of the amalgamating subgroup is identified with the product of the involutions which generate the factors of μ and which is in $\phi = \mathrm{Ker}(u|_\mu)$.

Restriction from μ to Z induces an epimorphism from $H^1(\mu; Z^u)$ to $H^1(Z; Z)$, by Lemma 7, and so

$$H^2(\pi; Z^u) \cong H^2(\mu; Z^u) \oplus H^2(\nu; Z^u) = 0,$$

by the Mayer-Vietoris sequence with coefficients Z^u. Similarly,

$$H^2(\pi; \mathbb{F}_2) \cong H^2(\mu; \mathbb{F}_2) \oplus H^2(\nu; \mathbb{F}_2),$$

by the Mayer-Vietoris sequence with coefficients $\mathbb{F}_2$. Let $e_i \in H^2(\pi; \mathbb{F}_2) = Hom(H_2\pi; \mathbb{F}_2), \mathbb{F}_2)$ correspond to restriction to the ith cone point. Then $\{e_1, \dots, e_{2g+2}\}$ forms a basis for $H^2(\pi; \mathbb{F}_2) \cong \mathbb{F}_2^{2g+2}$, and Σe_i

is clearly the only element with nonzero restriction to all the cone point involutions. Since $H^2(\pi;Z^u)=0$ the u-twisted Bockstein maps $H^2(\pi;\mathbb{F}_2)$ isomorphically onto $H^3(\pi;Z^u)$, and so there is an unique possible k-invariant.

Suppose now that $r>0$. Then $B=r\mathbb{J}\cup B_o$, where B_o is a connected 2-orbifold with r boundary components and k cone points. Hence $\pi=\pi\mathcal{G}$, where $\mathcal{G}$ is a graph of groups with underlying graph a tree having one vertex of valency r with group $\nu=\pi_1^{orb}(B_o)$, r terminal vertices, with groups $\gamma_i\cong\pi_1^{orb}(\mathbb{J})=Z\oplus Z/2Z$, and r edge groups $\omega_i\cong Z$. If $k>0$ then restriction maps $H^1(\nu;Z^u)$ onto $\oplus H^1(\omega_i;Z)$ and then $H^2(\pi;Z^u)\cong\oplus H^2(\gamma_i;Z^u)\cong Z/2Z^r$. However if $k=0$ then $H^2(\pi;Z^u)\cong Z\oplus(Z/2Z)^{r-1}$.

The Mayer-Vietoris sequence with coefficients $\mathbb{F}_2$ gives an isomorphism $H^2(\pi;\mathbb{F}_2)\cong H^2(\nu;\mathbb{F}_2)\oplus(H^2(Z\oplus Z/2Z;\mathbb{F}_2))^r\cong\mathbb{F}_2^{2r+k}$. The generator of the second summand of $H^2(Z\oplus Z/2Z;\mathbb{F}_2)$ is in the image of reduction modulo (2) from $H^2(Z\oplus Z/2Z;Z^u)$, and so is in the kernel of β^u. Therefore the image of β^u has a basis corresponding to the cone-points and reflector curves, and we again find an unique k-invariant. Since $\beta^u(U^2)$ restricts to the generator of $H^3(Z/2Z;Z^u)$ at each involution in π, we must have $k_1(M)=\beta^u(U^2)$. □

Corollary. *If M is a closed 4-manifold with $\pi_2(M)\cong Z$ and $\pi_1(M)\cong\pi^{orb}(B)$ then $P_2(M)\simeq P_2(M_{st})$, where M_{st} is the standard geometric 4-manifold with the same fundamental group.* □

Theorem 4. *If $\chi(M)$ is even then $v_2(M)=0$ or U^2. If the base orbifold has cone points then $v_2(M)\neq 0$.*

Proof. If $\chi(\kappa=\chi(M)$ is even then $\alpha^2=0$ for all $\alpha\in H^1(\kappa)$. Therefore if $A\in H^1(\pi)$ then $Res_\pi^\kappa(A^2)=0$, and so $A^2=U\cup B$, for some $B\in H^1(\pi)$, by Theorem 1. Hence $(U\cup A)^2=U^2\cup A^2=U^3\cup B=0$. It follows easily that $v_2(M)=0$ or U^2.

The exceptional fibre over a cone point of order 2 has nonzero self-intersection, by Lemma 10.14 of [2]. Thus if the base orbifold has cone points $v_2(M)\neq 0$. □

This was proven by hand for the cases with $\chi(M)=0$ in Theorem 10.16 of [2].

Whereas regular fibres in an S^2-orbifold bundle over a connected base are isotopic, exceptional fibres over distinct components of the singular locus of B are usually not even homologous. An arc γ in B connecting two such components is in fact a reflector interval, and the restriction of the fibration over γ has total space $RP^3\#RP^3$. Thus it should not be surprising that fibres over reflector curves have self-intersection 0, whereas fibres over cone points have self-intersection 1.

3. Local models for orbifold bundles

A cone point of order 2 in a 2-orbifold has a regular neighbourhood which is orbifold-homeomorphic to $D(2) = D^2/w \sim -w$. Let $\mathbb{J} = [[0,1] = [-1,1]/x \sim -x$ be the compact connected 1-orbifold with one reflector point. A reflector curve (with no corner points) in a 2-orbifold has a regular neighbourhood which is orbifold-homeomorphic to $\mathbb{J} \times S^1$. However this is the quotient of a nonsingular surface in two ways, as there are two possible surjections $u : \pi^{orb}(\mathbb{J} \times S^1) \to Z/2Z$ with torsion-free kernel. If the cover is the Möbius band $Mb = [-1,1] \times S^1/(x,u) \sim (-x,-u)$ with involution $[x,u] \mapsto [-x,u] = [x,-u]$ we shall say that the curve is *u-twisted*; if the cover is $[-1,1] \times S^1$ with involution $(x,u) \mapsto (-x,u)$ we shall say that the curve is untwisted. (Note that this notion involves both the reflector curve and the action.)

For example, as the quotient of an involution of the torus T the "silvered annulus" $\mathbb{A} = S^1 \times S^1/(u,v) \sim (u,\bar{v})$ has two untwisted reflector curves. However it is also the quotient of an involution of the Klein bottle Kb, and the reflector curves are then both twisted. On the other hand, the "silvered Möbius band" $\mathbb{M}b = S^1 \times S^1/(u,v) \sim (v,u)$ has two distinct (but isomorphic) nonsingular covers, but in both cases the reflector curve is untwisted.

Models for regular neighbourhoods of the exceptional fibres of such orbifold bundles may be constructed as follows. Let

$$E(2) = S^2 \times D^2/(z,w) \sim (a(z),-w),$$

$$\mathbb{E} = S^2 \times [-1,1] \times S^1/(z,x,u) \sim (a(z),-x,u)$$

and

$$\mathbb{E}' = S^2 \times [-1,1] \times S^1/(z,x,u) \sim (a(z),-x,u) \sim (z,-x,-u).$$

Then $p_2([z,w]) = [w]$, $p_{\mathbb{E}}([z,x,u]) = [u,x]$ and $p_{\mathbb{E}'}([z,x,u]) = [x,u]$ define bundle projections $p_2 : E(2) \to D(2)$, $p_{\mathbb{E}} : \mathbb{E} \to \mathbb{J} \times S^1$ (with untwisted reflector curve) and $p_{\mathbb{E}'} : \mathbb{E}' \to \mathbb{J} \times S^1$ (with twisted reflector curve). Any S^2-bundle over $\mathbb{J} \times S^1$ or $D(2)$ with nonsingular total space must be of this form. The other local models for nontrivial actions on the fibre have base Mb and total space $S^2 \times Mb$ (non-orientable) or $S^2 \times [-1,1] \times [0,1]/(z,t,0) \sim (a(z),-t,1)$ (orientable).

It is also convenient to let $D(2,2) = [-1,1] \times S^1/(x,u) \sim (-x,\bar{u})$ be the disc with two cone points of order 2 and

$$E(2,2) = S^2 \times [-1,1] \times S^1/(z,x,u) \sim (a(z),-x,\bar{u}),$$

with projection $p_{2,2}([z,x,u]) = [x,u]$. Then $D(2,2)$ is the boundary-connected-sum of two copies of $D(2)$, and $E(2,2)$ is the corresponding fibre sum of two copies of $E(2)$.

The manifolds $E(2)$ and $\mathbb{E}'$ have boundary $S^2\tilde{\times}S^1$, and $p_2|_{\partial E(2)}$ and $p|_{\partial\mathbb{E}'}$ are nontrivial S^2-bundles over S^1. In all the other cases the restriction of the fibration over the boundary of the base orbifold is trivial. (When the base is $B = Mb$ or $D(2,2)$ this can be seen by noting that ∂B is homotopic to the product of two generators of $\pi_1^{orb}(B)$, and considering the action on $\pi_2(E) \cong Z$.) For later uses we may need to choose homeomorphisms $\partial E \cong S^2 \times S^1$.

Let α, β and τ be the self-homeomorphisms of $S^2 \times S^1$ defined by $\alpha(z,u) = (a(z),u)$, $\beta(z,u) = (z,\bar{u})$ and $\tau(z,u) = (uz,u)$, for all $(z,u) \in S^2 \times S^1$. The images of α, β and τ generate $\pi_0(Homeo(S^2 \times S^1)) \cong (Z/2Z)^3$. The group $\pi_0(Homeo(S^2\tilde{\times}S^1)) \cong (Z/2Z)^2$ is generated by the involution $\tilde{\beta}([z,u]) = [z,\bar{u}]$ and the twist $\xi([z,u]) = [uz,u]$.

Lemma 5. *(1) The self-homeomorphisms α and β of $S^1 \times S^2$ extend to fibre-preserving self-homeomorphisms of $S^2 \times D^2$ and $E(2,2)$.*

(2) Every self-homeomorphism of $S^1 \times S^2$ extends to a fibre-preserving self-homeomorphism of $\mathbb{E}$.

(3) The self-homeomorphism $\tilde{\beta}$ of $S^2\tilde{\times}S^1$ extends to fibre-preserving self-homeomorphisms of $E(2)$ and $\mathbb{E}'$.

Proof. It is sufficient to check that the above representatives of the isotopy classes extend, which in each case is clear. □

However τ does not extend across $S^2 \times D^2$ or $E(2,2)$, as we shall see. Nor does ξ extend across $E(2)$ or $\mathbb{E}'$.

4. General results on orbifold bundles

Let M be a closed 4-manifold which is the total space of an orbifold bundle $p : M \to B$ with regular fibre $F \cong S^2$ over the 2-orbifold B. Then $\pi_1^{orb}(B) \cong \pi_1(M)$. Let ΣB be the singular locus of B. For brevity, we shall say that M is an S^2*-orbifold bundle space* and p is an S^2*-orbifold bundle*.

Lemma 6. *The singular locus ΣB consists of cone points of order* 2 *and reflector curves (with no corner points). The number of cone points plus the number of u-twisted reflector curves is even. In particular, the base orbifold must be good.*

Proof. The first assertion holds since the stabilizer of a point in the base orbifold must act freely on the fibre S^2.

Let N be a regular neighbourhood of ΣB, and let V be the restriction of U to $B \setminus N$. Then $V(\partial N) = 0$. The action u is trivial on boundary components of N parallel to untwisted reflector curves, but is nontrivial on all other boundary components. Therefore $V(\partial N)$ is the sum of the number of cone points and the number of u-twisted reflector curves, modulo (2). Thus this number must be even, and B cannot be $S(2)$, which is the only bad orbifold in which all point stabilizers have order at most 2. □

If B is spherical then $\widetilde{M} \cong S^2 \times S^2$; otherwise $\widetilde{M} \cong S^2 \times R^2$.

Lemma 7. *Let $q : E \to F$ be an S^2-bundle over a surface with nonempty boundary. If q is nontrivial but $q|_{\partial E}$ is trivial then there is a non-separating simple closed curve γ in the interior of F such that the restriction of the bundle over $F \setminus \gamma$ is trivial.*

Proof. The bundle is determined by the action of $\pi_1(F)$ on $\pi_2(E)$, and thus by a class $u \in H^1(F; \mathbb{F}_2)$. Since $u|_{\partial F} = 0$ and $u \neq 0$ the Poincaré-Lefshetz dual of u is represented by a simple closed curve γ in the interior of F, and u restricts to 0 on $F \setminus \gamma$. □

The restrictions to each fibre of a bundle automorphism of an S^2-bundle over a connected base must either all be orientation-preserving or all be orientation-reversing.

Lemma 8. *Let $q : E \to F$ be an S^2-bundle over a surface such that $q|_{\partial E}$ is trivial. If ∂E has boundary components $\{C_i \mid 1 \leq i \leq d\}$ for some $d > 0$ and if ϕ_i is an orientation-preserving bundle automorphism of $q|_{C_i}$ for $i < d$ then there is a bundle automorphism ϕ of q such that $\phi|_{q^{-1}(C_i)} = \phi_i$ for $i < d$.*

Proof. We may clearly assume that $d \geq 2$. Suppose first that q is trivial. We may obtain F by identifying in pairs $2k$ sides of a $(2k + d)$-gon P. (The remaining sides corresponding to the boundary components C_i.) A bundle automorphism of a trivial S^2-bundle over X is determined by a map from X to $Homeo(S^2)$. Let $[\phi_i]$ be the image of ϕ_i in $\pi_1(Homeo(S^2)) = Z/2Z$, for $i < d$, and define ϕ_d on $q^{-1}(C_d)$ so that $[\phi_d] = \Sigma_{i<d}[\phi_i]$. Let ϕ be the identity on the images of the other sides of P. Then $[\phi|_{\partial P}] = 0$ and so $\phi|_{\partial P}$ extends across P. This clearly induces a bundle automorphism ϕ of q compatible with the data.

If q is nontrivial let γ be a simple closed curve in F as in the previous lemma, and let N be an open regular neighbourhood of γ. If q is trivial let $N = \emptyset$. Then the restriction of q over $F' = F \setminus N$ is trivial, and so $E' = q^{-1}(F') \cong F' \times S^2$. If $N \cong \gamma \times (-1, 1)$ then $\partial E'$ has $d + 2$ components; if $N \cong Mb$ and $\partial E'$ has $d + 1$ components. In either case, we let ϕ be the identity on the new boundary components, and proceed as before. □

The two exceptional fibres in $E(2,2)$ have regular neighbourhoods equivalent to $E(2)$. If we delete the interiors of two such neighbourhoods we obtain the S^2-bundle over the thrice-punctured sphere $B = S^2 \setminus 3intD^2$ which is trivial over exactly one component of ∂E. (There is one such bundle up to isomorphism, since $B \simeq S^1 \vee S^1$.)

Lemma 9. *Let $q : E \to B$ be the S^2-bundle over $B = S^2 \setminus 3intD^2$ which is trivial over exactly one component of ∂E. Then there is a bundle automorphism which restricts to τ on the orientable component, to the identity on one non-orientable boundary component and to ξ on the other.*

Proof. Let $L = S^2 \times [0,1]^2/\sim$, where $(z,x,0) \sim (a(z),x,1)$ for all $s \in S^2$ and $0 \leq x \leq 1$. Then L is the total space of an S^2-bundle over the annulus $A = [0,1] \times S^1$, with projection $p : L \to A$ given by $p([z,x,y]) = (x, e^{2\pi iy})$. The boundary components of L are each homeomorphic to $S^2 \tilde{\times} S^1$. Let $k = (\frac{1}{2}, 1) \in A$, $D = \{(x,u) \in A \mid d((x,u),K) < \frac{1}{4}\}$, $B = A \setminus D$ and $E = L \setminus p^{-1}(D)$. Then $p|_E$ is a model for q.

Let $P = (0,-1)$, $Q = (1,-1)$ $R = (\frac{3}{4}, 1)$ and $S = (1,1)$ be points in B and let $B' = B \setminus (PQ \cup RS) \times (-\varepsilon, \varepsilon)$. Then $B' \cong D^2$, and so the restriction $q' = q|_{B'}$ is trivial. We may clearly define a bundle automorphism of q' which rotates the fibre once as we go along each of the arcs corresponding to $\{1\} \times S^1$ and ∂D and is the identity over the rest of the boundary. Since the automorphisms agree along the pairs of arcs corresponding to PQ and RS, we obtain the desired automorphism of q. □

Let $j : S^2 \times D^2 \to M$ be a fibre-preserving embedding of a closed regular neighbourhood of a regular fibre of p, and let N be the image of j. The *Gluck reconstruction* of p is the orbifold bundle $p^\tau : M^\tau \to B$ with total space $M^\tau = M \setminus intN \cup_{j\tau} S^2 \times D^2$ and projection given by p on $M \setminus intN$ and by projection to the second factor on $S^2 \times D^2$.

Theorem 10. *Let $p : M \to B$ and $p' : M' \to B$ be S^2-orbifold bundles over the same base B and with the same action $u : \pi_1^{orb}(B) \to \{\pm 1\}$. If ΣB is nonempty then p' is isomorphic to p or p^τ, and so $M' \cong M$ or M^τ.*

Proof. The base B has a suborbifold N which contains ΣB and is a disjoint union of copies of regular neighbourhoods of reflector curves and copies of $D(2,2)$, by Lemma 6. If C is a reflector curve, with regular neighbourhood $N(C) \cong J \times S^1$, then $p^{-1}(N(C)) \cong \mathbb{E}$ or $\mathbb{E}'$, while if $D(2,2) \subset B$ then $p^{-1}(D(2,2)) \cong E(2,2)$.

Since N is nonempty and the restrictions of p and p' over $B\setminus N$ are S^2 bundles with the same data they are isomorphic. Moreover the bundles are trivial over the boundary components of $B \setminus N$. After composing with a fibrewise involution, if necessary, we may assume that the bundle isomorphism restricts to orientation-preserving homeomorphisms of these boundary components. Let R be a regular neighbourhood of a regular fibre S^2. Using Lemma 8 we may construct a fibre-preserving homeomorphism h from $M \setminus p^{-1}(R)$ to $M' \setminus p'^{-1}(R)$. If $h|_{\partial R}$ extends across R then $p' \cong p$; otherwise $p' \cong p^\tau$. □

If u is nontrivial the standard geometric 4-manifold M_{st} realizing $\pi = \pi_1^{orb}(B)$ is the total space of an orbifold bundle p_{st} with regular fibre S^2, base B and action u.

Corollary (A). *Every S^2-orbifold bundle is either geometric or is the Gluck reconstruction of a standard geometric orbifold bundle.* □

Corollary (B). *If ΣB contains an untwisted reflector curve then every S^2-orbifold bundle over B is a standard geometric bundle.* □

We may also realize actions with base a non-compact hyperbolic 2-orbifold by geometric orbifold bundles.

Corollary (C). *If B has a nontrivial decomposition into hyperbolic pieces then M has a proper geometric decomposition.* □

In particular, if B is hyperbolic then either M is geometric or it has a proper geometric decomposition.

5. SPHERICAL BASE ORBIFOLD

If the base orbifold is spherical then it must be one of S^2, RP^2, $S(2,2)$, $\mathbb{D}$ or $\mathbb{D}(2)$, by Lemma 6. There are two S^2-bundle spaces over S^2, and four over RP^2. The latter are quotients of $S^2 \times S^2$ by involutions of the form $(A, -I)$, where $A \in GL(3,\mathbb{Z})$ is a diagonal matrix, and projection to the quotient of the second factor by the antipodal map induces the bundle projection.

If $A = diag[-1,-1,1] = R_\pi$ or $diag[1,1,-1] = aR_\pi$ then projection to the first factor induces an orbifold bundle (over $S(2,2)$ or $\mathbb{D}$, respectively) with general fibre S^2. The geometric orbifold bundle over $S(2,2)$ has total space $E(2,2) \cup S^2 \times D^2$. There is one other such orbifold bundle over $S(2,2)$, with total space $RP^4 \#_{S^1} RP^4 = E(2,2) \cup_\tau S^2 \times D^2$. (Note that by Lemma 5 there is a bundle automorphism of $E(2,2) \setminus E(2)$ which is the twist τ on $\partial E(2,2)$ and the twist ξ on $\partial E(2)$. Hence

$E(2,2)\cup_\tau S^2\times D^2 \cong E(2)\cup_\xi E(2)$. The latter model for $RP^4\#_{S^1}RP^4$ is used in [3].) These spaces are not homotopy equivalent, since the values of the q-invariant of [3] differ. Thus τ does not extend to a homeomorphism of $E(2,2)$.

The S^2-orbifold bundle over $\mathbb{D} = S^2/z \sim aR_\pi(z)$ given by this construction is the unique such bundle, by Lemma 6. (The reflector curve is untwisted.) The total space is orientable and has $v_2 = 0$.

Finally, $\mathbb{D}(2)$ is the quotient of S^2 by the group $(Z/2Z)^2$ generated by a and R_π. Since these generators commute, R_π induces an involution of RP^2 which fixes RP^1 and a disjoint point. The corresponding S^2-orbifold bundle space is $S^2\times S^2/(x,y)\sim(x,-y)\sim(-x,R_\pi(y))$. This is also the total space of the nontrivial RP^2-bundle over RP^2.

6. The image of $[M]$ in $H_4(P_2(M);\mathbb{F}_2)$

If u is trivial then π is a PD_2-group, and so $k_1(M) = 0$. Let F be a closed surface with $\pi_1(F) = \pi$, and let $P = CP^\infty\times F \simeq \Omega K(Z,3)\times F$. The natural inclusion $f_{st} : M_{st} = S^2\times F \to P$ is 3-connected, and so it is the second stage of the Postnikov tower for M_{st}.

The nontrivial bundle space with this group and action is the Gluck reconstruction M^τ_{st}. We may assume that the neighbourhood N of a fibre is a product $S^2\times D^2$, where $D^2\subset F$. Let $h : M^\tau \to CP^2\times F \subset P$ be the map defined by $h(m) = f_{st}(m)$ for all $m\in M\setminus N$ and $h([z_0 : z_1], d) = ([dz_0 : z_1 : (1-|d|)z_0], d)$ for all $[z_0 : z_1]\in S^2 = CP^1$ and $d\in D^2$. (The two definitions agree on $S^2\times S^1$, since $\tau([z_0 : z_1], u) = ([uz_0 : z_1], u)$ for $u\in S^1$.) Then h is 3-connected, and so is the second stage of the Postnikov tower for M^τ_{st}.

By the Künneth Theorem,

$$H_4(P;\mathbb{F}_2)\cong H_4(CP^\infty;\mathbb{F}_2)\oplus(H_2(CP^\infty;\mathbb{F}_2)\otimes H_2(F;\mathbb{F}_2))\cong\mathbb{F}_2^2.$$

Homotopy classes of self-maps of P which induce the identity on π and π_2 are represented by maps $(c,f)\mapsto(c.s(f),f)$, where $s : F\to \Omega K(Z,3)$ and we use the loop space multiplication on $\Omega K(Z,3)$. It is not hard to see that these act trivially on $H_4(P;\mathbb{F}_2)$. Since automorphisms of π and π_2 are realized by self-homeomorphisms of F and CP^∞, respectively, $Aut(P)$ acts trivially on $H_4(P;\mathbb{F}_2)$.

Let $q : P\to CP^\infty$ be the projection to the first factor. Then qf_{st} factors through the inclusion of CP^1, and so has degree 0. On the other hand, if (w,d) is in the open subset $U = \mathbb{C}\times intD^2$ with $z_0\neq 0$ and $|d| < 1$ then $qh(w,d) = [d : w : 1-|d|]$, and $(qh)^{-1}([a : b : 1]) = (b/(1+|a|), a/(1+|a|))$. Hence qh maps U bijectively onto the dense open subset $CP^2\setminus CP^1$, and collapses $M^\tau_{st}\setminus h(U) = M\setminus intN$ onto CP^1. Therefore $qh : M^\tau_{st}\to CP^2$ has degree 1. Thus the images of

$[M_{st}]$ and $[M_{st}^\tau]$ in $H_4(P_2(M);\mathbb{F}_2)$ are not equivalent under the action of $Aut(P)$.

This is not surprising, as $v_2(M_{st})=0$, but twisting the neighbourhood of a regular fibre changes the *mod*-(2) self-intersection number of a section to the bundle, and so $v_2(M_{st}^\tau)\neq 0$.

If M is an S^2-orbifold bundle space with exceptional fibres then the image of a regular fibre in $H_2(M;\mathbb{F}_2)$ is trivial, since the inclusion factors through the covering $S^2\to RP^2$, up to homotopy. Therefore the *mod*-(2) Hurewicz homomorphism is trivial, and Gluck reconstruction does not change the *mod*-(2) self-intersection pairing. In particular, $H^2(\pi;\mathbb{F}_2)\cong H^2(M;\mathbb{F}_2)$, and $v_2(M_{st}^\tau)=v_2(M_{st})$.

Although we cannot expect to detect the effect of twisting through the Wu class, we may adapt the argument above to S^2-orbifold bundles with $u\neq 1$. Then

$$K(\pi,1)\simeq S^\infty\times K(\kappa,1)/(s,k)\sim(-s,\zeta(k)).$$

(If π is torsion-free we do not need the S^∞ factor.) The antipodal map of $CP^1=S^2$ extends to involutions on CP^n given by

$$[z_0:z_1:z_2:\dots:z_n]\mapsto[-\overline{z_1}:\overline{z_0}:\overline{z_2}:\dots:\overline{z_n}].$$

(Here only the first two harmonic coordinates change position or sign.) Since these are compatible with the inclusions of CP^n into CP^{n+1} given by $[z_0:\dots:z_n]\mapsto[z_0:\dots:z_n:0]$, they give rise to an involution σ on $CP^\infty=\varinjlim CP^n$. Let

$$P=CP^\infty\times S^\infty\times K(\kappa,1)/(z,s,k)\sim(\sigma(z),-s,\zeta(k)).$$

Then $\pi_1(P)\cong\pi$, $\pi_2(P)\cong Z^u$ and $\pi_j(P)=0$ for $j>2$. We shall exclude the case of RP^2-bundle spaces, with $\pi\cong\kappa\times Z/2Z$, as these are well understood. (The self-intersection number argument does apply in this case.)

Theorem 11. *Let π be a group with an epimorphism $u:\pi\to Z/2Z$ such that $\kappa=\mathrm{Ker}(u)$ is a PD_2-group, and suppose that π is not a direct product $\kappa\times Z/2Z$. Let M_{st} be the standard geometric 4-manifold corresponding to (π,u) and $P=P_2(M_{st})$. Then the images of $[M_{st}]$ and $[M_{st}^\tau]$ in $H_4(P;\mathbb{F}_2)$ are distinct.*

Proof. The diagonal map from S^2 to $S^2\times S^2=CP^1\times S^2$ determines a 3-connected map $f_{st}:M_{st}\to P$ by $f_{st}([s,k])=[s,s,k]$. This is the second stage of the Postnikov tower for M_{st}, and embeds M_{st} as a submanifold of $CP^1\times S^2\times K(\kappa,1)/\sim$ in P. We again have $H_4(P;\mathbb{F}_2)\cong\mathbb{F}_2^2$, with generators the images of $[M_{st}]$ and $[CP^2]$.

The projection of $CP^\infty\times S^\infty\times K(\kappa,1)$ onto its first two factors induces a map $g:P\to Q=CP^\infty\times S^\infty/(z,s)\sim(\sigma(z),-s)$ which is in fact a bundle projection with fibre $K(\kappa,1)$. Since gf_{st} factors through S^2 the image of $[M_{st}]$ in $H_4(Q;\mathbb{F}_2)$ is trivial.

Since π is not a direct product, M_{st} is the total space of an S^2-orbifold bundle p_{st}. Let $v : S^2 \times D^2 \to V \subset M_{st}$ be a fibre-preserving homeomorphism onto a regular neighbourhood of a regular fibre. Since V is 1-connected $f_{st}|_V$ factors through $CP^\infty \times S^\infty \times K(\kappa,1)$. Let f_1 and f_2 be the composites of a fixed lift of $f_{st}v\tau : S^2 \times S^1 \to P$ with the projections to CP^∞ and S^∞, respectively. Let F_1 be the extension of f_2 given by

$$F_2([z_0 : z_1], d) = [dz_0 : z_1 : (1 - |d|)z_0]$$

for all $[z_0 : z_1] \in S^2 = CP^1$ and $d \in D^2$. Since f_2 maps $S^2 \times S^1$ to S^2 it is nullhomotopic in S^3, and so extends to a map $F_2 : S^2 \times D^2 \to S^3$. Then the map $F : M_{st}^\tau \to P$ given by f_{st} on $M_{st} \setminus N$ and $F(s, d) = [F_1(s), F_2(s), d]$ for all $(s, d) \in S^2 \times D^2$ is 3-connected, and so it is the second stage of the Postnikov tower for M_{st}^τ.

Now F_1 maps the open subset $U = \mathbb{C} \times intD^2$ with $z_0 \neq 0$ bijectively onto its image in CP^2, and maps V onto CP^2. Let Δ be the image of CP^1 under the diagonal embedding in $CP^1 \times CP^1 \subset CP^2 \times S^3$. Then (F_1, F_2) carries $[V, \partial V]$ to the image of $[CP^2, CP^1]$ in $H_4(CP^2 \times S^3, \Delta; \mathbb{F}_2)$. The image of $[V, \partial V]$ generates $H_4(M, M \setminus U; \mathbb{F}_2)$. A diagram chase now shows that $[M_{st}^\tau]$ and $[CP^2]$ have the same image in $H_4(Q;\mathbb{F}_2)$. Since this is nonzero it follows that M_{st}^τ is not homotopy equivalent to M_{st}. Thus every homotopy type in this algebraic 2-type is realized by an S^2-orbifold bundle space. □

Is there a more explicit invariant? The q-invariant of [3] is 0 for every orbifold bundle with regular fibre S^2 over an aspherical base.

Corollary. *Let M be a closed 4-manifold with $\pi_2(M) \cong Z$. Assume that $\pi = \pi_1(M)$ acts nontrivially on $\pi_2(M)$, but is not a product, and let M_κ be the double cover associated to the kernel κ of the action $u : \pi \to \{\pm 1\}$. Then M is homotopy equivalent to an S^2-bundle space, and $M_\kappa \simeq S^2 \times K(\kappa, 1)$.*

Proof. The first assertion is an immediate consequence of Theorems 3 and 11, since M_{st} and M_{st}^τ exhaust the possibilities allowed by Theorem 10.6 of [2].

The double cover of M_{st} is $S^2 \times K(\kappa, 1)$, and the double cover of M_{st}^τ may be obtained from this by two Gluck reconstructions. Hence these covers are homeomorphic. The second assertion follows. □

The quotient of the total space of any S^2-bundle over a closed surface F by the fibrewise antipodal involution is an RP^2-bundle over F. Thus the condition that π be not a product is necessary for this corollary.

Since $S^2 \times D^2 = (D^2 \times D^2) \cup (D^2 \times D^2) = (D^2 \times D^2) \cup D^4$, we may obtain each of M_{st} and M_{st}^τ from $M_{st} \setminus N$ (up to homotopy) by first adding a 2-cell and then a 4-cell. The attaching maps for the 2-cells are the inclusions $u \mapsto (1, u)$ and $u \mapsto (u, u)$ of S^1 into $\partial N = S^2 \times S^1$, respectively. Since these are clearly homotopic, M_{st}^τ may be obtained from M_{st} by changing the attaching map for the top cell of $M_{st} = M_o \cup D^4$. (It can be shown that the attaching maps differ by the image of the Hopf map η in $\pi_3(M_o)$.)

The inclusion of M_o induces isomorphisms of cohomology with coefficients $\mathbb{F}_2$ in degrees ≤ 3. Since $U^3 = 0$ in M_{st} it follows that $U^3 = 0$ in general. (Can we see this more directly?)

7. $\mathbb{S}^2 \times \mathbb{H}^2$-manifolds

Let M be the total space of an S^2-orbifold bundle with hyperbolic base orbifold B. If B has an untwisted reflector curve then M must be diffeomorphic to M_{st}. If B has no reflector curves and $\pi = \pi^{orb}(B)$ is generated by involutions then $B = S(2_{2k})$ is the quotient of the orientable surface of genus $k - 1$ by the hyperelliptic involution. By Lemma 7, there is an unique k-invariant realized by 4-manifolds with this fundamental group and $\pi_2 \cong Z$. Moreover, $H^2(\pi; Z^u) = 0$, and so there are two distinct homotopy types of such manifolds. Each is represented by an S^2-orbifold bundle space over B, but only one one is geometric. However, if B has no reflector curves but π is not generated by involutions, or if all reflector curves are twisted then it is not clear whether M_{st}^τ is also geometric.

8. $\mathbb{S}^2 \times \mathbb{E}^2$-manifolds

In this section we shall assume that M is a closed 4-manifold with $\chi(M) = 0$ and $\pi_2(M) \cong Z$ (equivalently, that π is virtually Z^2). In Chapter 10 of [2] it is shown that there are between 21 and 24 possible homotopy types of such 4-manifolds. Ten are total spaces of S^2-bundles over T or Kb, four are total spaces of RP^2-bundles, and four are mapping tori of self-homeomorphisms of $RP^3 \# RP^3$. These bundle spaces are all $\mathbb{S}^2 \times \mathbb{E}^2$-manifolds, and their homotopy types are detected by the fundamental groups and Stiefel-Whitney classes.

The uncertainty relates to the three possible fundamental groups with finite abelianization. Each such group is realized by at most two homotopy types. In each case, the Stiefel-Whitney classes are determined by the group, and the orientation character w is non-trivial. There is one geometric manifold for each of the groups $D *_Z D$ and $(Z \oplus (Z/2Z)) *_Z D$, and two for $Z *_Z D$. We shall show that there is another (non-geometric) orbifold bundle over $S(2,2,2,2)$ (with group $D *_Z D$), and that these five homotopy types are distinct.

If M is an orbifold bundle over a flat base then it follows from Lemma 2 that either

(1) M is an S^2- or RP^2-bundle over T or Kb; or
(2) $B = \mathbb{A}$ or $\mathbb{M}b$; or
(3) $B = S(2,2,2,2)$, $P(2,2)$ or $\mathbb{D}(2,2)$.

When the base is $S(2,2,2,2) = D(2,2) \cup D(2,2)$ there are two possible S^2-orbifold bundles. The total spaces are not homotopy equivalent, since $H^2(D *_Z D; Z^u) = 0$, by Lemma 7. The double of $E(2,2)$ is geometric, whereas $E(2,2) \cup_\tau E(2,2)$ is not.

There is just one S^2-orbifold bundle with base $\mathbb{D}(2,2) = J \times S^1 \cup D(2,2)$, by Lemma 6. It has geometric total space.

The orbifold $P(2,2) = D(2,2) \cup Mb$ is the quotient of the plane $\mathbb{R}^2$ by the group of euclidean isometries generated by $t = (\frac{1}{2}\mathbf{j}, \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right))$ and $x = (\frac{1}{2}(\mathbf{i}+\mathbf{j}), -I)$. There are two possible S^2-orbifold bundles with base $P(2,2)$. If we fix identifications of ∂Mb with S^1 and $\partial E(2,2)$ with $S^2 \times S^1$ then one has total space $M = E(2,2) \cup S^2 \times Mb$ and the other has total space $M' = E(2,2) \cup_\tau S^2 \times Mb$. (The bundles with total space $E(2,2) \cup_{(\tau)} S^2 \tilde{\times} Mb$ are each equivalent to one of these via the automorphism of the base induced by reflection of $\mathbb{R}^2$ across the principal diagonal.)

The total spaces of these two S^2-orbifold bundles are the two affinely distinct $\mathbb{S}^2 \times \mathbb{E}^2$-manifolds with fundamental group $Z *_Z D \cong \pi_1^{orb}(P(2,2))$. Let $T = (U, t)$ and $X = (a, x)$, where $U = \pm 1 \in S^1$. (Equivalently, $U = I_3$ or $R_\pi = diag[-1,-1,1] \in GL(3,\mathbb{Z})$.) Then $\{t, x\}$ generates a free, discrete, cocompact isometric action of $Z *_Z D$ on $S^2 \times R^2$. The subgroup $\kappa \cong Z \rtimes_{-1} Z$ is generated by T and $(XT)^2$.

Let $\widetilde{A} = S^2 \times [-\epsilon, \epsilon] \times \mathbb{R}$ and $\widetilde{B} = S^2 \times [\epsilon, \frac{1}{2} - \epsilon] \times \mathbb{R}$, where $0 < \epsilon < \frac{1}{2}$. Then in each case the total space $M = \widetilde{A}/\langle T \rangle \cup \widetilde{B}/\langle X, T^2 \rangle$. Note that $\widetilde{B}/\langle X, T^2 \rangle$ is independent of the choice of U. If $U = id$ then there is an obvious homeomorphism $\widetilde{A}/\langle T \rangle = S^2 \times Mb$. If $U = R_\pi$ then we may define a homeomorphism $h : \widetilde{A}/\langle T \rangle \to S^2 \times Mb$ by $h([z, x, y] = (e^{iy}z, [x, y])$. It is then clear that these S^2-orbifold bundles differ by the twist τ. These two manifolds are not homotopy equivalent, since $H^2(Z *_Z D; Z^u) = 0$, by Theorem 9.

The example constructed using $U = id$ fibres over RP^2, with fibre Kb. Does the other example also fibre over RP^2?

9. SURGERY

If $\chi = 0$ the relevant surgery obstruction groups can be computed (or shown to be not finitely generated) in most cases, via the Shaneson-Wall exact sequences and the results of [1] on $L_n(D, w)$. Lück has settled the one case in which such reductions do not easily apply [5]. (The groups $L(\pi) \otimes \mathbb{Z}[\frac{1}{2}]$ are computed for all aspherical 2-orbifold groups π when w is trivial in [6].)

Let σ be the automorphism of $D = Z/2Z * Z/2Z$ which interchanges the factors. Let $I_\pi : \pi/\pi' \to L_1(\pi)$ be the natural transformation described in §6.2 of [2].

(a) Z^2 and $Z \rtimes_{-1} Z$.

(b) $\kappa \times (Z/2Z)^-$. $L_1(\pi, w) \cong (Z/2Z)^2$.

(c) $L_1(D \times Z) \cong \mathrm{Cok}(: L_0(1) \to L_0(Z/2Z)^2) \cong Z^3$. The direct summand $L_1(Z) \cong Z$ is the image of I_π.

(d) $D \times Z^-$. $L_1(\pi, w) = 0$. $L_0(\pi, w)$ has exponent 2 and infinite rank.

(e) $L_5(D \rtimes_\sigma Z) \cong \mathrm{Ker}(1 - L_0(\sigma)) \cong Z^2$. The direct summand $L_1(Z) \cong Z$ is the image of I_π.

(f) $D \rtimes_\sigma Z^-$. $L_1(\pi, w) \cong \mathrm{Ker}(1 + L_0(\sigma)) \cong Z$.

The remaining three groups $L_1(\pi, w)$ are not finitely generated.

(g) $D *_Z D$ retracts onto $D(-,-) = Z/2Z^- * Z/2Z^-$, compatibly with w_1.

(h) $(Z \oplus (Z/2Z)) *_Z D$ retracts onto $D(-,-) = Z/2Z^- * Z/2Z^-$, compatibly with w_1.

(i) $Z *_Z D$ does not surject to D. However $L_1(Z *_Z D, w)$ has an infinite $UNil$ summand, of exponent 4 [5].

For closed 4-manifolds with $\chi = 0$ and π of type (a) homotopy type implies homeomorphism. If $\pi \cong Z^2 \times Z/2Z$ then $|S_{TOP}(M)| = 8$, while if $\pi \cong Z \rtimes_{-1} Z \times Z/2Z$ then $8 \leq |S_{TOP}(M)| \leq 32$. If M is in type d then $|S_{TOP}(M)| \leq 16$.

In each of the remaining cases the structure sets are infinite. In order to estimate the number of homeomorphism types within each homotopy type we must consider the actions of the groups $E(M)$ of homotopy classes of self-homotopy equivalences.

Let M be a closed 4-manifold with $\widetilde{M} \simeq S^2$. As observed above, if M is the total space of an orbifold bundle then $Aut(\pi)$ and $Aut(\pi_2(M))$ act on M via homeomorphisms. Thus in order to understand the action of $E(M)$ on $S_{TOP}(M)$ it is sufficient to consider the action of the subgroup

$K_\pi(M)$ of self-homotopy equivalences which induce the identity on π and $\pi_2(M)$. (Note also that if $f : M \to M$ is a self-map such that $\pi_2(f) = id$ then lifts of f to $\widetilde{M}$ are homotopic to the identity, and so $\pi_k(f) = id$ for all $k \geq 2$.)

We may assume that $M_o = M \setminus intD^4$ is homotopy equivalent to a 3-complex. Fix a basepoint $* \in M_o$. Let $P_3(M) = M \cup e^{\geq 5}$ be the 3-stage of the Postnikov tower for M. (Thus $\pi_i(M) \cong \pi_i(P_3(M))$ for $i \leq 3$ and $\pi_j(P_3(M)) = 0$ for all $j > 3$). If $(X, *)$ is a based space let $E_*(X)$ be the group of based homotopy classes of based self-homotopy equivalences. If $f \in E_*(M)$ is in the kernel of the natural homomorphism from $E_*(M)$ to $E_*(P_3(M))$ then we may assume that $f|_{M_o}$ is the identity, by cellular approximation. Thus f differs from id_M by at most a pinch map corresponding to $\eta S\eta \in \pi_4(\widetilde{M}) = Z/2Z$.

Let $K_\#$ be the kernel of the natural homomorphism from $E_*(P_3(M))$ to $\Pi_{j\leq 3}Aut(\pi_j)$. Let $P = P_2(M)$ be the 2-stage of the Postnikov tower for M. Then $K_\#(M)$ maps onto $K_\#$, with kernel of order ≤ 2. There is an exact sequence

$$H^1(\pi; Z^u) \xrightarrow{\Delta} H^3(P;\mathbb{Z}) \to K_\# \to H^2(\pi; Z^u) \xrightarrow{\rho} H^3(P;\mathbb{Z}),$$

and the image of $H^3(P;\mathbb{Z})$ under the second homomorphism is central. The homomorphism Δ involves the second k-invariant $k_2(M) \in H^4(P;\mathbb{Z})$ and factors through the finite group $H^3(\pi;\mathbb{Z})$. The kernel of ρ is the isotropy subgroup of $k_2(M)$ under the action of $H^2(\pi; Z^u)$ on P. (See Corollary 2.9 of [7].)

Since $v.c.d.\pi = 2$ spectral sequence arguments show that $H^i(\pi; Z^u)$ is commensurable with $H^0(Z/2Z; H^i(\kappa;\mathbb{Z}) \otimes Z^u)$, for all i, and $H^3(P;\mathbb{Z})$ is commensurable with $H^1(\pi;\mathbb{Z})$. Thus $K_\#$ is a finitely generated, nilpotent group. In particular, if π/π' is finite then $K_\#$ is finite, and so there are infinitely many homeomorphism types within each such homotopy type.

However, if $\pi \cong D \times Z$ or $D \rtimes Z$ then $K_\#$ is infinite, and it is not clear how this group acts on $S_{TOP}(M)$.

10. SURFACE BUNDLES OVER RP^2

Let F be a closed aspherical surface and $p : M \to RP^2$ be a bundle with fibre F, and such that $\pi_2(M) \cong Z$. (This condition is automatic if $\chi(F) < 0$.) Then $\pi = \pi_1(M)$ acts nontrivially on $\pi_2(M)$. The covering space M_κ associated to the kernel κ of the action is an F-bundle over S^2, and so $M_\kappa \cong S^2 \times F$, since all such bundles are trivial. The projection admits a section if and only if $\pi \cong \kappa \rtimes Z/2Z$.

The product $RP^2 \times F$ is easily characterized.

Theorem 12. *Let M be a closed 4-manifold with fundamental group π, and let F be an aspherical closed surface. Then the following are equivalent.*

(1) $M \simeq RP^2 \times F$;
(2) $\pi \cong Z/2Z \times \pi_1(F)$, $\chi(M) = \chi(F)$ and $v_2(M) = 0$;
(3) $\pi \cong Z/2Z \times \pi_1(F)$, $\chi(M) = \chi(F)$ and $M \simeq E$, where E is the total space of an F-bundle over RP^2.

Proof. Clearly (1) $\Rightarrow$ (2) and (3). If (2) holds then M is homotopy equivalent to the total space of an RP^2-bundle over F, by Theorem 5.16 of [2]. This bundle must be trivial since $v_2(M) = 0$. If (3) holds then there are maps $q : M \to F$ and $p : M \to RP^2$ such that $\pi_1(p)$ and $\pi_1(q)$ are the projections of π onto its factors and $\pi_2(p)$ is surjective. The map $(p, q) : M \to RP^2 \times F$ is then a homotopy equivalence. □

The implication (3) $\Rightarrow$ (1) fails if $F = RP^2$ or S^2.

We may assume henceforth that π is not a product.

Theorem 13. *A closed orientable 4-manifold M is homotopy equivalent to the total space of an F-bundle over RP^2 with a section if and only if $\pi = \pi_1(M)$ has an element of order 2, $\pi_2(M) \cong Z$ and $\kappa = \mathrm{Ker}(u) \cong \pi_1(F)$, where u is the natural action of π on $\pi_2(M)$.*

Proof. The conditions are clearly necessary. If they hold, then M is homotopy equivalent to an S^2-orbifold bundle space (since it is not homotopy equivalent to an RP^2-bundle space). The double cover M_κ is an S^2-bundle over F. Since M is orientable and κ acts trivially, F must also be orientable. Hence the base orbifold must have untwisted reflector curves. Therefore $M \simeq M_{st}$, which is the total space of an F-bundle over RP^2 with a section. □

Orientability is used here mainly to ensure that π is the group of a 2-orbifold with an untwisted reflector curve. (Note that the S^2-orbifold bundle over $\mathbb{D}(2)$ is also an RP^2-bundle over RP^2.)

When π is torsion-free M is homotopy equivalent to the total space of an S^2-bundle over a surface B, with $\pi = \pi_1(B)$ acting nontrivially on the fibre. Inspection of the geometric models for such bundle spaces shows that if also $v_2(M) \neq 0$ then the bundle space fibres over RP^2. (See Theorems 10.8 and 10.9 of [2].) Is the condition $v_2(M) \neq 0$ necessary?

References

[1] Connolly, F. and Davis, D. The surgery obstruction groups of the infinite dihedral group, Geometry and Topology 8 (2004), 1043–1087
[2] Hillman, J. A. *Four-Manifolds, Geometries and Knots*, Geometry and Topology Monographs 5, Geometry and Topology Publications (2002). (Revision 2007).
[3] Kim, M.H., Kojima, S. and Raymond, F. Homotopy invariants of nonorientable 4-manifolds, Trans. Amer. Math. Soc. 333 (1992), 71–83.
[4] Kim, M.H., and Raymond, F. The diffeotopy group of the twisted 2-sphere bundle over the circle, Trans. Amer. Math. Soc. 322 (1990), 159–168.
[5] Lück, W. personal communication, July 2010.
[6] Lück, W. and Stamm, R. Computations of K- and L-theory of cocompact planar groups, K-Theory 21 (2000), 249–292.
[7] Rutter, J.W. The group of self-homotopy equivalences of non-simply connected spaces using Postnikov decompositions, Proc. Roy. Soc. Edinburgh Ser. A. 120 (1992), 47–60.
[8] Vogt, E. Foliations of codimension 2 with all leaves compact on closed 3-, 4- and 5-manifolds, Math. Z. 157 (1977), 201–223.

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