A Pricing Formula for Delayed Claims

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Abstract

We consider the valuation of contingent claims with delayed dynamics in a Black & Scholes complete market model. We find a pricing formula that can be decomposed into terms reflecting the market values of the past and the present, showing how the valuation of future cashflows cannot abstract away from the contribution of the past. As a practical application, we provide an explicit expression for the market value of human capital in a setting with wage rigidity.

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AMS Classification— 34K50, 91B25, 91G80

1 Introduction

Consider a financial market living in a filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, \mathbf{P})$. It is a standard result in asset pricing theory that the absence of arbitrage opportunities is essentially equivalent to the existence of an equivalent probability measure $\mathbf{Q}$, under which the price of any contingent claim is a local martingale after deflation by the money market account; see [18, 19, 12]. Practical applications and theoretical extensions of the theory have focused on different frictions, ranging from risks unspanned

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by tradable securities, to discontinuous asset price dynamics, trading constraints, and transaction costs (see [21], [13], [17], and references therein). Some recent contributions study instead the interesting case of delayed price dynamics in an otherwise standard, complete market model driven by a Brownian motion; see [1, 25], among others. In Arriojas *et al.* [1], for example, the authors consider a market with a riskless asset yielding the instantaneous risk free rate $r > 0$, and a stock, whose price process S is driven by a Brownian motion, and satisfies a stochastic functional differential equation (SFDE) with fixed or variable delay in the drift and volatility terms. The authors define a local martingale measure $\mathbf{Q}$, equivalent to the reference measure $\mathbf{P}$, via a Girsanov transformation depending on both the delayed drift and the volatility coefficient of the stock price. They then prove completeness of the market, and hence uniqueness of the no-arbitrage price $V_H(t)$ of a generic contingent claim $H \in L^1(\mathcal{F}_T)$, defined via the pricing formula $V_H(t) = e^{-r(T-t)} E^{\mathbf{Q}} [H|\mathcal{F}_t]$. The latter can be equivalently written as $V_H(t) = \xi(t)^{-1} E^{\mathbf{P}} [\xi(T)H|\mathcal{F}_t]$, by using the state price process ξ related to the measure $\mathbf{Q}$ via Girsanov's Theorem in the usual way.

In this paper, we take a different path in exploring the implications of delayed dynamics in the traditional arbitrage free valuation framework. We work with a standard, complete market model of securities with prices evolving as geometric Brownian motions (GBM), but consider contingent claims that have dynamics described by an SFDE. Interestingly, the no-arbitrage pricing machinery results in a valuation formula that can be decomposed into a term related to the 'current market value of the past' (in a sense to be made precise below), and a term reflecting the 'market value of the present'. Both terms are scaled by an appropriate annuity factor to yield the current market value of the future flow of contingent payments. The message is that the market consistent valuation of future cashflows cannot abstract away from the contribution of the past, which in our setting is represented by the portion of a contingent claim's past trajectory that shapes its dynamics going forward.[1]

A practical example we have in mind is the case of stochastic labor income, and the valuation of human capital.[2] It is well known that when labor income is spanned by tradable assets, the market value of human capital can be easily derived via risk-neutral valuation. In [15], this result is extended to settings of greater generality, including endogenous retirement and borrowing constraints. It is in general difficult

[1]The importance of the past in understanding the qualitative feature of a model with delay was also emphasized in Fabbri and Gozzi [16], although in a deterministic setting, when solving the endogenous growth model with vintage capital of Boucekkine *et al.* [6].

[2]An alternative example of delayed contingent claims is represented by the moving average options considered by Bernhart *et al.* [4], who approximate the infinite-dimensional price dynamics with truncated Laguerre series expansions.

to allow for richer dynamics of labor income, including unspanned sources of risk (e.g., [29]), or state variables capturing wage rigidity (e.g., [15], section 6). The empirical evidence on wage rigidity (see [26], [22], [11], [2], and [24], for example) suggests that there may be value in introducing delay terms in labor income dynamics, even in the case of a partial equilibrium, complete market model. We study this situation by introducing delayed drift and volatility coefficients[3] in a GBM model, to obtain an example of income dynamics that adjusts slowly to financial market shocks. The empirical literature on labor income dynamics widely relies on autoregressive moving average (ARMA) processes: Reiss [31], Lorenz [23], and Dunsmuir *et al.* [14] show how SFDEs can be understood as the weak limit of discrete time ARMA processes. We obtain a closed form solution for human capital, which makes explicit the contributions of the market value of the past and the present. In the following, we develop our analysis within the human capital application, as it is quite intuitive. The extension to alternative applications should be immediate.

The paper is organized as follows. In the following section, we introduce the setup, and state our main result. Section 3 presents mathematical tools used to deal with the non-Markovian nature of a setting with delayed dynamics. In particular, we embed our problem in an infinite dimensional Hilbert space, on which the state variable process is Markovian. In section 4, we prove our results by following a chain of five lemmas. Section 5 concludes.

2 Setup and Main Result

Consider a Black-Scholes complete market model defined on our filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, \mathbf{P})$. Available for trade are a money market account, S_0, and n risky assets with price vector process S. Prices have dynamics described by

$$\begin{cases} \mathrm{d}S_0(t) = & S_0(t) r \mathrm{d}t, \\ \mathrm{d}S(t) = & \mathrm{diag}(S(t)) \left\{ \mu \mathrm{d}t + \sigma \mathrm{d}Z(t) \right\}, \\ S_0(0) = & 1, \qquad S(0) \in \mathbf{R}^n_{>0}, \end{cases} \tag{1}$$

where Z is an n-dimensional Brownian motion, $\mu \in \mathbf{R}^n$, and $\sigma \in \mathbf{R}^n \otimes \mathbf{R}^n$, such that $\sigma\sigma^\top > 0$. Here and in what follows, we use the notation $\mathbf{R}^n_{>0}$ for the set

[3] Meghir and Pistaferri [27] demonstrate the importance of properly modeling the conditional variance of income shocks, and find strong evidence of ARCH effects in data from the Michigan Panel Study of Income Dynamics (PSID).

$\{(x_i) \in \mathbf{R}^n : x_i > 0,\ i = 1, \dots, n\}$. We assume that $\mathbf{F} := (\mathcal{F}_t)_{t \geq 0}$ is the filtration generated by the Brownian Motion Z, and enlarged with the $\mathbf{P}$-null sets. We consider an agent who receives $\mathbf{F}$-adapted stochastic income X_0, and can invest her wealth in the financial market. Defining the market price of risk as

$$\kappa := (\sigma^\top)^{-1}(\mu - r\mathbf{1}), \tag{2}$$

the stochastic discount factor ξ can be shown to evolve as follows in our setting (see [13]):

$$\begin{cases} \mathrm{d}\xi(t) &= -\xi(t) r \mathrm{d}t - \xi(t)\kappa^\top \mathrm{d}Z(t) \\ \xi(0) &= 1. \end{cases} \tag{3}$$

A natural constraint for the agent is that her portfolio allocation can be supported by the sum of her current wealth and (a fraction of) future wealth, depending on the extent to which human capital can be pledged to raise funds (see [15]). Our aim is to give an explicit expression for human capital, which is given by the following expectation:

$$\xi(t_0)^{-1}\mathbf{E}\left(\int_{t_0}^{+\infty} \xi(t) X_0(t) \mathrm{d}t \middle| \mathcal{F}_{t_0}\right). \tag{4}$$

The introduction of the expression above to a bounded horizon would allow us to model permanent exit from the labor market (e.g., death, irreversible unemployment or retirement). Our results can be extended to this setting along the lines indicated in Remark 2.2 below.

In line with the empirical evidence on wage rigidity, we assume labor income to obey the following SFDE, which introduces slow adjustment of labor income to market shocks via delay terms in the drift and volatility coefficients of a GBM model:

$$\begin{cases} \mathrm{d}X_0(t) = & \left[X_0(t)\mu_0 + \int_{-d}^0 X_0(t+s)\phi(\mathrm{d}s)\right] \mathrm{d}t \\ & + \left[X_0(t)\sigma_0^\top + \begin{pmatrix} \int_{-d}^0 X_0(t+s)\varphi_1(\mathrm{d}s) \\ \vdots \\ \int_{-d}^0 X_0(t+s)\varphi_n(\mathrm{d}s) \end{pmatrix}^\top\right] \mathrm{d}Z(t), \\ X_0(0) = & x_0, \\ X_0(s) = & x_1(s) \quad \text{for } s \in [-d, 0), \end{cases} \tag{5}$$

where $\mu_0 \in \mathbf{R}_{>0}$, $\sigma_0 \in \mathbf{R}^n$, and ϕ, φ_i are signed measures of bounded variation on $[-d, 0]$, with $i = 1, \ldots, n$, and $x_0 \in \mathbf{R}_{>0}$ and $x_1 \in L^2\big([-d, 0]; \mathbf{R}_{>0}\big)$. Equation (5) admits a unique strong solution, as ensured by Theorem I.1 and Remark 4 Section I.3 in [28], and provides a simple, tractable example of income dynamics adjusting slowly to financial market shocks.

To provide an explicit expression for (4), and formulate the main result of this paper, we define the function

$$K(\lambda) := \lambda - (\mu_0 - \sigma_0^\top \kappa) - \int_{-d}^{0} e^{\lambda\tau} \Phi(\mathrm{d}\tau), \quad \lambda \in \mathbf{C}, \tag{6}$$

and the measure Φ on $[-d, 0]$

$$\Phi(\tau) := \left[\phi(\tau) - \begin{pmatrix} \varphi_1(\tau) \\ \vdots \\ \varphi_n(\tau) \end{pmatrix}^\top \kappa \right]. \tag{7}$$

We also define the constant

$$K := K(r) = r - \mu_0 + \sigma_0^\top \kappa - \int_{-d}^{0} e^{r\tau} \Phi(\mathrm{d}\tau), \tag{8}$$

and assume the following conditions to hold throughout the paper.

Hypothesis 1. *The function Φ is non-negative, and the constant K is strictly positive:*

$$\Phi(\tau) \geq 0, \quad \tau \in [-d, 0], \tag{9}$$

$$K > 0. \tag{10}$$

We are now ready to state our main result, which provides an explicit decomposition of the market value of human capital in our setting.

Theorem 2.1. *Let ξ be defined as in (3), and X_0 evolve as in (5). Then, under Hypothesis 1, for any $t_0 \geq 0$ we can write*

$$\mathbf{E}\left(\int_{t_0}^{+\infty} \frac{\xi(t)}{\xi(t_0)} X_0(t)\, dt \middle| \mathcal{F}_{t_0} \right) = \frac{1}{K} \left(X_0(t_0) + \int_{-d}^{0} G(s) X_0(t_0 + s)\, ds \right), \quad \mathbf{P} - a.s., \tag{11}$$

where $X_0(t)$ denotes the solution at time t of equation (5), K is defined in (8), and G is given by

$$G(s) := \int_{-d}^{s} e^{-r(s-\tau)} \Phi(d\tau). \tag{12}$$

In expression (11), we recognize an annuity factor, K^{-1}, multiplying a term representing current labor income, and a term representing the current market value of the past trajectory of labor income over the time window $(t_0 - d, t_0)$. The 'market value of the past' trades off the returns on labor income against its exposure to financial risk, as can be seen from expression (7). When the delay terms in the drift and volatility coefficients of labor income vanish, human capital reduces to $K^{-1}X_0(t_0)$, in line with [15], for example.

Remark 2.2. *The setup can be extended to the case of payments over a bounded horizon in some interesting situations. When labor income is received until an exogenous Poisson stopping time τ (representing death or irreversible unemployment, for example), expression* (11) *still applies, provided discounting is carried out at rate $r+\delta$ instead of r, where $\delta > 0$ represents the Poisson parameter. When labor income is received until a fixed retirement date $T > 0$, a result analogous to* (11) *can be derived at the price of some technical complications; see [5]. The case of an endogenous retirement time, as in [15], is an open problem in our setting.*

Remark 2.3. *The solution of equation* (5) *is not always positive. A sufficient condition for almost sure positivity of X_0 is that $\varphi_i = 0$ for all i, so that the delay term in the volatility coefficient of* (5) *vanishes, and hence Φ coincides with ϕ. Defining*

$$\mathcal{E}(t) := e^{(\mu_0 - \frac{1}{2}\sigma_0^\top \sigma_0)t + \sigma_0 Z(t)},$$
$$\mathcal{I}(t) := \int_0^t \mathcal{E}^{-1}(u) \int_{-d}^0 \phi(s)\, X_0(s+u) ds\, du,$$

the variation of constants formula yields

$$X_0(t) = \mathcal{E}(t)\big(x_0 + \mathcal{I}(t)\big). \tag{13}$$

The first statement of Hypothesis 1 ensures the positivity of the labor income X_0 in this special case.

Remark 2.4. *As a simple example of when the solution of equation* (5) *can take positive and negative values, consider the case of $n = 1$, $\mu_0 = 0$, $\phi = 0$, $\sigma_0 = 0$, and $\varphi(s) = \delta_{-d}(s)$, where $\delta_a(s)$ indicates the delta-Dirac measure at a, so that equation* (5) *reads*

$$dX_0(t) = X_0(t-d)\, dZ(t). \tag{14}$$

Then, for $t \in [0, d)$ we have

$$X_0(t) = x_0 + \int_0^t X_0(s-d)\, dZ(s) = x_0 + \int_{-d}^{t-d} x_1(\tau)\, dZ(\tau + d). \tag{15}$$

In this case $X_0(t)$ is Gaussian, and dynamics (14) *could be used to model, for example, the variation margin of an over-the-counter swap, when the collateralization procedure relies on a delayed mark-to-market value of the instrument (see [8], for example).*

3 Mathematical tools

It will be convenient to embed the labor income X_0 in the infinite dimensional Hilbert space $\mathcal{H}$

$$\mathcal{H} := \mathbf{R} \times L^2\big([-d, 0]; \mathbf{R}\big),$$

endowed with an inner product for $x = (x_0, x_1), y = (y_0, y_1) \in \mathcal{H}$ defined as

$$\langle x, y \rangle_{\mathcal{H}} := x_0 y_0 + \langle x_1, y_1 \rangle_{L^2},$$

where

$$\langle x_1, y_1 \rangle_{L^2} := \int_{-d}^{0} x_1(s) y_1(s)\, \mathrm{d}s.$$

In what follows we omit the subscript L^2 in the inner product notation.
Let us define two operators, A and C, that act on the domain $\mathcal{D}(A)$ as follows:[4]

$$\mathcal{D}(A) = \mathcal{D}(C) := \{(x_0, x_1) \in \mathcal{H} : x_1 \in W^{1,2}\big([-d, 0];\, \mathbf{R}\big), x_0 = x_1(0)\},$$

and

$$A : \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H},$$
$$A(x_0, x_1) := \Big(\mu_0 x_0 + \int_{-d}^{0} x_1(s) \phi(\mathrm{d}s), \frac{\mathrm{d}x_1}{\mathrm{d}s}\Big),$$

with μ_0 and ϕ as in (5), and

$$C : \mathcal{D}(A) \subset \mathcal{H} \to \mathbf{R}^n \times L^2([-d, 0]; \mathbf{R}^n),$$
$$C\,(x_0, x_1) := \left(\sigma_0 x_0 + \begin{pmatrix} \int_{-d}^{0} x_1(s) \varphi_1(\mathrm{d}s) \\ \vdots \\ \int_{-d}^{0} x_1(s) \varphi_n(\mathrm{d}s) \end{pmatrix}, 0\right),$$

[4] The Sobolev space $W^{1,2}\big([-d, 0]; \mathbf{R}\big)$ is defined as

$$W^{1,2}\big([-d, 0]; \mathbf{R}\big) := \big\{u \in L^2([-d, 0]) : \ \exists\, g \in L^2([-d, 0]) \text{ such that } u(\theta) = c + \int_{-d}^{\theta} g(s)\, \mathrm{d}s\big\}.$$

with σ_0 and φ_i as in (5). The following, well known fact (see [10]) is crucial for the rest of the paper.

Lemma 3.1. *The operator A generates a strongly continuous semigroup in $\mathcal{H}$.*

Proof. The operator A can be written in the form

$$A\,(x_0, x_1) = \left(\int_{-d}^{0} x_1(\theta) a(\mathrm{d}\theta), \frac{\mathrm{d}x_1}{\mathrm{d}s} \right) , \tag{16}$$

with

$$a(\mathrm{d}\theta) = \mu_0 \delta_0(\mathrm{d}\theta) + \phi(\mathrm{d}\theta) \,.$$

Therefore the lemma follows immediately from Proposition A.25 in [10]. $\square$

The labor income in (5) can be equivalently defined as the first component of the solution in $\mathcal{H}$ of the following equation (see [9])

$$\begin{cases} \mathrm{d}X(t) & = AX(t)\mathrm{d}t + (CX(t))^\top \mathrm{d}Z(t), \\ X_0(0) & = x_0, \\ X_1(0,s) & = x_1(s) \text{ for } s \in [-d, 0), \end{cases} \tag{17}$$

with A and C defined above, and x_0, x_1 as in (5).

4 Proof of the Main Result

The proof of Theorem 2.1 will follow by a chain of five lemmas stated below. To prove the theorem we will consider the conditional mean of the labor income X_0 under an equivalent probability measure. We will show that this quantity obeys a deterministic differential equation described in terms of the operator A_1 defined below. Let

$$\mathcal{D}\,(A_1) := \left\{ (x_0, x_1) \in \mathcal{H} : x_1(\cdot) \in W^{1,2}\big([-d, 0]; \mathbf{R}\big),\ x_0 = x_1(0) \right\} ,$$

and

$$\begin{aligned} A_1 : \mathcal{D}\,(A_1) \subset \mathcal{H} &\longrightarrow \mathcal{H} \\ A_1(x_0, x_1) &:= \Big((\mu_0 - \sigma_0^\top \kappa) x_0 + \int_{-d}^{0} x_1(s) \Phi(\mathrm{d}s), \frac{\mathrm{d}x_1}{\mathrm{d}s} \Big), \end{aligned} \tag{18}$$

with $(\mu_0 - \sigma_0^\top \kappa) \in \mathbf{R}$ and Φ defined in (7). Replacing μ_0 with $\mu_0 - \sigma_0^\top \kappa$ and ϕ with Φ we infer from Lemma 3.1 that A_1 generates a strongly continuous semigroup

$(S(t))$ in $\mathcal{H}$. Let $\big(M_0(t;0,m_0,m_1),\ M_1(t,s;0,m_0,m_1)\big)$ be the solution at time t of the following differential equation

$$\begin{cases} \dfrac{\mathrm{d}M(t)}{\mathrm{d}t} = A_1 M(t), \\ \\ M_0(0) = m_0, \\ \\ M_1(0,s) = m_1(s), \quad s \in [-d,0), \end{cases} \tag{19}$$

with $m_0 \in \mathbf{R}_{>0}$ and $m_1 \in L^2\big([-d,0];\mathbf{R}_{>0}\big)$. Then by definition

$$S(t)\begin{pmatrix} m_0 \\ m_1 \end{pmatrix} = \begin{pmatrix} M_0(t;0,m_0,m_1) \\ M_1(t,s;0,m_0,m_1) \end{pmatrix}. \tag{20}$$

Denote by $\rho(A_1)$ and $R(\lambda, A_1) = (\lambda - A_1)^{-1}$, the resolvent set and the resolvent of A_1 respectively and by $\sigma(A_1)$ the spectrum of A_1. It is known, see for example Proposition 2.13 on p. 126 of [3], that the spectrum of A_1 is given by

$$\sigma(A_1) = \{\lambda \in \mathbf{C} :\ K(\lambda) = 0\},$$

where $K(\cdot)$ is defined in (6). Moreover it is known that $\sigma(A_1)$ is a countable set and every $\lambda \in \sigma(A_1)$ is an isolated eigenvalue of finite multiplicity. Let

$$\lambda_0 = \sup\{\operatorname{Re}\lambda :\ K(\lambda) = 0\} \tag{21}$$

be the spectral bound of A_1.

Lemma 4.1. *The function*

$$\mathbf{R} \ni \xi \longrightarrow K(\xi) \in \mathbf{R},$$

is strictly increasing and the spectral bound λ_0 is the only real root of the equation $K(\xi) = 0$. In particular, K defined by (8) is positive if and only if $r > \lambda_0$.

Proof. The function $K(\cdot) : \mathbf{R} \to \mathbf{R}$ is differentiable. We have

$$K'(\xi) = 1 + \int_{-d}^{0} e^{\xi\tau}|\tau|\Phi(\mathrm{d}\tau) > 0, \quad \xi \in \mathbf{R},$$

where positivity follows from the fact that Φ is nonnegative by Hypothesis 1. It is easy to see that

$$\lim_{\xi \to \pm\infty} K(\xi) = \pm\infty,$$

and therefore the equation $K(\xi) = 0$ has exactly one real solution ξ_0. Clearly, we have $\xi_0 \le \lambda_0$. To show that $\xi_0 = \lambda_0$ consider an arbitrary $\lambda = x + iy$ such that $K(\lambda) = 0$. Then

$$\begin{aligned} 0 &= x - \mu_0 + \sigma_0^\top \kappa - \int_{-d}^0 e^{x\tau} \cos(x\tau)\, \Phi(\mathrm{d}\tau) \\ &\ge x - \mu_0 + \sigma_0^\top \kappa - \int_{-d}^0 e^{x\tau}\, \Phi(\mathrm{d}\tau) \\ &= K(x)\,. \end{aligned}$$

Therefore, $K(x) \le 0$ which yields $x = \operatorname{Re} \lambda \le \xi_0$, hence $\lambda_0 \le \xi_0$. Finally, $\lambda_0 < r$ if and only if $K(r) > 0$. □

Lemma 4.2. *Let $\lambda \in \mathbf{R} \cap \rho\,(A_1)$. Then the resolvent $(\lambda - A_1)^{-1}$ is given by*

$$R(\lambda, A_1) \begin{pmatrix} m_0 \\ m_1 \end{pmatrix} = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \tag{22}$$

with

$$\begin{aligned} u_0 &= \frac{1}{K(\lambda)} \big[m_0 + \int_{-d}^0 \int_{-d}^{s_1} e^{\lambda(\tau - s_1)} \Phi(d\tau)\, m_1(s_1) ds_1 \big], \\ u_1(s) &= \frac{e^{\lambda s}}{K(\lambda)} \left(m_0 + \int_{-d}^0 \int_{-d}^{s_1} e^{\lambda(\tau - s_1)} \Phi(d\tau)\, m_1(s_1) ds_1 \right) + \int_s^0 e^{-\lambda(s_1 - s)} m_1(s_1)\, ds_1. \end{aligned} \tag{23}$$

Proof. To compute $R(\lambda, A_1)$, we will consider for a fixed $\begin{pmatrix} m_0 \\ m_1 \end{pmatrix} \in \mathcal{H}$ the equation

$$(\lambda - A_1) \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} m_0 \\ m_1 \end{pmatrix}, \tag{24}$$

that by definition of A_1 is equivalent to

$$\begin{cases} (\lambda - (\mu_0 - \sigma_0^\top \kappa)) u_0 - \displaystyle\int_{-d}^0 u_1(\tau) \Phi(\mathrm{d}\tau) = m_0 \\ \lambda u_1 - \dfrac{\mathrm{d}u_1}{\mathrm{d}s} = m_1. \end{cases}$$

Then

$$u_1(s) = e^{\lambda s} u_0 + \int_s^0 e^{-\lambda(s_1 - s)} m_1(s_1)\, \mathrm{d}s_1, \quad s \in [-d, 0],$$

and u_0 is determined by the equation

$$\big(\lambda - (\mu_0 - \sigma_0^\top \kappa)\big) u_0 = \Big[m_0 + \int_{-d}^{0} \left(e^{\lambda\tau} u_0 + \int_{\tau}^{0} e^{-\lambda(s_1-\tau)} m_1(s_1)\, \mathrm{d}s_1 \right) \Phi(\mathrm{d}\tau) \Big]$$

or equivalently, u_0 is given by the equation

$$K(\lambda) u_0 = m_0 + \int_{-d}^{0} \int_{-d}^{s_1} e^{\lambda(\tau - s_1)} \Phi(\mathrm{d}\tau)\, m_1(s_1) \mathrm{d}s_1,$$

with $K(\lambda)$ defined in (6). Thus for $K(\lambda) \neq 0$ the equation (24) is invertible and the result follows. □

The following fact is well known.

Lemma 4.3. *For any $\lambda > \lambda_0$ we have*

$$\int_0^\infty e^{-\lambda t} S(t) \begin{pmatrix} m_0 \\ m_1 \end{pmatrix} dt = R(\lambda, A_1) \begin{pmatrix} m_0 \\ m_1 \end{pmatrix}. \tag{25}$$

Proof. Formula (25) is standard for any strongly continuous semigroup provided λ is big enough. To show that we can take $\lambda > \lambda_0$ we invoke the fact that the semigroup $S(t)$ is eventually compact, hence for the generators of the delay semigroups the growth bound and the spectral bound λ_0 coincide, see Corollary 2.5 on p. 121 of [3]. □

For $\lambda \in \mathbf{R}$ such that $K(\lambda) \neq 0$, let $\big(f(\lambda), g(\lambda)\big)$ be defined as

$$\begin{aligned} f(\lambda) &:= \frac{1}{K(\lambda)}, \\ g(\lambda, s) &:= \frac{1}{K(\lambda)} \int_{-d}^{s} e^{-\lambda(s-\tau)} \Phi(\mathrm{d}\tau). \end{aligned} \tag{26}$$

Lemma 4.4. *Fix $t_0 \geq 0$. Let $M = (M_0, M_1) \in \mathcal{H}$ be a solution to the following differential equation*

$$\begin{cases} \dfrac{dM(t)}{dt} = A_1 M(t), \\ M_0(t_0) = m_0, \\ M_1(t_0, s) = m_1(s), \quad s \in [-d, 0). \end{cases} \tag{27}$$

with $(m_0, m_1) \in \mathbf{R} \times L^2([-d,0];\mathbf{R})$. *Then for any* $\lambda > \lambda_0$ *we have*

$$\int_{t_0}^{+\infty} e^{-\lambda t} M_0(t)\,dt = e^{-\lambda t_0}\langle (f(\lambda), g(\lambda)), (m_0, m_1)\rangle.$$

Proof. We first prove the result for $t_0 = 0$. Recalling Lemma 4.2 and Lemma 4.3, we have

$$\begin{aligned}\int_0^{\infty} e^{-\lambda t} M_0(t)\mathrm{d}t =& \frac{m_0 + \int_{-d}^0 \int_{-d}^{s_1} e^{\lambda(\tau - s_1)}\Phi(\mathrm{d}\tau)\, m_1(s_1)\mathrm{d}s_1}{K(\lambda)} \\ =& \langle (f(\lambda), g(\lambda)), (m_0, m_1)\rangle.\end{aligned} \tag{28}$$

Now, consider $t_0 \geq 0$, and let $\big(M_0(t; t_0, m_0, m_1), M_1(t; t_0, m_0, m_1)\big)$ be a solution to equation (27) starting at time t_0 from (m_0, m_1). Then we have

$$M_0(t; t_0, m_0, m_1) = M_0(t - t_0; 0, m_0, m_1)\,.$$

By (28), it holds

$$\begin{aligned}\int_{t_0}^{+\infty} e^{-\lambda t} M_0(t; t_0, m_0, m_1)\mathrm{d}t &= \int_0^{+\infty} e^{-\lambda(s+t_0)} M_0(s; 0, m_0, m_1)\mathrm{d}s \\ = e^{-\lambda t_0}\int_0^{+\infty} e^{-\lambda s} M_0(s; 0, m_0, m_1)\mathrm{d}s &= e^{-\lambda t_0}\langle (f(\lambda), g(\lambda)), (m_0, m_1)\rangle.\end{aligned}$$

□

In order to prove Theorem 2.1 we also need the following technical lemma.

Lemma 4.5. *It holds that*

$$\mathbf{E}\Big(\int_{t_0}^t \Big\| X_0(s)\sigma_0 + \begin{pmatrix} \int_{-d}^0 X_0(s+\tau)\varphi_1(d\tau) \\ \vdots \\ \int_{-d}^0 X_0(s+\tau)\varphi_n(d\tau)\end{pmatrix} \Big\|_{\mathbf{R}^n}^2 ds\Big) < +\infty\,.$$

Proof. Let us denote with σ_0^i the i-th component of σ_0, and let us show that

$$\mathbf{E}\Big(\int_{t_0}^t \big[X_0(s)\sigma_0^i + \int_{-d}^0 X_0(s+\tau)\varphi_i(\mathrm{d}\tau)\big]^2 \mathrm{d}s\Big) < +\infty.$$

By the trivial inequality $(a+b)^2 \leq 2(a^2 + b^2)$, it is sufficient to show that

$$\mathbf{E}\Big(\int_{t_0}^t X_0^2(s)(\sigma_0^i)^2\mathrm{d}s\Big) < +\infty, \tag{29}$$

and

$$\mathbf{E}\Big(\int_{t_0}^{t}\big[\int_{-d}^{0}X_0(s+\tau)\varphi_i(\mathrm{d}\tau)\big]^2\mathrm{d}s\Big)<+\infty. \tag{30}$$

To show (29), by Theorem 7.4 in [10] we can write

$$\mathbf{E}\Big(\int_{t_0}^{t}X_0^2(s)(\sigma_0^i)^2\mathrm{d}s\Big)\le(\sigma_0^i)^2(t-t_0)\mathbf{E}\big(\sup_{s\in[t_0,t]}X_0^2(s)\big)<+\infty.$$

To show (30), by the Hölder inequality

$$\left(\int_{-d}^{0}X_0(s+\tau)\,\varphi_i(\mathrm{d}\tau)\right)^2\le\left(\int_{-d}^{0}|X_0(s+\tau)|^2\,\varphi_i(\mathrm{d}\tau)\right)\left(\int_{-d}^{0}\varphi_i(\mathrm{d}\tau)\right)$$

$$=\varphi_i([-d,0])\left(\int_{-d}^{0}|X_0(s+\tau)|^2\,\varphi_i(\mathrm{d}\tau)\right).$$

Thus

$$\begin{aligned}&\mathbf{E}\Big(\int_{t_0}^{t}\big(\int_{-d}^{0}X_0(s+\tau)\varphi_i(\mathrm{d}\tau)\big)^2\mathrm{d}s\Big)\\ &\le\varphi_i([-d,0])\int_{t_0}^{t}\int_{-d}^{0}\mathbf{E}\Big(|X_0(s+\tau)|^2\Big)\,\varphi_i(\mathrm{d}\tau)\mathrm{d}s\\ &\le\Big(\varphi_i([-d,0])\Big)^2(t-t_0)\sup\mathbf{E}\Big(|X_0(s+\tau)|^2\Big).\end{aligned}$$

By Theorem 7.4 in [10], the expression above is finite. □

We can now provide the proof of Theorem 2.1.

Proof. We have

$$\mathbf{E}\Big(\int_{t_0}^{+\infty}\xi(s)X_0(s)\mathrm{d}s\mid\mathcal{F}_{t_0}\Big)=\int_{t_0}^{+\infty}\mathbf{E}\big(\xi(s)X_0(s)\mid\mathcal{F}_{t_0}\big)\mathrm{d}s\qquad\mathbf{P}\text{-a.s.} \tag{31}$$

In fact, using the characteristic property of the conditional mean, and Fubini's Theorem together with Theorem 7.4 in [10], for any $F\in\mathcal{F}_{t_0}$ we have

$$\begin{aligned}\int_F\mathbf{E}\Big(\int_{t_0}^{+\infty}\xi(s)X_0(s)\mathrm{d}s\mid\mathcal{F}_{t_0}\Big)\mathrm{d}\mathbf{P}&=\int_F\int_{t_0}^{+\infty}\xi(s)X_0(s)\mathrm{d}s\,\mathrm{d}\mathbf{P}\\ =\int_{t_0}^{+\infty}\int_F\xi(s)X_0(s)\mathrm{d}\mathbf{P}\,\mathrm{d}s&=\int_{t_0}^{+\infty}\int_F\mathbf{E}\big(\xi(s)X_0(s)\mid\mathcal{F}_{t_0}\big)\mathrm{d}\mathbf{P}\,\mathrm{d}s\\ &=\int_F\int_{t_0}^{+\infty}\mathbf{E}\big(\xi(s)X_0(s)\mid\mathcal{F}_{t_0}\big)\mathrm{d}s\,\mathrm{d}\mathbf{P}.\end{aligned}$$

To compute $\mathbf{E}\big(\xi(s)X_0(s) \mid \mathcal{F}_{t_0}\big)$, let us consider the equivalent probability measure

$$\mathrm{d}\tilde{\mathbf{P}}(s) := e^{-\frac{1}{2}|\kappa|^2 s-\kappa^\top Z_s}\mathrm{d}\mathbf{P}\,,$$

defined on $\mathcal{F}_s$. Note that

$$\frac{\mathrm{d}\tilde{\mathbf{P}}(s)}{\mathrm{d}\mathbf{P}} = e^{-\frac{1}{2}|\kappa|^2 s-\kappa^\top Z_s} = e^{rs}\xi(s)\,,$$

and hence by Lemma 3.5.3 in [20] we can write

$$\mathbf{E}\Big(\xi(s)X_0(s) \mid \mathcal{F}_{t_0}\Big) = \xi(t_0)e^{-r(s-t_0)}\tilde{\mathbf{E}}\Big(X_0(s) \mid \mathcal{F}_{t_0}\Big),$$

where $\tilde{\mathbf{E}}$ denotes the mean under the measure $\tilde{\mathbf{P}}(s)$. Our aim is to evaluate

$$\int_{t_0}^{+\infty}\mathbf{E}\Big(\xi(s)X_0(s) \mid \mathcal{F}_{t_0}\Big)\mathrm{d}s = \xi(t_0)e^{rt_0}\int_{t_0}^{+\infty}e^{-rs}\tilde{\mathbf{E}}\Big(X_0(s) \mid \mathcal{F}_{t_0}\Big)\mathrm{d}s. \tag{32}$$

Let $\tilde{\mathbf{P}}$ denote the measure, such that $\tilde{\mathbf{P}}\Big|_{\mathcal{F}_s} = \tilde{\mathbf{P}}(s)$ for all $s \geq 0$. By the Girsanov Theorem, the process

$$\tilde{Z}(t) = Z(t) + \kappa t \tag{33}$$

is an n-dimensional Brownian motion under the measure $\tilde{\mathbf{P}}$, and the dynamics of X_0 under $\tilde{\mathbf{P}}$ is

$$\begin{aligned} dX_0(s) = \quad & \big[(\mu_0 - \sigma_0^\top\kappa)X_0(s) + \textstyle\int_{-d}^0 X_0(s+\tau)\,\Phi(\mathrm{d}\tau)\big]\mathrm{d}s \\ & + \left[X_0(t)\sigma_0^\top + \begin{pmatrix} \int_{-d}^0 X_0(s+\tau)\varphi_1(\mathrm{d}\tau) \\ \vdots \\ \int_{-d}^0 X_0(s+\tau)\varphi_n(\mathrm{d}\tau)\end{pmatrix}^\top\right]\mathrm{d}\tilde{Z}(s), \end{aligned}$$

where Φ is defined in (7). Integrating between t_0 and t we obtain

$$\begin{aligned} X_0(t) = X_0(t_0) &+ \int_{t_0}^t(\mu_0 - \sigma_0^\top\kappa)X_0(s)\mathrm{d}s + \int_{t_0}^t\int_{-d}^0 X_0(s+\tau)\Phi(\mathrm{d}\tau)\mathrm{d}s \\ &+ \int_{t_0}^t\left[X_0(s)\sigma_0^\top + \begin{pmatrix} \int_{-d}^0 X_0(s+\tau)\varphi_1(\mathrm{d}\tau) \\ \vdots \\ \int_{-d}^0 X_0(s+\tau)\varphi_n(\mathrm{d}\tau)\end{pmatrix}^\top\right]\mathrm{d}\tilde{Z}(s)\,, \end{aligned} \tag{34}$$

and therefore

$$\begin{aligned}\tilde{\mathbf{E}}\Big(X_0(t) \mid \mathcal{F}_{t_0}\Big) =& X_0(t_0) + (\mu_0 - \sigma_0^\top \kappa)\tilde{\mathbf{E}}\Big(\int_{t_0}^t X_0(s)\mathrm{d}s \mid \mathcal{F}_{t_0}\Big) \\ &+ \tilde{\mathbf{E}}\Big(\int_{t_0}^t \int_{-d}^0 X_0(s+\tau)\Phi(\mathrm{d}\tau)\mathrm{d}s \mid \mathcal{F}_{t_0}\Big) \\ &+ \tilde{\mathbf{E}}\Big(\int_{t_0}^t \left[X_0(s)\sigma_0^\top + \begin{pmatrix}\int_{-d}^0 X_0(s+\tau)\varphi_1(\mathrm{d}\tau) \\ \vdots \\ \int_{-d}^0 X_0(s+\tau)\varphi_n(\mathrm{d}\tau)\end{pmatrix}^\top\right] \mathrm{d}\tilde{Z}(s) \mid \mathcal{F}_{t_0}\Big).\end{aligned} \tag{35}$$

By Lemma 4.5, which still applies after the change of measure, the stochastic integral with respect to $\tilde{Z}$ is a martingale, and has zero mean. By definition of conditional mean and by Fubini's Theorem, the expression in (35) gives

$$\begin{aligned}\tilde{\mathbf{E}}\Big(X_0(t) \mid \mathcal{F}_{t_0}\Big) =& X_0(t_0) + (\mu_0 - \sigma_0^\top \kappa)\int_{t_0}^t \tilde{\mathbf{E}}\big(X_0(s) \mid \mathcal{F}_{t_0}\big)\,\mathrm{d}s \\ &+ \int_{t_0}^t \int_{-d}^0 \tilde{\mathbf{E}}\big(X_0(s+\tau) \mid \mathcal{F}_{t_0}\big)\,\Phi(\mathrm{d}\tau)\mathrm{d}s.\end{aligned} \tag{36}$$

Deriving (36) with respect to t, we obtain the following, for $t > t_0$:

$$\frac{\mathrm{d}\tilde{\mathbf{E}}\Big(X_0(t) \mid \mathcal{F}_{t_0}\Big)}{\mathrm{d}t} = (\mu_0 - \sigma_0^\top \kappa)\tilde{\mathbf{E}}\Big(X_0(t) \mid \mathcal{F}_{t_0}\Big) + \int_{-d}^0 \tilde{\mathbf{E}}\Big(X_0(t+\tau) \mid \mathcal{F}_{t_0}\Big)\,\Phi(\mathrm{d}\tau). \tag{37}$$

We then see that $\tilde{\mathbf{E}}\big(X_0(t) \mid \mathcal{F}_{t_0}\big)$ must be a solution of

$$\begin{cases}\dfrac{\mathrm{d}M_0}{\mathrm{d}t}(t) = (\mu_0 - \sigma_0^\top \kappa)M_0(t) + \int_{-d}^0 M_0(t+s)\,\Phi(\mathrm{d}s), & t > 0, \\ M_0(t_0) = m_0, & \\ M_1(t_0, s) = m_1(s), & s \in [-d, 0).\end{cases} \tag{38}$$

By Hypothesis 1 and Lemma 4.1 we have $r > \lambda_0$, hence invoking Lemma 4.4 we obtain

$$\int_{t_0}^{+\infty} e^{-rt}\tilde{\mathbf{E}}\big(X_0(t) \mid \mathcal{F}_{t_0}\big)\mathrm{d}t = e^{-rt_0}\langle\big(f(r), g(r)\big), (m_0, m_1)\rangle.$$

Recalling (31) and (32), we can write

$$\begin{aligned}\mathbf{E}\Big(\int_{t_0}^{+\infty}\xi(s)X_0(s)\mathrm{d}s \mid \mathcal{F}_{t_0}\Big) &= \xi(t_0)e^{rt_0}\int_{t_0}^{+\infty}e^{-rs}\tilde{\mathbf{E}}\Big(X_0(s)\mid\mathcal{F}_{t_0}\Big)\mathrm{d}s\\ &= \xi(t_0)\langle\big(f(r),g(r)\big),(m_0,m_1)\rangle.\end{aligned}$$

Note that $\big(f(r),g(r)\big) = (\frac{1}{K},\frac{1}{K}G)$, with (f,g) defined in (26), K in (8), and G in (12). The proof is thus complete. □

5 Conclusion

In this paper, we have provided a valuation formula for streams of payments with delayed dynamics in an otherwise standard, complete market model with risky assets driven by a GBM. In particular, we have developed our analysis with a focus on human capital valuation in a setting with sticky wages. We have allowed for rigidity in labor income dynamics by introducing delay terms in the drift and volatility coefficients of a GBM driven by market risk. Our valuation formula results in an explicit expression of human capital demonstrating the importance of appreciating the past to quantify the current market value of future labor income. More generally, the approach followed in this paper shows how tools from infinite-dimensional analysis can be successfully used to address valuation problems that are non-Markovian, and hence beyond the reach of coventional approaches.

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