Plünnecke inequalities for measure graphs with applications

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Abstract

We generalize Petridis's new proof of Plünnecke's graph inequality [6] to graphs whose vertex set is a measure space. Consequently, by a recent work of Björklund and Fish [2], this gives new Plünnecke inequalities for measure preserving actions which enable us to deduce, via a Furstenberg correspondence principle, Banach density estimates in countable abelian groups that improve on those given by Jin in [5].

1 Introduction

1.1 Background and summary of results

Given an abelian group G with subsets $A, B \subset G$, it is of great interest to estimate the size of the *product set* (commonly referred to as the *sumset* when additive notation is employed) defined by

$$AB = \{ab | a \in A, b \in B\}.$$

In particular, one is also interested in the sizes of iterated product sets B^k, which may be recursively defined by $B^1 = B$ and $B^k = B^{k-1}B$ for positive integers $k > 1$. General inequalities regarding the cardinalities of these were given by Plünnecke and Ruzsa, a comprehensive treatment of which may be found in [8]. In particular, it was shown in [7] that if one defines, for finite sets $A \subset G$ and $B \subset G$, the magnification ratios

$$D_k = \min_{B' \subset B, B' \neq \emptyset} \frac{|A^k B'|}{|B'|}$$

then $D_k^{1/k}$ is a decreasing sequence in k. For infinite subsets we can no longer use cardinalities. It is natural, instead, to use the notion of an invariant density. This notion can be defined in countable amenable groups - the groups which possess a Følner sequence. By a Følner sequence, we mean a sequence of finite subsets $F_n \subset G$ such that for each $g \in G$ we have

$$\lim_{n \to \infty} \frac{|F_n \cap gF_n|}{|F_n|} = 1.$$

A countable group for which a Følner sequence exists is called amenable. It is well known that all countable abelian groups are amenable. Each Følner sequence $(F_n)_{n\in\mathbb{N}}$ gives rise to corresponding upper and lower densities for $A \subset G$ given by

$$\overline{d}_{(F_n)}(A) = \limsup_{n\to\infty} \frac{|F_n \cap A|}{|F_n|}$$

and

$$\underline{d}_{(F_n)}(A) = \liminf_{n\to\infty} \frac{|F_n \cap A|}{|F_n|},$$

respectively. For example, the Følner sequence $F_n = [1, n] \cap \mathbb{Z}$ in the additive group $\mathbb{Z}$ gives rise to the classical upper and lower *asymptotic* densities in the natural numbers, which are of particular interest in Number Theory. Of interest to us is the *upper* (resp. *lower*) *Banach density*, denoted by $d^*(A)$ (resp. $d_*(A)$), which may be defined as the maximum (resp. minimum) of the set

$$\{\overline{d}_{(F_n)}(A) | (F_n)_{n\in\mathbb{N}} \text{ is Følner}\}.$$

The fact that these extrema are attained in any given countable amenable group may be verified by a simple diagonalisation argument. In fact one can show (see Lemma 3.3 of [1]) the stronger assertion that given any Følner sequence $(F_n)_{n\in\mathbb{N}}$ and $A \subset G$, there exist $t_n \in G$ such that

$$d^*(A) = d_{(t_n F_N)}\overline{d}(A),$$

and likewise for lower Banach density. In particular, in the additive group $\mathbb{Z}^d$ one only needs to look at sequences of finite cubes of strictly increasing cardinality. One may now ask whether Banach densities of sumsets satisfy analogous Plünnecke-Ruzsa type inequalities. Indeed, Jin has proven in [5] such inequalities for the additive semi-group $\mathbb{Z}_{\geq 0}$.

Theorem 1.1 (Jin)**.** Suppose $A, B \subset \mathbb{Z}_{\geq 0}$, then

$$d^*(A + B) \geq d^*(kA)^{1/k} d^*(B)^{1-\frac{1}{k}}$$

and

$$d_*(A + B) \geq d^*(kA)^{1/k} d_*(B)^{1-\frac{1}{k}}.$$

In [2], Björklund and Fish introduced a new dynamical approach for Plünnecke type inequalities. As a result, they extended Jin's theorem to any countable abelian group.

Theorem 1.2 (Björklund, Fish)**.** Suppose G is a countable abelian group and $A, B \subset G$. Then we have

$$d^*(AB) \geq d^*(A^k)^{1/k} d^*(B)^{1-\frac{1}{k}}$$

and

$$d_*(AB) \geq d^*(A^k)^{1/k} d_*(B)^{1-\frac{1}{k}}.$$

Our work extends [2] and provides analogous lower bounds for $d^*(A^jB)$ and $d_*(A^jB)$. More precisely, we prove following.

Theorem 1.3. Suppose that G is a countable abelian group and $A, B \subset G$. Then for integers $0 < j < k$ we have

$$d^*(A^jB) \geq d^*(A^k)^{j/k} d^*(B)^{1-j/k}$$

and

$$d_*(A^jB) \geq d^*(A^k)^{j/k} d_*(B)^{1-j/k}.$$

It seems unclear whether or not our results, even for just $G = (\mathbb{Z}, +)$, are attainable from an application of Jin's techniques to the inequality $D_k^{1/k} \leq D_j^{1/j}$. We are also able to obtain a result involving multiple different factors, which is a generalization of Theorem 1.2.

Theorem 1.4. Suppose G is a countable abelian group and $B, A_1, \ldots, A_k \subset G$. Then

$$d^*(A_1 \ldots A_k) \leq d^*(B)^{1-k} \prod_{i=1}^{k} d^*(A_iB)$$

and

$$d^*(A_1 \ldots A_k) \leq d_*(B)^{1-k} \prod_{i=1}^{k} d_*(A_iB).$$

The strategy of the proofs of the main theorems involves employing an ergodic approach. This approach was developed by Björklund and the second author in [2]. First, we prove a Plünnecke inequality for measure preserving actions and then we combine it with a Furstenberg correspondence principle for product sets. Next, we recall the magnification ratios defined for the dynamical setting in [2].

Definition 1.5 (G acting on a measure space)**.** We say that a group G acts on a measure space $(X, \mathcal{B}, \mu)$ if, for each $g \in G$, the map $x \mapsto g.x$ is measure preserving, i.e., it is measurable and $\mu(gB) = \mu(B)$ for each $B \in \mathcal{B}$.

Definition 1.6 ([2])**.** Given a countable abelian group G acting on a measure space $(X, \mathcal{B}, \mu)$, let us define, for $A \subset G$ and $B \in \mathcal{B}$ of finite positive measure, the magnification ratio

$$c(A, B) = \inf \left\{ \frac{\mu(AB')}{\mu(B')} | B' \subset B, \mu(B') > 0 \right\}.$$

The following is an extension to general $j < k$ of the result in [2].

Theorem 1.7. If A is finite, then for positive integers $j < k$ we have

$$c(A^k, B)^{1/k} \leq c(A^j, B)^{1/j}.$$

Note that the classical Plünnecke-Ruzsa inequality is the case where $X = G$, μ is the counting measure and the action is by multiplication. It follows from the techniques developed in [2] that Theorem 1.7 implies Plünnecke inequalities for $A \subset G$ not necessarily finite.

Theorem 1.8 (Plünnecke inequalities)**.** If $\mu(X) = 1$ and $L^2(X, \mathcal{B}, \mu)$ is separable, then for positive integers $j < k$ we have

$$c(A^k, B)^{1/k} \leq c(A^j, B)^{1/j}.$$

Next, we recall Furstenberg's correspondence principle for product sets. This correspondence principle can be derived from the seminal work of Furstenberg [4]. Nevertheless, it was noticed much later in [1]. The following version of the correspondence principle (the third and fourth inequalities) for product sets is due to Björklund and the second author [3].

Proposition 1.9 (Furstenberg's correspondence principle)**.** Suppose that G is a countable abelian group and $A, B \subset G$. Then there exists a compact metrizable space X on which G acts by homeomorphisms such that there exist G-invariant ergodic Borel probability measures μ, ν on X together with a clopen $\widetilde{B} \subset X$ such that

$$\begin{aligned} d^*(B) &= \mu(\widetilde{B}) \\ d^*(AB) &\geq \mu(A\widetilde{B}) \\ d_*(B) &\leq \nu(\widetilde{B}) \\ d_*(AB) &\geq \nu(A\widetilde{B}). \end{aligned}$$

Next, we demonstrate how Theorem 1.8, through Furstenberg's correspondence principle, implies Theorem 1.3 and Theorem 1.4.

Proof of Theorem 1.3 and Theorem 1.4: Let (X, μ) and $\widetilde{B}$ be as in the correspondence principle. Note that we may assume that $d^*(B) > 0$ as the result is trivial otherwise. Note also that in Section 7 (see Lemma 7.3) we show that $d^*(A^k) \leq \mu(A^k\widetilde{B})$. Altogether, this gives

$$\begin{aligned} \left(\frac{d^*(A^jB)}{d^*(B)}\right)^{1/j} &\geq \left(\frac{\mu(A^j\widetilde{B})}{\mu(\widetilde{B})}\right)^{1/j} \\ &\geq c(A^j, \widetilde{B})^{1/j} \\ &\geq c(A^k, \widetilde{B})^{1/k} \\ &\geq \left(\frac{d^*(A^k)}{d^*(B)}\right)^{1/k}, \end{aligned}$$

which shows the first inequality. The second one may be deduced from the same argument applied to the measure ν instead of μ from the correspondence principle. Moreover, Theorem 1.4 may be obtained from applying the correspondence principle to the Plünnecke inequality for different summands (Proposition 6.2). ■

1.2 Outline of paper

The main object introduced in this paper is, what we call, a *measure graph*. Section 2 provides all the relevant definitions and basic properties. Intuitively, a measure graph is a directed edge-labelled graph equipped with a measure on the vertex set that mimics certain elementary combinatorial properties of the classical graph-theoretic notion of a *matching*. The aim is to prove a measure-theoretic version of the classical Plünnecke inequality for commutative graphs. The classical approach employs Menger's theorem, which has no obvious measure theoretic analogue. However, Petridis [6] has recently found a new proof of this inequality that avoids the use of Menger's theorem. In Section 3 we generalize this proof to measure graphs. Immediate corollaries concerning measure preserving actions are given in Section 4. We then, in Section 5, turn to extending the results regarding $c(A, B)$ with A finite, to ones where A is countable. Section 6 is devoted to a proof of a measure-theoretic analogue of Ruzsa's Plünnecke inequality involving different summands. Finally, we prove the correspondence principle for products sets in Section 7.

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2 Definitions

By a labelled directed graph we mean a tuple (V, E, A) where V and A are sets and $E \subset V \times V \times A$. We regard an element $(v, w, a) \in E$ as an edge directed from v to w and labelled a. For subsets $W \subset V$ and labels $a \in A$ the a-image and a-preimage are defined, respectively, as

$$Im_a^+(W) = \{v \in V | (w, v, a) \in E \text{ for some } w \in W\}$$

and

$$Im_a^-(W) = \{v \in V | (v, w, a) \in E \text{ for some } w \in W\}.$$

That is, the a-image of W consists of the vertices that may be approached to from W by walking, in the direction of the orientation, along an edge labelled a. Moreover we define for $W \subset V$ the (pre)image $Im^{\pm}(W) = \bigcup_{a \in A} Im_a^{\pm}(W)$. For each integer h we have the h-fold image $Im^h(W)$ defined recursively by $Im^0(W) = W$ and $Im^h(W) = Im^+(Im^{h-1}(W))$ for $h > 0$ and $Im^h(W) = Im^-(Im^{h+1}(W))$ for $h < 0$. In other words, $Im^h(W)$ consists of all end points of walks with $|h|$ steps that begin at W and agree (resp. disagree) with the orientation of each edge if $h > 0$ (resp. $h < 0$). Define also the incoming and outgoing degrees of a vetex v as $d^-(v) = |\{(x, y, a) \in E | y = v\}|$ and $d^+(v) = |\{(x, y, a) \in E | x = v\}|$ respectively. Note that $|Im^{\pm}(\{v\})| \leq d^{\pm}(v)$ with strict inequality possible in case of multiple edges between two vertices (of course any two such edges would have different labels). Given an edge e from v to w, we will call v the tail, denoted $tail(e)$, and w the head, denoted by $head(e)$. Let $E^+(v)$ denote the edges whose tail is v and $E^-(v)$ those edges whose head is v.

Definition 2.1. A *measure graph* is a tuple $\Gamma = (V, \mathcal{B}, \mu, A, E)$ where $(V, \mathcal{B}, \mu)$ is a finite measure space (that is, $\mu(V) < \infty$), A is a finite set and (V, E, A) is a labelled directed graph such that

1. For each $a \in A$ the sets

   $$L_a^+(\Gamma) = L_a^+ = \{x \in V | \text{ There exists } y \in V \text{ such that } (x, y, a) \in E\}$$

   and

   $$L_a^-(\Gamma) = L_a^- = \{x \in V | \text{ There exists } y \in V \text{ such that } (y, x, a) \in E\}$$

   are measurable.

2. For $a \in A$ and measurable $W \subset L_a^\pm$ we have that $Im_a^\pm(W)$ is measurable and $\mu(W) = \mu(Im_a^\pm(W))$.

3. For each label $a \in A$ and vertex $x \in V$ there is at most one outgoing and at most one incoming a-labelled edge incident to x. That is, $|Im_a^\pm(x)| \leq 1$.

Example 2.2 (The (A, Y, h)-orbit graph)**.** Given an abelian group G acting on a measure space $(X, \mathcal{B}, \mu)$ one may form for each integer $h > 0$, finite $A \subset G$ and $Y \in \mathcal{B}$ of finite measure, a measure graph whose underlying vertex set is $\bigsqcup_{k=0}^{h} A^k Y \times \{k\}$ with edge set

$$\{((x, k), (a.x, k + 1), a) | a \in A, x \in A^k Y, k = 0, 1 \ldots, h - 1\}.$$

The measure is the restriction of the natural product measure on $X \times \{0, 1, \ldots, h\}$.

Given a labelled graph $\Gamma = (V, E, A)$ and $W \subset V$, the subgraph *induced* by W is the directed labelled graph (W, E_W, A) where $E_W = \{(w_1, w_2, a) | w_1, w_2 \in W, a \in A \text{ and } (w_1, w_2, a) \in E\}$. We say that a subgraph of Γ is an *induced subgraph* if it is induced by some subset of V.

Example 2.3. If $\Gamma = (V, \mathcal{B}, \mu, A, E)$ is a measure graph and $W \subset V$ is measurable then the subgraph of Γ induced by W is a measure graph (with the restricted measure, restricted σ-algebra and the same edge-label set A). Note that the set of vertices with an outgoing edge labelled $a \in A$ is $L_a^+ \cap W \cap Im_a^-(L_a^- \cap W)$ and thus is measurable as required.

Note that the (A, Y, h)-orbit graph defined above is a generalization of the commutative addition graph studied in classical Additive Combinatorics, see for example [8], [9]. It is also an example of what is known as a commutative, or Plünnecke, graph which may be defined as follows.

Definition 2.4. A *layered-graph* is a directed labelled graph (V, E, A) together with a partition $V = V_0 \sqcup V_1 \sqcup \ldots \sqcup V_h$ such that if $e = (x, y, a) \in E$ is a directed edge then $x \in V_i$ and $y \in V_{i+1}$

for some $i \in \{0, \ldots h-1\}$. We call V_k the k-th layer and we say that (V, E, A) is a *h-layered* graph (we regard the partition as part of the data of a layered graph). A *semi-commutative* (or *semi-Plünnecke*) graph is a layered graph (V, E, A) such that if $(x, y, a) \in E$ is an edge then there is an injection $\phi : E^+(y) \to E^+(x)$ such that $(head(\phi(e)), head(e), a) \in E$ for all $e \in E^+(y)$. A *commutative*, or *Plünnecke*, graph Γ is a directed layered graph such that both Γ and the dual graph (the layered graph obtained by reversing edges and the ordering of the layers) Γ^* are semi-commutative.

Example 2.5. The (A, Y, h)-orbit graph defined above is a commutative graph with layering $V = \bigsqcup_{j=0}^{h} A^j Y \times \{j\}$. To check semicommutativity, take a typical edge $((x, j), (ax, j+1), a)$ running from $A^j Y \times \{j\}$ to $A^{j+1} Y \times \{j+1\}$ where $0 \le j \le h-1$. Then for edges

$$e = ((ax, j+1), (a'ax, j+2), a') \in E^+((ax, j+1))$$

we may choose $\phi(e) = ((x, j), (a'x, j+1), a')$ since, by commutativity of G, $a.(a'.x) = (a'a.x)$ and thus there is an a-labelled edge from $head(\phi(e))$ to $head(e)$ as required. The semi-commutativity of the dual can be similairly verified.

The following is an easy exercise in commutative graphs (see [8]).

Proposition 2.6. Suppose that (V, E) is a h-layered commutative graph with layers $V = V_0 \ldots \sqcup V_h$. Then for $S \subset V_j$ and $T \subset V_k$, where $0 \le j < k \le h$, we have that the channel between S and T (that is, the subgraph consisting of all directed paths from S to T) is a commutative graph. We denote this subgraph $ch(S, T)$.

We will be interested in studying channels of an (A, Y, h)-orbit graph, it turns out these are measurable.

Lemma 2.7. Given an h-layered measure graph $\Gamma = (V, \mathcal{B}, \mu, A, E)$ with layering[1] $V = V_0 \sqcup \ldots \sqcup V_h$ and measurable $S \subset V_i$, $T \subset V_j$ where $0 \le i < j \le h$ we have that the channel $ch(S, T)$ has measurable vertex set.

Proof: Let us denote the vertex set of a subgraph Γ' as $V(\Gamma')$. We use induction on $j - i$. The base case $j = i+1$ holds since then $ch(S, T)$ has vertex set $V(ch(S, T)) = S \cap Im^-(T) \bigsqcup T \cap Im(S)$. Now suppose that $j - i > 1$. Then by the induction hypothesis we have that $ch(S, Im^-(T))$ has measurable vertex set. By the base case, $ch(Im^- T, T)$ has measurable vertex set. Now let $U = V(ch(S, Im^- T)) \cap V(ch(Im^- T, T)) \subset V_{j-1}$. Finally we have $V(ch(S, T)) = V(ch(S, U)) \cup V(ch(U, T))$ which is measurable again by the induction hypothesis. ■

Note that the previous Lemma and Example 2.3 demonstrate that the channel between two measurable sets may be naturally viewed as a measure graph (as channels are induced subgraphs).

We now turn to generalizing the notion of the number of edges in a bipartite graph.

[1] We always assume implicitly that each layer is measurable.

Definition 2.8. Fix a 1-layered commutative measure graph $(U, \mathcal{B}, \mu, A, E)$ with layering $U = U_0 \sqcup U_1$. Define the *flow* of Γ to be the quantity

$$Flow(\Gamma) = \int_{U_0} d^+(v) d\mu(v).$$

We now show that the flow behaves nicely and that $d^{\pm}$ is a measurable function.

Proposition 2.9. Under the setting of the previous definition, the map $d^+ : U \to \mathbb{R}$ is measurable and $Flow(\Gamma) = Flow(\Gamma^*)$, that is

$$\int_{U_0} d^+(v) d\mu(v) = \int_{U_1} d^-(v) d\mu(v).$$

Proof: Since (by definition of a measure graph) $|Im_a^{\pm}(\{v\})| \leq 1$, we may express

$$d^{\pm} = \sum_{a \in A} \chi_{L_a^{\pm}}$$

and thus $d^{\pm}$ is measurable. Consequently we have that

$$Flow(\Gamma) = \sum_{a \in A} \mu(L_a^+) = \sum_{a \in A} \mu(Im_a^+(L_a^+)) = \sum_{a \in A} \mu(L_a^-) = Flow(\Gamma^*)$$

as required. ■

3 Plünnecke's inequality for measure graphs

Definition 3.1. Given a commutative measure graph $\Gamma = (V, \mathcal{B}, \mu, A, E)$ with layering $V = V_0 \sqcup \ldots \sqcup V_h$, the magnification ratio of order j, where $j \in \{1, \ldots h\}$, is

$$D_j = \inf_{Y \subset V_0, \mu(Y) > 0} \frac{\mu(Im^j(Y))}{\mu(Y)}.$$

Moreover, for $C > 0$, define the *weight* (corresponding to C) to be the measure on $\mathcal{B}$ given by

$$w(S) = \sum_{j=0}^{h} C^{-j} \mu(S \cap V_j)$$

for $S \in \mathcal{B}$. Furthermore, we say that $S \in \mathcal{B}$ is a *cutset* if any path from V_0 to V_h intersects S and that S is an ϵ-*minimal cutset* if S is a cutset such that

$$w(S) \leq m_0 + \epsilon$$

where $m_0 = \inf\{w(Y) | Y \in \mathcal{B} \text{ is a cutset}\}$.

Lemma 3.2. Fix a 2-layered commutative measure graph $(U, \mathcal{B}, \mu, A, E)$ with layering $U = U_0 \sqcup U_1 \sqcup U_2$ and $C > 0$. Then if U_1 is an ϵ-minimal cutset (with respect to the weight corresponding to C), then U_0 is an $f(\epsilon)$-minimal cutset where

$$f(\epsilon) = \epsilon + 4|A|^2 C\epsilon + 4|A|^2\epsilon.$$

Proof: Let $m_0 = \inf\{w(S) | S \in \mathcal{B} \text{ is a cutset}\}$. Firstly note that for measurable $S \subset U_1$ we have that $Im(S) \sqcup (U_1 \setminus S)$ is a cutset and thus $m_0 \leq w(Im(S)) + w(U_1 \setminus S)$. On the other hand, since U_1 is ϵ-minimal we have that $w(S) + w(U_1 \setminus S) - \epsilon \leq m_0$ and thus $w(S) \leq w(Im(S)) + \epsilon$. A similair argument yields that $w(S) \leq w(Im^{-1}(S)) + \epsilon$. Thus

$$C\mu(S) \leq \mu(Im(S)) + C^2\epsilon \tag{$\star$}$$

and

$$C^{-1}\mu(S) \leq \mu(Im^{-1}(S)) + \epsilon \tag{$\dagger$}$$

for measurable $S \subset U_1$.

For each integer $i \geq 0$ let

$$\begin{aligned} X_i &= \{u \in U_1 | d^-(v) = i\} \\ Y_i &= \{u \in U_2 | d^-(v) = i\} \\ X_i' &= \{u \in U_1 | d^+(v) = i\} \\ Y_i' &= \{u \in U_0 | d^+(v) = i\}. \end{aligned}$$

The X_i are measurable and partition U_1. Let $k = |A|$. Define now inductively $T_k = Im(X_k)$ and $T_i = Im(X_i) \setminus T_{i+1}$ for $i = k-1, k-2, \ldots, 1$. Note that the T_i partition the set of vertices in U_2 that have at least one incoming edge. Moreover, by the definition of a commutativity we have that each vertex in T_i has inwards degree at least i (specifically, this is by the semicommutativity of the dual). Thus we obtain

$$\sum_{i=1}^{k} i\mu(T_i) \leq \sum_{i=1}^{k} Flow(ch(U_1, T_i)) = Flow(ch(U_1, U_2)) \tag{1}$$

where the right hand side is well defined since induced subgraphs with measurable vertex sets are measure graphs. From now on we will use the shorthand notation $Flow(U_i, U_j) := Flow(ch(U_i, U_j))$.

Moreover, since $Im(X_j \sqcup \ldots \sqcup X_k) = T_j \sqcup \ldots \sqcup T_k$ we have by ($\star$)

$$C \sum_{i=j}^{k} \mu(X_i) \leq \sum_{i=j}^{k} \mu(T_i) + C^2\epsilon. \tag{2}$$

Adding these inequalities for $j = 1, \dots k$ we obtain

$$C.Flow(U_0, U_1) = C \sum_{i=1}^{k} i\mu(X_i) \le \sum_{i=1}^{k} i\mu(T_i) + kC^2\epsilon \tag{3}$$

which implies, together with the preceding inequality, that

$$C.Flow(U_0, U_1) \le Flow(U_1, U_2) + kC^2\epsilon. \tag{4}$$

We will now apply the same argument to the dual graph to obtain an inequality of the form

$$Flow(U_1, U_2) \le C.Flow(U_0, U_1) + O_{k,C}(\epsilon).$$

To do this, inductively define $T'_k = Im^{-1}(X'_k)$ and $T'_i = Im^{-}(X'_i) \setminus T'_{i+1}$ for $i = k-1, k-2, \dots, 1$. This time we have, by the duals of the previous arguments (with (†) in place of ($\star$)) and the fact that flows are the same for duals (Proposition 2.9), that

$$\sum_{i=1}^{k} i\mu(T'_i) \le Flow(ch_{\Gamma^*}(U_1, U_0)) = Flow(U_0, U_1) \tag{5}$$

and, for each $\ell \in \{1, 2 \dots k\}$,

$$C^{-1} \sum_{i=\ell}^{k} \mu(X'_i) \le \sum_{i=\ell}^{k} \mu(T'_i) + \epsilon \tag{6}$$

which, by summing as before, gives

$$C^{-1} Flow(U_1, U_2) \le Flow(U_0, U_1) + k\epsilon.$$

Thus $Flow(U_1, U_2)$ is close to $C.Flow(U_0, U_1)$. This means that the inequalities above must have been close to being equalities. We will now explicitly estimate how close. Let us start with the inequality (6). We obtain from it, (4), and (5) that for each $j \in \{1, \dots k\}$ we have

$$\begin{aligned} C^{-1} \sum_{i=j}^{k} \mu(X'_i) - \sum_{i=j}^{k} \mu(T'_i) + (k-1)\epsilon &\ge C^{-1} \sum_{i=1}^{k} i\mu(X'_i) - \sum_{i=1}^{k} i\mu(T'_i) \\ &\ge C^{-1}.Flow(U_1, U_2) - Flow(U_0, U_1) \\ &\ge -kC\epsilon \end{aligned}$$

where the first inequality is obtained by summing (6) for $\ell \in \{1, 2, \dots, k\} \setminus \{j\}$. This finally gives

$$C^{-1} \sum_{i=j}^{k} \mu(X'_i) - \sum_{i=j}^{k} \mu(T'_i) \ge -kC\epsilon - (k-1)\epsilon \tag{7}$$

and so

$$|C^{-1}\sum_{i=j}^{k}\mu(X_i') - \sum_{i=j}^{k}\mu(T_i')| \leq \max\{kC\epsilon + (k-1)\epsilon, \epsilon\} \leq kC\epsilon + k\epsilon. \quad (8)$$

Thus the triangle inequality gives

$$|C^{-1}\mu(X_i') - \mu(T_i')| \leq 2kC\epsilon + 2k\epsilon. \quad (9)$$

Now we wish to show that T_i' is approximately Y_i' (that is, they have small symmetric difference). Note that $T_j' \sqcup \ldots \sqcup T_k' \subset Y_j' \sqcup \ldots \sqcup Y_k'$ since, as before, the definition of a commutative graph implies that each vertex in T_i' has outwards degree at least i. Thus

$$\sum_{i=j}^{k}\mu(T_i') - \sum_{i=j}^{k}\mu(Y_i') \leq 0. \quad (10)$$

Combining this with (6) and (4) we have that for each $j \in \{1, \ldots k\}$ we have

$$\begin{aligned}
\sum_{i=j}^{k}\mu(T_i') - \sum_{i=j}^{k}\mu(Y_i') &\geq \sum_{i=1}^{k} i\mu(T_i') - \sum_{i=1}^{k} i\mu(Y_i') \\
&\geq C^{-1}\sum_{i=1}^{k} i\mu(X_i') - k\epsilon - \sum_{i=1}^{k} i\mu(Y_i') \\
&= C^{-1}.Flow(U_1, U_2) - Flow(U_0, U_1) - k\epsilon \\
&\geq -kC\epsilon - k\epsilon.
\end{aligned}$$

Thus

$$|\sum_{i=j}^{k}\mu(T_i') - \sum_{i=j}^{k}\mu(Y_i')| \leq kC\epsilon + k\epsilon$$

from which the trianlge inequality implies

$$|\mu(T_i') - \mu(Y_i')| \leq 2kC\epsilon + 2k\epsilon. \quad (11)$$

Combining this with (9) yields

$$|C^{-1}\mu(X_i') - \mu(Y_i')| \leq 4kC\epsilon + 4k\epsilon.$$

Finally we get

$$|w(U_1) - w(U_0)| = |C^{-1}\sum_{i=1}^{k}\mu(X_i') - \sum_{i=1}^{k}\mu(Y_i')| \leq 4k^2C\epsilon + 4k^2\epsilon$$

and so in fact U_0 is $(\epsilon + 4k^2C\epsilon + 4k^2\epsilon)$-minimal. ■

We will now inductively apply the above lemma to construct ϵ-minimal cutsets that lie in the union of the top and bottom layers.

Lemma 3.3. Suppose that $\Gamma = (V, \mathcal{B}, \mu, A, E)$ is a h-layered commutative measure graph with layering $X = X_0 \sqcup \ldots \sqcup X_h$. Fix $C > 0$ and let w be the weight on Γ corresponding to C. Then for each $\epsilon > 0$ there exists an ϵ-minimal cutset $S \in \mathcal{B}$ such that $S \subset X_0 \sqcup X_h$.

Proof: We will prove, by induction on $j \in \{h-1, \ldots, 1, 0\}$, that there exists an ϵ-minimal cutset contained in $V_0 \sqcup V_1 \sqcup \ldots \sqcup V_j \sqcup V_h$. The base case $j = h-1$ is clear. Thus fix $\delta > 0$ and suppose that $j \in \{1, \ldots, h-1\}$ and $S \subset V_0 \sqcup V_1 \sqcup \ldots \sqcup V_j \sqcup V_h$ is a δ-minimal cutset. Let $S_i = S \cap X_i$ for $i \in \{1, \ldots, h\}$. Let $U_0 \subset V_{j-1}$ be those vertices that may be approached to V_{j-1} from V_0 along a path that does not intersect S. Let $U_2 = V_{j+1} \cap ch(V_{j+1}, V_h \setminus S_h)$ be the set of vertices in V_{j+1} that may be approached to $V_h \setminus S_h$. We know that U_2 is measurable by the measurability of channels and similairly U_0 is measurable by an application of the measurability of channels to the subgraph induced by $\bigsqcup_{i=0}^{h} V_i \setminus S_i$. Let $H = ch(U_0, U_2)$ and let $U_1 \subset V_j$ by the vertices in H that lie in V_j. Thus H is a 2-layered measure subgraph of Γ that is also commutative. Let us equip H with the measure $C^{-j+1}\mu$ instead of μ, since then the weight function on H corresponding to C agrees with the that of Γ.

Subclaim: The middle layer U_1 is a δ-minimal cutset of H.

To see this, firstly note that $U_1 \subset S_j$ (see Figure 1). If U_1 is not δ-minimal, then there exists a cutset T in H of weight $w(T) < w(U_1) - \delta \le w(S_j) - \delta$. But then the set $S' = S_0 \cup \ldots S_{j-1} \cup T \cup S_h$ is a cutset of Γ of weight

$$w(S') \le \sum_{i=0}^{j-1} w(S_i) + w(T) + w(S_h) < \sum_{i=0}^{j-1} w(S_i) + w(S_j) - \delta + w(S_h) = w(S) - \delta,$$

contradicting S being δ-minimal. This proves the subclaim.

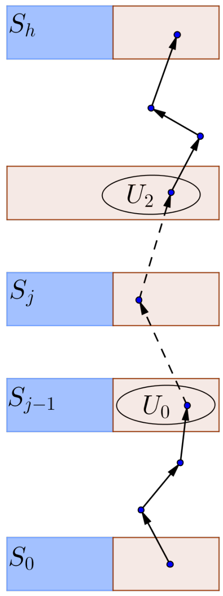

Figure 1: A dotted 2-length path as shown cannot exist as this gives rise to a path from V_0 to V_h which avoids S, as shown, by the definition of U_0 and U_2. Thus $U_1 \subset S_j$. One can similairly argue that the S' given in the proof of the subclaim is a cutset.

Hence we get by Lemma 3.2 that $(S \cup U_0) \setminus S_j$ is a $(\delta + f(\delta))$-minimal cutset, where f is as in the respective lemma (which we may take with the parameters of Γ, i.e: we consider H as having labelling set A and thus this f does not depend on H). Taking $\delta \to 0$ finishes the induction step and hence the proof of this lemma. ■

We are now ready to show that in the case $C = D_h^{1/h}$ the bottom layer is in fact a cutset of minimal weight.

Corollary 3.4. Suppose that $\Gamma = (X, \mathcal{B}, \mu, A, E)$ is a h-layered commutative measure graph with layering $X = X_0 \sqcup \ldots \sqcup X_h$. Suppose that $D_h^{1/h} > 0$ and let w be the weight corresponding to $C = D_h^{1/h}$. Then X_0 is a cutset of minimal weight.

Proof: We want to show that X_0 is ϵ-minimal for all $\epsilon > 0$. Choose $\epsilon > 0$ and by the above lemma an ϵ-minimal cutset $S \subset X_0 \sqcup X_h$. Write $S_i = X_i \cap S$. As S is a cutset we have $Im^h(X_0 \setminus S_0) \subset S_h$ and so

$$\mu(S_h) \geq \mu(Im^h(X_0 \setminus S_0)) \geq D_h \mu(X_0 \setminus S_0) = C^h \mu(X_0) - C^h \mu(S_0).$$

Thus

$$w(S) = \mu(S_0) + C^{-h} \mu(S_h) \geq \mu(X_0) = w(X_0)$$

and so $w(X_0)$ is ϵ-minimal. ■

We may now finally prove the Plünnecke inequality for measure graphs.

Theorem 3.5 (Plünnecke inequality for measure graphs)**.** Suppose that $\Gamma = (X, \mathcal{B}, \mu, A, E)$ is a h-layered commutative measure graph with layering $X = X_0 \sqcup \ldots \sqcup X_h$. Then for $j \in \{1, \ldots, h\}$ we have

$$D_j^h \geq D_h^j.$$

Proof: If $D_h = 0$ then we are done. If $D_h > 0$ then we may set $C = D_h^{1/h}$ and apply the above corollary as follows. For each non-null measurable $Z \subset X_0$ we have that $(X_0 \setminus Z) \sqcup Im^j(Z)$ is a cutset and thus by minimality of X_0 we have

$$\mu(X_0) = w(X_0) \leq w((X_0 \setminus Z) \sqcup Im^j(Z)) = \mu(X_0) - \mu(Z) + D_h^{-j/h}\mu(Im^j(Z))$$

and so

$$D_h^{j/h} \leq \frac{\mu(Im^j(Z))}{\mu(Z)}$$

which completes the proof as $Z \subset X_0$ was arbitrary. ■

4 Applications to measure preserving systems

First, we recall the notions of the magnification ratios for the dynamical setting introduced by Björklund and the second author in [2].

Definition 4.1. Suppose that G is a countable abelian group acting on a measure space $(X, \mathcal{B}, \mu)$. Define for $A \subset G$ and $B \in \mathcal{B}$ of positive finite measure the magnification ratio

$$c(A, B) = \inf\{\frac{\mu(AB')}{\mu(B')} | B' \subset B, \mu(B') > 0\}.$$

Moreover, for $\delta > 0$ we may define the δ-heavy magnification ratio

$$c_\delta(A, B) = \inf\{\frac{\mu(AB')}{\mu(B')} | B' \subset B, \mu(B') \geq \delta.\mu(B)\}.$$

Furthermore, if $E \subset X$ is measurable then we may define the restricted magnification ratio

$$c(A, B, E) = \inf\{\frac{\mu(AB' \setminus E)}{\mu(B')} | B' \subset B, \mu(B') > 0\}.$$

By applying the Plünnecke inequality for measure graphs to the case of orbit graphs we obtain the following Plünnecke inequality for measure preserving systems.

Theorem 4.2. Suppose that G is a countable abelian group acting on a measure space $(X, \mathcal{B}, \mu)$. Then for $A \subset G$ finite and measurable $B \in \mathcal{B}$ of positive finite measure, we have

$$c(A^j, B)^{1/j} \geq c(A^k, B)^{1/k}$$

for positive integers $j < k$.

We may also obtain the G-system analogue of a classical restricted addition result.

Theorem 4.3. Suppose that G is a countable abelian group acting on a measure space $(X, \mathcal{B}, \mu)$. For finite $A \subset G$, measurable $B \subset X$ of positive finite measure and measurable $E \subset X$ we have

$$c(A^j, B, A^{j-1}E)^{1/j} \geq c(A^k, B, A^{k-1}E)^{1/k}$$

for positive integer $j < k$.

Proof: Consider the subgraph of the (A, B, k)-orbit graph induced by the subset

$$B \times \{0\} \sqcup \bigsqcup_{j=1}^{k} (A^j B \setminus A^{j-1}E) \times \{j\}.$$

One may check that this subgraph is indeed commutative (see [8]). ■

5 Countable set of translates

The inequalities established in Section 4 required the set of translates $A \subset G$ to be finite. We now turn to extending Theorem 4.2 to the case where A is countable. We use the techniques developed by Björklund and the second author in [2].

The following proposition is analogous to Proposition 2.2 in [2].

Proposition 5.1. Suppose that G is an abelian group acting on a probability space $(X, \mathcal{B}, \mu)$ and fix a finite $A \subset G$ and non-null $B \in \mathcal{B}$ together with a $0 < \delta < 1$ and positive integers $j \leq k$. If $B' \subset B$ is measurable and satisfies

$$\left(\frac{\mu(A^k B')}{\mu(B')}\right)^{1/k} \leq (1-\delta)^{-1/j} \left(\frac{\mu(A^j B)}{\mu(B)}\right)^{1/j}. \tag{12}$$

Then $\mu(B') \geq \delta.\mu(B)$ or there exists $B' \subset B'' \subset B$ such that $\mu(B'' \setminus B) > 0$ and B'' satisfies (12).

Proof: Firstly we note that if the hypothesis holds for $B' = B_1$ and $B' = B_2$ with B_1 and B_2 disjoint, then it holds for $B_1 \sqcup B_2$ since the hypothesis may be rewritten as the inequality

$$\mu(A^k B') \leq (1-\delta)^{-k/j} \left(\frac{\mu(A^j B)}{\mu(B)}\right)^{k/j} \mu(B').$$

By Theorem 4.2 we know that there exists non-null measurable $B' \subset B$ such that (12) is satisfied. Suppose that $\mu(B') < \delta.\mu(B)$, thus we wish to construct a strictly larger $B'' \supset B$ that satisfies (12) and is contained in B. Set $B_0 = B \setminus B'$. We have that

$$\frac{\mu(B_0)}{\mu(B)}(1-\delta)^{-1} > 1$$

and thus there exists $B_0' \subset B_0$ such that

$$\left(\frac{\mu(A^k B_0')}{\mu(B_0')}\right)^{1/k} \le \left(\frac{\mu(B_0)}{\mu(B)}(1-\delta)^{-1}\right)^{1/j} \left(\frac{\mu(A^j B_0)}{\mu(B_0)}\right)^{1/j}$$
$$= (1-\delta)^{-1/j} \left(\frac{\mu(A^j B_0)}{\mu(B)}\right)^{1/j}$$
$$\le (1-\delta)^{-1/j} \left(\frac{\mu(A^j B)}{\mu(B)}\right)^{1/j}$$

and thus we may set $B'' = B' \sqcup B_0'$. ■

We will now apply the above lemma to construct a set $B' \subset B$ such that $\mu(B') \ge \delta.\mu(B)$ and (12) holds. The idea is to choose a set $B' \subset B$ that satisfies (12) and that is maximal in the sense that B does not contain any measurable $B'' \supset B'$ of strictly larger measure that satisfies (12). Such a set would have to necessarily satisfy $\mu(B') \ge \delta.\mu(B)$. The existence of such a maximal B' follows from the continuity of measure together with the following easy lemma on monotone classes.

Lemma 5.2. Suppose that $(X, \mathcal{B}, \mu)$ is a finite measure space and $\mathcal{M} \subset \mathcal{B}$ is non-empty and closed under countable nested unions (that is, if $M_i \in \mathcal{M}$ with $M_i \subset M_{i+1}$ then $\bigcup_{i=1}^{\infty} M_i \in \mathcal{M}$). Then there exists $M \in \mathcal{M}$ such that $\mu(M) = \mu(M')$ for all $M' \in \mathcal{M}$ with $M \subset M'$.

Proof: For $M \in \mathcal{M}$ let $s(M) = \sup\{\mu(M') | M' \in \mathcal{M}, M \subset M'\}$. Choose $M_1 \in \mathcal{M}$. Now inductively choose $M_{n+1} \in \mathcal{M}$ such that $M_n \subset M_{n+1}$ and $\mu(M_{n+1}) \ge \frac{\mu(M_n)+s(M_n)}{2}$. Let $M = \bigcup_{n=1}^{\infty} M_n$. We claim that $\mu(M) = s(M)$. To see this, note that $\mu(M_n) \to \mu(M)$ and $s(M_n) \ge s(M)$. Thus

$$\mu(M) = \lim_{n\to\infty} \mu(M_{n+1}) \ge \limsup_{n\to\infty} \frac{\mu(M_n)+s(M_n)}{2} \ge \limsup_{n\to\infty} \frac{\mu(M_n)+s(M)}{2} = \frac{\mu(M)+s(M)}{2}$$

and thus $\mu(M) \ge s(M)$ as required. ■

If we set $\mathcal{M} = \{B' \subset B | B' \text{ satisfies } (12)\}$ then we see that $\mathcal{M}$ is non-empty by Theorem 4.2 and is closed under countable nested unions by the continuity of measure. Thus by the discussion above we obtain a $B' \subset B$ such that $\mu(B') \ge \delta.\mu(B)$ and (12) holds. Consequently we have shown

Lemma 5.3. Suppose that G is an abelian group acting on a probability space $(X, \mathcal{B}, \mu)$ and fix a finite $A \subset G$ and non-null $B \in \mathcal{B}$ together with a $0 < \delta < 1$ and positive integers $j \le k$. Then

$$c_\delta(A^k, B)^{1/k} \le (1-\delta)^{-1/j} \left(\frac{\mu(A^j B)}{\mu(B)}\right)^{1/j}.$$

We may now obtain our first result about the case where $A \subset G$ is not necessarily finite.

Lemma 5.4. Suppose that G is an abelian group acting on a probability space $(X, \mathcal{B}, \mu)$ and fix a (not necessarily finite) set $A \subset G$ and non-null $B \in \mathcal{B}$ together with a $0 < \delta < 1$ and positive integers $j \le k$. Then

$$\sup\{c_\delta(A', B)^{1/k} | A' \subset A^k,\ A' \text{ is finite}\} \le (1-\delta)^{-1/j} \left(\frac{\mu(A^j B)}{\mu(B)}\right)^{1/j}.$$

Proof: If $A' \subset A^k$ is finite then one may choose a finite $A_0 \subset A$ such that $A' \subset A_0^k$. Consequently

$$c_\delta(A', B)^{1/k} \le c_\delta(A_0^k, B)^{1/k} \le (1-\delta)^{-1/j} \left(\frac{\mu(A_0^j B)}{\mu(B)}\right)^{1/j} \le (1-\delta)^{-1/j} \left(\frac{\mu(A^j B)}{\mu(B)}\right)^{1/j}$$

and so as A' was arbitrary this completes the proof. ■

The next non-trivial result due to Björklund and Fish allows us to extend the Plünnecke inequalities for a finite set of translates (Theorem 4.2) to the case of an infinite set of translates.

Theorem 5.5 (Proposition 4.1 of [2])**.** Suppose that G is a countable group acting on a probability space $(X, \mathcal{B}, \mu)$ such that $L^2(X, \mathcal{B}, \mu)$ is separable and fix a (not necessarily finite) set $A \subset G$ and non-null $B \in \mathcal{B}$ together with a $0 < \delta < 1$. Then

$$c(A, B) \le \sup\{c_\delta(A', B) | A' \subset A,\ A' \text{ is finite}\}.$$

Theorem 5.6. (Plünnecke inequalities for an infinite set of translates) Suppose that G is a countable abelian group acting on a probability space $(X, \mathcal{B}, \mu)$ such that $L^2(X, \mathcal{B}, \mu)$ is separable and fix a (not necessarily finite) set $A \subset G$ and non-null $B \in \mathcal{B}$ together with a $0 < \delta < 1$ and positive integers $j \le k$. Then

$$c(A^k, B)^{1/k} \le c(A^j, B)^{1/j} \le \left(\frac{\mu(A^j B)}{\mu(B)}\right)^{1/j}.$$

Proof: By the previous two results we obtain for each $\delta > 0$ the inequalities

$$c(A^k, B)^{1/k} \le \sup\{c_\delta(A', B)^{1/k} | A' \subset A^k,\ A' \text{ is finite}\} \le (1-\delta)^{-1/j} \left(\frac{\mu(A^j B)}{\mu(B)}\right)^{1/j}.$$

Taking $\delta \to 0$ gives

$$c(A^k, B)^{1/k} \le \left(\frac{\mu(A^j B)}{\mu(B)}\right)^{1/j}.$$

Now applying this to non-null $B_i \subset B$ such that

$$\frac{\mu(A^j B_i)}{\mu(B_i)} \to c(A^j, B)$$

gives

$$c(A^k, B)^{1/k} \le c(A^k, B_i)^{1/k} \le \left(\frac{\mu(A^j B_i)}{\mu(B_i)}\right)^{1/j} \to c(A^j, B)^{1/j}$$

as desired. ■

6 Different summands

Given measure preserving actions $G \curvearrowright (X, \mathcal{B}, \mu)$ and $G' \curvearrowright (X', \mathcal{B}', \mu')$ one can form the measure preserving product action $G \oplus G \curvearrowright (X \times X, \mathcal{B} \otimes \mathcal{B}', \mu \times \mu')$ given by $(g, g').(x, x') = (g.x, g'.x')$. We will now verify that the corresponding multiplication ratios are multiplicative.

Lemma 6.1. Suppose that G and G' are countable groups acting on probability spaces $(X, \mathcal{B}, \mu)$ and $(X', \mathcal{B}', \mu')$ respectively. Then for $A \subset G$, $A' \subset G'$ and non-null $B \subset X$, $B' \subset X'$ we have

$$c(A, B)c(A', B') = c(A \times A', B \times B').$$

Proof: For non-null $B_0 \subset B$ and $B'_0 \subset B'$ we have

$$\begin{aligned} c(A \times A', B \times B') &\leq \frac{\mu \times \mu'(A \times A'.B_0 \times B'_0)}{\mu \times \mu'(B_0 \times B'_0)} \\ &= \frac{\mu(A.B_0)}{\mu(B_0)} \frac{\mu(A'.B'_0)}{\mu(B'_0)}. \end{aligned}$$

By selecting appropriate B_0 and B'_0, the right hand side may be made arbitrarily close to $c(A, B)c(A', B')$ and thus

$$c(A, B)c(A', B') \geq c(A \times A', B \times B').$$

We now aim to show the reverse inequality. For $U \subset X \times X'$ and $(x_0, x'_0) \in X \times X'$ let

$$U_{x_0} = \{x' \in X' | (x_0, x') \in U\}$$

and

$$U^{x'_0} = \{x \in X | (x, x'_0) \in U\}.$$

Also, let $\mathcal{B}(V)$ denote the measurable subsets of V where V is any subset of a measurable space. Define

$$\phi : \mathcal{B}(X \times B') \to \mathcal{B}(X \times X')$$

by

$$\phi(U) = \bigsqcup_{x \in X} \{x\} \times A'U_x = (\{1_G\} \times A').U.$$

By Fubini's theorem we have

$$\begin{aligned} \mu \times \mu'(\phi(U)) &= \int_X \mu'(A'U_x) d\mu(x) \\ &\geq \int_X c(A', B')\mu'(U_x) d\mu(x) \\ &= c(A', B') \int_X \mu'(U_x) d\mu(x) \\ &= c(A', B') \cdot \mu \times \mu'(U). \end{aligned}$$

Thus $\mu \times \mu'(\phi(U)) \geq c(A', B') \cdot \mu \times \mu'(U)$ for $U \subset X \times B'$. We may reverse the role of co-ordinates to obtain a similair inequality, from which we finally get that

$$\begin{aligned}\mu \times \mu'((A \times A').U) &= \mu \times \mu'((\{1_G\} \times A')(A \times \{1_{G'}\}).U)) \\ &\geq c(A', B') \cdot \mu \times \mu'((A \times \{1_{G'}\}).U)) \\ &\geq c(A', B')c(A, B) \cdot \mu \times \mu'(U)\end{aligned}$$

for $U \subset B \times B'$. This implies that $c(A, B)c(A', B') \leq c(A \times A', B \times B')$, as required. ■

Proposition 6.2. Suppose that G is a countable abelian group acting on a probability space $(X, \mathcal{B}, \mu)$. Then for $A_1, A_2, \ldots A_k \subset G$ and non-null $B \in \mathcal{B}$ we have

$$c(A_1 \ldots A_k, B) \leq \prod_{i=1}^{k} \frac{\mu(A_i B)}{\mu(B)}.$$

Proof: Choose rational numbers

$$\alpha_i > \frac{\mu(A_i B)}{\mu(B)}$$

and choose $n \in \mathbb{Z}_{>0}$ such that for each $i \in \{1, \ldots k\}$ we have

$$n_i := \frac{n}{\alpha_i} \in \mathbb{Z}_{>0}.$$

Suppose that there exists $T_i \subset G$ with $|T_i| = n_i$ such that the map

$$\begin{aligned}T_1 \times \ldots \times T_k \times (A_1 A_2 \ldots A_k B) &\to X \\ (t_1, \ldots, t_k, y) &\mapsto t_1 \ldots t_k y\end{aligned}$$

is injective. We may assume, without loss of generality, that we are in this case by naturally embedding $G \hookrightarrow G \oplus \mathbb{Z}/N\mathbb{Z}$ and $X \hookrightarrow X \times \mathbb{Z}/N\mathbb{Z}$ and replacing the measure preserving system $G \curvearrowright X$ with the product measure preserving system $G \oplus \mathbb{Z}/N\mathbb{Z} \curvearrowright X \times \mathbb{Z}/N\mathbb{Z}$, for large enough N. Let $A = \bigcup_{i=1}^{k} A_i T_i$ and notice that

$$\mu(AB) \leq \sum_{i=1}^{k} \mu(A_i T_i B) \leq \sum_{i=1}^{k} n_i \cdot \mu(A_i B) < \mu(B) \sum_{i=1}^{k} n_i \alpha_i = k \cdot n \cdot \mu(B)$$

and thus, by Theorem 5.6, we obtain non-null $B' \subset B$ such that

$$\mu(A^k B') \leq (k \cdot n)^k \, \mu(B').$$

However, by the injection above, we have

$$\mu(A^k B') \geq \mu(T_1 \ldots T_k A_1 A_2 \ldots A_k B') = \left(\prod_{i=1}^{k} n_i \right) \mu(A_1 \ldots A_k B').$$

Combining the previous two inequalities gives

$$c(A_1 \dots A_k, B) \leq \frac{\mu(A_1 \dots A_k B')}{\mu(B')} \leq k^k \prod_{i=1}^{k} \alpha_i.$$

Since the $\alpha_i > \frac{\mu(A_i B)}{\mu(B)}$ were arbitrary rational numbers, we obtain

$$c(A_1 \dots A_k, B) \leq k^k \prod_{i=1}^{k} \frac{\mu(A_i B)}{\mu(B)}. \tag{13}$$

We now wish to remove the k^k constant. This may be done by consdering a large cartesian power, as follows. Let us denote $V^{\times m} = V \times V \dots \times V$ and $V^{\oplus m} = V \oplus V \dots \oplus V$, where m factors are present. For each positive integer m, an application of (13) and Lemma 6.1 to the sets $A_1^{\oplus m} \dots A_k^{\oplus m} = (A_1 A_2 \dots A_k)^{\oplus m} \subset G^{\oplus m}$ and $B^{\times m} \subset X^{\times m}$ gives

$$c(A_1 \dots A_k, B) \leq k^{k/m} \prod_{i=1}^{k} \frac{\mu(A_i B)}{\mu(B)}.$$

Taking the limit $m \to \infty$ gives the desired result. ■

7 Correspondence principle for product sets

We will now establish a Furstenberg correspondence principle for product sets. The first appearance of the correspondence principle for product sets was in [1]. The principle appearing in this paper is due to Björklund and Fish, and appeared in [3]. Given a countable group G and $B \subset G$, we define the Furstenberg *G-system corresponding to B* to be the topological G-system $G \curvearrowright X$, i.e., G acts on compact metric space X by homeomorphisms, given by the following construction. Let $X_0 = \{0,1\}^G$ be the space of all sequences indexed by G equipped with the product topology. Let $z \in \{0,1\}^G$ be the indicator function of B, that is $z_g = 1$ if and only if $g \in B$. Note that there is a natural action of G on X_0 given by

$$(g.x)_h = x_{gh}$$

for $g, h \in G$ and $(x_k)_{k \in G} \in X_0$.

Let $X = \overline{Gz} = \overline{\{gz \mid g \in G\}}$ be the closure of the orbit of z. Note that X is G-invariant. This defines the system corresponding to B. Moreover, we define the *clopen set corresponding to B* to be the set

$$\widetilde{B} = \{x \in X \mid x_1 = 1\},$$

which is a clopen subset of X.

Lemma 7.1. Suppose that G is a countable abelian group, $B \subset G$ and X is the G-system corresponding to B. Suppose that μ is a G-invariant Borel probability measure on X. Then for finite $A_0 \subset G$ we have that

$$d^*(A_0B) \geq \mu(A_0\widetilde{B}) \geq d_*(A_0B).$$

Proof: Note that it suffices to prove this for μ ergodic by either of the following arguments. Suppose that the result holds for all ergodic μ, and thus holds for all convex combinations of ergodic measures. It is a well known fact that the extreme G-invariant measures are precisely the ergodic measures. The Krein-Milman theorem therefore implies that any G-invariant probability measure is in the weak* closure of the set of all convex combinations of ergodic measures. Since the map $\nu \mapsto \nu(A_0\widetilde{B})$ is weak* continuous (since $A_0\widetilde{B}$ is clopen), we obtain the result. Alternatively, one may use Bauer's maximum principle (instead of the Krein-Milman) which says that the maximum (resp. minimum) of the the map $\nu \mapsto \nu(A_0\widetilde{B})$ is attained at an extremal (hence ergodic) measure, say μ^* (resp. μ_*), and thus for any G-invariant μ we have

$$d_*(A_0B) \leq \mu_*(A_0\widetilde{B}) \leq \mu(A_0\widetilde{B}) \leq \mu^*(A_0\widetilde{B}) \leq d^*(A_0B).$$

Now we turn to the proof of the Lemma under the assumption that μ is ergodic. Given any Følner sequence $(F_n)_{n\in\mathbb{N}}$ and continuous $f \in C(X)$ we have, by the Von Neumann mean ergodic theorem, that

$$\frac{1}{|F_N|}\sum_{g\in F_N} f \circ g \to \int f d\mu \tag{14}$$

in the L^2-norm. We may then, by the Borel-Cantelli lemma, pass to a subsequence of $(F_n)_{n\in\mathbb{N}}$ to obtain almost everywhere pointwise convergence in (14). In particular we may apply this result to $f = \chi_{A_0\widetilde{B}}$ and get a Følner sequence $(F_n)_{n\in\mathbb{N}}$ such that

$$\frac{1}{|F_N|}\sum_{g\in F_N} \chi_{A_0\widetilde{B}}(gx) \to \mu(A_0\widetilde{B})$$

for some $x \in X$. Now fix $N \in \mathbb{N}$ and note that since $X = \overline{Gz}$ we have $h_i z \to x$ for some $h_i \in G$ and thus $\chi_{A_0\widetilde{B}}(gh_i z) \to \chi_{A_0\widetilde{B}}(gx)$ for each $g \in F_N$. Therefore, for some large M, we have for $q_N := h_M$ the equality

$$\frac{1}{|F_N|}\sum_{g\in F_N} \chi_{A_0\widetilde{B}}(gx) = \frac{1}{|F_N|}\sum_{g\in F_N} \chi_{A_0\widetilde{B}}(gq_N z) = \frac{|q_N F_N \cap A_0B|}{|q_N F_N|}$$

where the second equality is obtained from the fact that, by construction of the corresponding system and clopen set, we have $gq_N z \in A_0\widetilde{B}$ if and only if $gq_N \in A_0B$. Since $(q_N F_N)_{N\in\mathbb{N}}$ is a Følner sequence, the limit (as $N \to \infty$) of this quantity must be between $d_*(A_0B)$ and $d^*(A_0B)$. ■

In fact, notice that the inequality $d^*(A_0B) \geq \mu(A_0\widetilde{B})$ in the previous lemma is also true for infinite A_0 since we can always write A_0 as an increasing union $A_1 \subset A_2 \ldots$ of finite sets and thus

$$d^*(A_0B) \geq d^*(A_kB) \geq \mu(A_k\widetilde{B}) \to \mu(A\widetilde{B}) \text{ as } k \to \infty.$$

Proposition 7.2 (Correspondence principle for product sets [2])**.** Suppose that G is a countable abelian group and $A, B \subset G$. Then there exists a compact metrizable space X on which G acts by homeomorphisms such that there exist G-invariant ergodic Borel probability measures μ, ν on X together with a clopen $\widetilde{B} \subset X$ such that

$$\begin{aligned} d^*(B) &= \mu(\widetilde{B}) \\ d^*(AB) &\geq \mu(A\widetilde{B}) \\ d_*(B) &\leq \nu(\widetilde{B}) \\ d_*(AB) &\geq \nu(A\widetilde{B}). \end{aligned}$$

Proof: The space X and clopen set $\widetilde{B}$ will be those coming from the correspondence. Note that the second and third inequalities are satisfied for all μ, ν by the lemma above. Moreover, this lemma shows that $d^*(B) \geq \mu(\widetilde{B})$, for all μ. Therefore to construct μ satisfying the first equality, it is enough to construct a not necessarily ergodic μ and then apply Bauer's maximum principle. Let z be as in the construction of the correspondence. Choose a Følner sequence $F_n \subset G$ such that

$$\frac{|F_n \cap B|}{|F_n|} \to d^*(B).$$

Consider now the following averages of point mass measures

$$\mu_n = \frac{1}{|F_n|} \sum_{g \in F_n} \delta_{g.z}$$

and let $\mu = \lim_{k\to\infty} \mu_{n_k}$ be a weak* limit of a subsequence of these. Since $(F_n)_{n\in\mathbb{N}}$ is Følner, we have the μ is G-invariant. Note that for $C \subset G$

$$\mu_n(C\widetilde{B}) = \frac{|F_n \cap CB|}{|F_n|}.$$

In particular, the $C = \{1\}$ case shows that the choice of Følner sequence, together with the fact that $\widetilde{B}$ is clopen, implies that $d^*(B) = \mu(\widetilde{B})$. Now we turn to dealing with the final inequality. To construct such a ν, it is enough to construct such a not necessarily ergodic ν by the following argument. The map $\nu \mapsto \nu(A\widetilde{B})$ is weak* lower semicontinuous since $A\widetilde{B}$ is open. Thus Bauer's minimum (but not maximum) principle applies and thus if at least one not necessarily ergodic ν satisfies the final inequality, then some ergodic minimizer does too. Write A as a union of an increasing sequence of finite sets $A_1 \subset A_2 \subset \ldots$ and choose a Følner sequence $E_m \subset G$ such that

$$\frac{|E_m \cap AB|}{|E_m|} \to d_*(AB) \text{ as } m \to \infty.$$

As before, we have that the averages

$$\nu_m = \frac{1}{|E_m|} \sum_{g \in E_n} \delta_{g.z}$$

have a weak* convergent subsequence $\nu_{m_j} \to \nu$. Since each $A_k\widetilde{B}$ is clopen, we have

$$d_*(AB) = \lim_{j\to\infty} \frac{|E_{m_j} \cap AB|}{|E_{m_j}|} \geq \lim_{j\to\infty} \frac{|E_{m_j} \cap A_kB|}{|E_{m_j}|} = \lim_{j\to\infty} \nu_{m_j}(A_k\widetilde{B}) = \nu(A_kB) \to \nu(A\widetilde{B}) \text{ as } k \to \infty.$$

■

The following statement was proven in [2].

Lemma 7.3 ([2])**.** Suppose that G is a countable abelian group that acts ergodically on a probability space $(X, \mathcal{B}, \mu)$ such that $L^2 = L^2(X, \mathcal{B}, \mu)$ is separable (for instance, a Borel probability space). Then

$$d^*(A) \leq \mu(AB)$$

for $A \subset G$ and non-null $B \in \mathcal{B}$.

Proof: Choose a Følner sequence $(F_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} \frac{|F_n \cap A|}{|F_n|} = d^*(A).$$

Define

$$f_N = \frac{1}{|F_N|} \sum_{g\in F_N} \chi_{g^{-1}AB} \in L^2.$$

As $\|f_N\|_2 \leq 1$ and L^2 is separable (and thus has unit ball compact metrizable in the weak topology), we may pass to a subsequence of $(F_n)_{n\in\mathbb{N}}$ such that f_N converges weakly to some $f \in L^2$. But f is G-invariant and thus constant by ergodicity. Therefore

$$\mu(AB) = \langle f_N, 1\rangle \to \langle f, 1\rangle = f$$

and so in fact f is the constant function $\mu(AB)$.

Notice that for $b \in B$ we have

$$\begin{aligned} f_N(b) &= \frac{1}{|F_N|}|\{g \in F_N | b \in g^{-1}AB\}| \\ &= \frac{1}{|F_N|}|\{g \in F_N | gb \in AB\}| \\ &\geq \frac{1}{|F_N|}|\{g \in F_N | g \in A\}| \end{aligned}$$

and so $d^*(A) \leq \liminf_{N\to\infty} f_N(b)$. In other words, we have

$$\chi_B d^*(A) \leq \chi_B \liminf_{N\to\infty} f_N.$$

Integrating this inequality and applying Fatou's lemma yields

$$\begin{aligned} \mu(B)d^*(A) &\leq \int \chi_B \liminf_{N\to\infty} f_N \\ &\leq \liminf_{N\to\infty} \int \chi_B f_N \\ &= \langle \chi_B, f\rangle \\ &= \mu(B)\mu(AB). \end{aligned}$$

As $\mu(B) > 0$, we obtain the desired inequality. ■

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