QUANTISATION AND NILPOTENT LIMITS OF MISHCHENKO–FOMENKO SUBALGEBRAS

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ABSTRACT. For any simple Lie algebra $\mathfrak{g}$ and an element $\mu \in \mathfrak{g}^*$, the corresponding commutative subalgebra $\mathcal{A}_\mu$ of $\mathcal{U}(\mathfrak{g})$ is defined as a homomorphic image of the Feigin–Frenkel centre associated with $\mathfrak{g}$. It is known that when μ is regular this subalgebra solves Vinberg's quantisation problem, as the graded image of $\mathcal{A}_\mu$ coincides with the Mishchenko–Fomenko subalgebra $\overline{\mathcal{A}}_\mu$ of $\mathcal{S}(\mathfrak{g})$. By a conjecture of Feigin, Frenkel and Toledano Laredo, this property extends to an arbitrary element μ. We give sufficient conditions which imply the property for certain choices of μ. In particular, this proves the conjecture in type C and gives a new proof in type A. We show that the algebra $\mathcal{A}_\mu$ is free in both cases and produce its generators in an explicit form. Moreover, we prove that in all classical types generators of $\mathcal{A}_\mu$ can be obtained via the canonical symmetrisation map from certain generators of $\overline{\mathcal{A}}_\mu$. The symmetrisation map is also used to produce free generators of nilpotent limits of the algebras $\mathcal{A}_\mu$ and give a positive solution of Vinberg's problem for these limit subalgebras.

INTRODUCTION

The universal enveloping algebra $\mathcal{U}(\mathfrak{q})$ of a Lie algebra $\mathfrak{q}$ is equipped with a canonical filtration so that the associated graded algebra is isomorphic to the symmetric algebra $\mathcal{S}(\mathfrak{q})$. The commutator on $\mathfrak{q}$ induces the Lie–Poisson bracket on $\mathcal{S}(\mathfrak{q})$ defined by taking $\{X, Y\}$ to be equal to the commutator of $X, Y \in \mathfrak{q}$ and then extending the bracket to the entire $\mathcal{S}(\mathfrak{q})$ by the Leibniz rule. If $\mathcal{A}$ is a commutative subalgebra of $\mathcal{U}(\mathfrak{q})$, then its graded image $\operatorname{gr}\mathcal{A}$ is a Poisson-commutative subalgebra of $\mathcal{S}(\mathfrak{q})$. The *quantisation problem* for a given Poisson-commutative subalgebra $\overline{\mathcal{A}}$ of $\mathcal{S}(\mathfrak{q})$ is to find a commutative subalgebra $\mathcal{A}$ of $\mathcal{U}(\mathfrak{q})$ with the property $\operatorname{gr}\mathcal{A} = \overline{\mathcal{A}}$.

In the case where $\mathfrak{q} = \mathfrak{g}$ is a finite-dimensional simple Lie algebra over $\mathbb{C}$, a family of commutative subalgebras of $\mathcal{U}(\mathfrak{g})$ can be constructed with the use of the associated *Feigin–Frenkel centre* $\mathfrak{z}(\widehat{\mathfrak{g}})$ which is a commutative subalgebra of $\mathcal{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)$. Given any $\mu \in \mathfrak{g}^*$ and a nonzero $z \in \mathbb{C}$, the image of $\mathfrak{z}(\widehat{\mathfrak{g}})$ with respect to the homomorphism

$$\varrho_{\mu,z} \colon \mathcal{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big) \to \mathcal{U}(\mathfrak{g}), \qquad X t^r \mapsto X z^r + \delta_{r,-1}\, \mu(X), \quad X \in \mathfrak{g},$$

is a commutative subalgebra $\mathcal{A}_\mu$ of $\mathcal{U}(\mathfrak{g})$ which is independent of z. This subalgebra was used by Rybnikov [R06] and Feigin, Frenkel and Toledano Laredo [FFTL] to give a positive solution of *Vinberg's quantisation problem* for regular μ. Namely, the graded image $\operatorname{gr}\mathcal{A}_\mu$ in the symmetric algebra $\mathcal{S}(\mathfrak{g})$ turns out to coincide with the *Mishchenko–Fomenko* *subalgebra* $\overline{\mathcal{A}}_\mu$ [MF78] which is generated by all *μ-shifts* of the $\mathfrak{g}$-invariants of $\mathcal{S}(\mathfrak{g})$; a precise definition is recalled in Section 2 below. It was conjectured in [FFTL, Conjecture 1] that the property $\operatorname{gr}\mathcal{A}_\mu = \overline{\mathcal{A}}_\mu$ extends to all $\mu \in \mathfrak{g}^*$ (it clearly holds for $\mu = 0$). The conjecture was confirmed in [FM15] for type A.

Our first main result is a proof of the FFTL-conjecture for type C. The same approach can be used in type A which leads to another proof of the conjecture. It is known by [FFTL, Proposition 3.12] that the inclusion $\overline{\mathcal{A}}_\mu \subset \operatorname{gr}\mathcal{A}_\mu$ holds for any simple Lie algebra $\mathfrak{g}$ and any $\mu \in \mathfrak{g}^*$. Our argument relies on this fact and is based on a general result establishing a maximality property of the Mishchenko–Fomenko subalgebra $\overline{\mathcal{A}}_\gamma$ associated with an arbitrary Lie algebra $\mathfrak{q}$ and an element $\gamma \in \mathfrak{q}^*$. In more detail, we show that under certain additional assumptions, $\overline{\mathcal{A}}_\gamma$ is a maximal Poisson-commutative subalgebra of the algebra of $\mathfrak{q}_\gamma$-invariants $\mathcal{S}(\mathfrak{q})^{\mathfrak{q}_\gamma}$, where $\mathfrak{q}_\gamma$ denotes the stabiliser of γ (see Theorem 2.3(ii) below). This property is quite analogous to the main result of [PY08] establishing the maximality of the Mishchenko–Fomenko subalgebra $\overline{\mathcal{A}}_\mu$ in $\mathcal{S}(\mathfrak{g})$ for regular μ.

Applying the results of [PPY], for any given $\mu \in \mathfrak{g}^*$ we then produce families of free generators of the algebra $\mathcal{A}_\mu$ in types A and C in an explicit form. This provides a new derivation of the corresponding results of [FM15] in type A.

As another principal result of the paper, we show that the free generators of $\mathcal{A}_\mu$ can be obtained via the canonical symmetrisation map; see (3·2) below. This map was used by Tarasov [T00] to construct a commutative subalgebra of $\mathcal{U}(\mathfrak{gl}_N)$ quantising the Mishchenko–Fomenko subalgebra $\overline{\mathcal{A}}_\mu \subset \mathcal{S}(\mathfrak{gl}_N)$. By another result of Tarasov [T03], that commutative subalgebra of $\mathcal{U}(\mathfrak{gl}_N)$ coincides with $\mathcal{A}_\mu$ if μ is regular semisimple. We extend these properties of the symmetrisation map to all classical Lie algebras $\mathfrak{g}$ by showing that the algebra of invariants $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ admits a family of free generators such that the images of their μ-shifts with respect to the symmetrisation map generate the algebra $\mathcal{A}_\mu$ for any μ. The respective generators of $\mathcal{A}_\mu$ are given explicitly in the form of symmetrised minors or permanents; see Theorems 3.2 and 3.3. We state as a conjecture that free generators of $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ with the same properties exist for all simple Lie algebras (Conjecture 3.1).

By the work of Vinberg [V91] and Shuvalov [Sh02], new families of Poisson-commutative subalgebras of $\mathcal{S}(\mathfrak{g})$ of maximal transcendence degree can be constructed by taking certain limits of the Mishchenko–Fomenko subalgebras; see also [V14]. For instance, the associated graded of the Gelfand–Tsetlin subalgebra $\mathcal{GT}(\mathfrak{gl}_N) \subset \mathcal{U}(\mathfrak{gl}_N)$ is a Poisson commutative subalgebra of $\mathcal{S}(\mathfrak{gl}_N)$ which does not occur as $\overline{\mathcal{A}}_\mu$ for any μ. However, it can be obtained by choosing a parameter-depending family $\mu(t)$ and taking an appropriate limit of $\overline{\mathcal{A}}_{\mu(t)}$ as $t \to 0$. We show that the Vinberg–Shuvalov limit subalgebras admit a quantisation. In particular, in the case of the symplectic Lie algebra $\mathfrak{g} = \mathfrak{sp}_{2n}$

this leads to a construction of a Gelfand–Tsetlin-type subalgebra $\mathcal{GT}(\mathfrak{sp}_{2n})$. This is a maximal commutative subalgebra of $\mathcal{U}(\mathfrak{sp}_{2n})$ which contains the centres of all universal enveloping algebras $\mathcal{U}(\mathfrak{sp}_{2k})$ with $k = 1, \dots, n$ associated with the subalgebras of the chain $\mathfrak{sp}_2 \subset \cdots \subset \mathfrak{sp}_{2n}$. We believe that this new subalgebra $\mathcal{GT}(\mathfrak{sp}_{2n})$ should be useful in the representation theory of $\mathfrak{sp}_{2n}$, in particular, for the *separation of multiplicity problem* in the reduction $\mathfrak{sp}_{2n} \downarrow \mathfrak{sp}_{2n-2}$ for finite-dimensional irreducible representations.

We also consider certain versions of the limit subalgebras which are different from those of [Sh02], but arise within the general scheme described in [V14]. We give a solution of the quantisation problem for these *nilpotent limit subalgebras* in types A and C; see Proposition 5.2.

It was already pointed out by Tarasov [T03] that the symmetrisation map commutes with taking limits thus allowing one to quantise the limit Poisson-commutative subalgebras of $\mathcal{S}(\mathfrak{g})$. Therefore, the quantisations can be obtained equivalently either by applying the symmetrisation map, or by taking the nilpotent limits of the subalgebras $\mathcal{A}_\mu$.

As a consequence of the nilpotent limit construction, we get a solution of Vinberg's quantisation problem for centralisers of nilpotent elements in types A and C. In their recent work [AP17] Arakawa and Premet extended the approach of [R06] and [FFTL] by replacing the Feigin–Frenkel centre with the centre of the affine $\mathcal{W}$-algebra associated with a simple Lie algebra $\mathfrak{g}$ and a nilpotent element $e \in \mathfrak{g}$. Under certain restrictions on the data, they gave a positive solution of Vinberg's problem for the centralisers $\mathfrak{g}_e$. It appears to be likely that their solution coincides with ours based on the nilpotent limits; see Conjecture 5.8.

Symmetric invariants of centralisers have been extensively studied at least since [PPY]. Certain polynomials ${}^eH \in \mathcal{S}(\mathfrak{g}_e)$ are defined in that paper via the restriction to a Slodowy slice. Notably, these elements are related to the e-shifts of H; see Lemma 1.5. Let $H_1, \dots, H_n \in \mathcal{S}(\mathfrak{g})$ with $n = \operatorname{rk}\mathfrak{g}$ be a set of homogeneous generating symmetric invariants. Then $\sum\limits_{i=1}^{n} \deg {}^eH_i \leqslant \mathbf{b}(\mathfrak{g}_e)$, where $\mathbf{b}(\mathfrak{g}_e)$ is a certain integer related to $\mathfrak{g}_e$. This inequality is one of the crucial points in [PPY] and it is proved via finite $\mathcal{W}$-algebras. We found a more direct line of argument, which works for Lie algebras of *Kostant type*; see Lemma 1.8.

Our ground field is $\mathbb{C}$. However, since semisimple Lie algebras are defined over $\mathbb{Z}$, it is not difficult to deduce that the main results are valid over any field of characteristic zero.

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1. Preliminaries on Lie-Poisson structures

Let Q be a non-Abelian connected algebraic group and $\mathfrak{q} = \operatorname{Lie} Q$ its Lie algebra. For $\gamma \in \mathfrak{q}^*$, let $\hat{\gamma}$ be the corresponding skew-symmetric form on $\mathfrak{q}$ given by $\hat{\gamma}(\xi, \eta) = \gamma\big([\xi, \eta]\big)$. Note that the kernel of $\hat{\gamma}$ is equal to the stabiliser

$$\mathfrak{q}_\gamma = \{\xi \in \mathfrak{q} \mid \operatorname{ad}^*(\xi)\gamma = 0\}.$$

We will identify the symmetric algebra $\mathcal{S}(\mathfrak{q})$ with the algebra $\mathbb{C}[\mathfrak{q}^*]$ of polynomial functions on $\mathfrak{q}^*$. Suppose that $\dim \mathfrak{q} = r$ and choose a basis $\{\xi_1, \ldots, \xi_r\}$ of $\mathfrak{q}$. Let $\{x_1, \ldots, x_r\}$ be the dual basis of $\mathfrak{q}^*$. Let

$$\pi = \sum_{i<j} [\xi_i, \xi_j]\, x_i \wedge x_j \tag{1·1}$$

be the Poisson tensor (bivector) of $\mathfrak{q}$. It is a global section of $\Lambda^2 T\mathfrak{q}^*$ and at each point $\gamma \in \mathfrak{q}^*$ we have $\pi(\gamma) = \hat{\gamma}$. Let dF denote the differential of $F \in \mathcal{S}(\mathfrak{q})$ and $d_\gamma F$ denote the differential of F at $\gamma \in \mathfrak{q}^*$. A well-known property of π is that

$$\{F_1, F_2\}(\gamma) = \pi(\gamma)(d_\gamma F_1, d_\gamma F_2)$$

for all $F_1, F_2 \in \mathcal{S}(\mathfrak{q})$. As defined by Dixmier, the *index* of $\mathfrak{q}$ is the number

$$\operatorname{ind} \mathfrak{q} = \min_{\gamma \in \mathfrak{q}^*} \dim \mathfrak{q}_\gamma = \dim \mathfrak{q} - \max_{\gamma \in \mathfrak{q}^*} \dim(Q\gamma) = \dim \mathfrak{q} - \max_{\gamma \in \mathfrak{q}^*} \operatorname{rk} \pi(\gamma).$$

For a reductive $\mathfrak{g}$, one has $\operatorname{ind} \mathfrak{g} = \operatorname{rk} \mathfrak{g}$. In this case, $(\dim \mathfrak{g} + \operatorname{rk} \mathfrak{g})/2$ is the dimension of a Borel subalgebra of $\mathfrak{g}$. For an arbitrary $\mathfrak{q}$, set $\boldsymbol{b}(\mathfrak{q}) = (\operatorname{ind} \mathfrak{q} + \dim \mathfrak{q})/2$. Observe that $\operatorname{ind} \mathfrak{q}_\gamma \geqslant \operatorname{ind} \mathfrak{q}$ for all $\gamma \in \mathfrak{q}^*$ by Vinberg's inequality [P03, Sect. 1].

One defines the *singular set* $\mathfrak{q}^*_{\rm sing}$ of $\mathfrak{q}^*$ by

$$\mathfrak{q}^*_{\rm sing} = \{\gamma \in \mathfrak{q}^* \mid \dim \mathfrak{q}_\gamma > \operatorname{ind} \mathfrak{q}\}.$$

Set also $\mathfrak{q}^*_{\rm reg} = \mathfrak{q}^* \setminus \mathfrak{q}^*_{\rm sing}$. Further, $\mathfrak{q}$ is said to have the codim-m property (or to satisfy the codim-m condition), if $\dim \mathfrak{q}^*_{\rm sing} \leqslant \dim \mathfrak{q} - m$. A reductive Lie algebra $\mathfrak{g}$ satisfies the codim-3 condition. Recall that an open subset is called *big* if its complement does not contain divisors. The codim-2 condition holds for $\mathfrak{q}$ if and only if $\mathfrak{q}^*_{\rm reg}$ is big.

Suppose that $\gamma \in \mathfrak{q}^*_{\rm reg}$. Then

$$\dim \mathfrak{q}_\gamma = \operatorname{ind} \mathfrak{q} \leqslant \operatorname{ind} \mathfrak{q}_\gamma.$$

Therefore $\operatorname{ind} \mathfrak{q}_\gamma = \dim \mathfrak{q}_\gamma = \operatorname{ind} \mathfrak{q}$ and $\mathfrak{q}_\gamma$ is a commutative Lie algebra.

1.1. **Transcendence degree bounds.** For any subalgebra $A \subset \mathcal{S}(\mathfrak{q})$ and any $x \in \mathfrak{q}^*$ set

$$d_x A = \langle d_x F \mid F \in A\rangle_{\mathbb{C}} \subset T_x^*\mathfrak{q}^*.$$

Then for the transcendence degree of A we have $\operatorname{tr.deg} A = \max\limits_{x\in\mathfrak{q}^*} \dim d_x A$. If A is Poisson-commutative, then $\hat{x}(d_x A, d_x A) = 0$ for each $x \in \mathfrak{q}^*$ and thereby

$$\operatorname{tr.deg} A \leqslant \frac{\dim\mathfrak{q} - \operatorname{ind}\mathfrak{q}}{2} + \operatorname{ind}\mathfrak{q} = \mathbf{b}(\mathfrak{q}).$$

Applying a result of Borho and Kraft [BK76, Satz 5.7], one obtains that $\operatorname{tr.deg}\mathcal{A} \leqslant \mathbf{b}(\mathfrak{q})$ for any commutative subalgebra $\mathcal{A} \subset \mathcal{U}(\mathfrak{q})$. We will prove a similar statement for a certain smaller class of Poisson-commutative subalgebras.

For any subalgebra $\mathfrak{l} \subset \mathfrak{q}$, let $\mathcal{S}(\mathfrak{q})^{\mathfrak{l}}$ denote the *Poisson centraliser* of $\mathfrak{l}$, i.e.,

$$\mathcal{S}(\mathfrak{q})^{\mathfrak{l}} = \{F \in \mathcal{S}(\mathfrak{q}) \mid \{\xi, F\} = 0 \quad \text{for all} \quad \xi \in \mathfrak{l}\}.$$

If $\mathfrak{l} = \operatorname{Lie} L$ and $L \subset Q$ is a connected subgroup, then $\mathcal{S}(\mathfrak{q})^{\mathfrak{l}}$ coincides with the subalgebra of L-invariants

$$\mathcal{S}(\mathfrak{q})^{L} = \{F \in \mathcal{S}(\mathfrak{q}) \mid gF = F \quad \text{for all} \quad g \in L\}.$$

Proposition 1.1. *Let $A \subset \mathcal{S}(\mathfrak{q})^{\mathfrak{l}}$ be a Poisson-commutative subalgebra. Then*

$$\operatorname{tr.deg} A \leqslant \frac{1}{2}\left(\dim\mathfrak{q} - \operatorname{ind}\mathfrak{q} - \dim\mathfrak{l} + \operatorname{ind}\mathfrak{l}\right) + \operatorname{ind}\mathfrak{q} = \mathbf{b}(\mathfrak{q}) - \mathbf{b}(\mathfrak{l}) + \operatorname{ind}\mathfrak{l}.$$

Proof. For any point $x \in \mathfrak{q}^*$, we have $\hat{x}(d_x A, d_x A) = 0$ and $\hat{x}(\mathfrak{l}, d_x A) = 0$. For a generic point x, the form $\hat{x}$ is of rank $\dim\mathfrak{q} - \operatorname{ind}\mathfrak{q}$ and $x|_{\mathfrak{l}} \in \mathfrak{l}^*_{\rm reg}$, therefore the restriction of $\hat{x}$ to $\mathfrak{l}$ is of rank $\dim\mathfrak{l} - \operatorname{ind}\mathfrak{l}$. The quotient $d_x A/(d_x A \cap \mathfrak{q}_x)$ is an isotropic subspace of $\mathfrak{q}/\mathfrak{q}_x$ orthogonal to $\mathfrak{l}/(\mathfrak{l} \cap \mathfrak{q}_x)$. Knowing that the rank of $\hat{x}$ on $\mathfrak{l}$ is equal to $\dim\mathfrak{l} - \operatorname{ind}\mathfrak{l}$, we can conclude that the dimension of this quotient is bounded by $\frac{1}{2}\dim(Qx) - \frac{1}{2}(\dim\mathfrak{l} - \operatorname{ind}\mathfrak{l})$. Thereby

$$\dim d_x A \leqslant \frac{1}{2}\big((\dim\mathfrak{q} - \operatorname{ind}\mathfrak{q}) - (\dim\mathfrak{l} - \operatorname{ind}\mathfrak{l})\big) + \operatorname{ind}\mathfrak{q}.$$

This completes the proof. □

The inequality $\operatorname{tr.deg} A \leqslant \mathbf{b}(\mathfrak{q}) - \mathbf{b}(\mathfrak{l}) + \operatorname{ind}\mathfrak{l}$ has the following explanation. According to [S04], there is a Poisson-commutative subalgebra ${}^{\mathfrak{l}}\!A \subset \mathcal{S}(\mathfrak{l})$ with $\operatorname{tr.deg}{}^{\mathfrak{l}}\!A = \mathbf{b}(\mathfrak{l})$. Clearly $\{A, {}^{\mathfrak{l}}\!A\} = 0$ and therefore $\dim(d_x A + d_x{}^{\mathfrak{l}}\!A) \leqslant \mathbf{b}(\mathfrak{q})$. In addition $\dim(d_x A \cap d_x{}^{\mathfrak{l}}\!A) \leqslant \operatorname{ind}\mathfrak{l}$ for generic $x \in \mathfrak{q}^*$.

Example 1.2. (i) Suppose that $\mathfrak{q} = \mathfrak{gl}_N$ and $\mathfrak{l} = \mathfrak{gl}_{N-1}$. Then $\operatorname{tr.deg} A \leqslant 2N - 1$ for any Poisson-commutative subalgbera $A \subset \mathcal{S}(\mathfrak{q})^{\mathfrak{l}}$. More generally, if $\mathfrak{l} = \mathfrak{gl}_m$ then

$$\operatorname{tr.deg} A \leqslant \frac{N(N{+}1)}{2} - \frac{m(m{+}1)}{2} + m.$$

(ii) Take $\mathfrak{q} = \mathfrak{sp}_{2n}$ and $\mathfrak{l} = \mathfrak{sp}_{2m}$. Then $\operatorname{tr.deg} A \leqslant n(n{+}1) - m(m{+}1) + m$.

Example 1.3. Suppose that $\mathfrak{l} = \mathfrak{q}_\gamma$. If $\operatorname{ind}\mathfrak{q}_\gamma = \operatorname{ind}\mathfrak{q}$, then the bound of Proposition 1.1 is simpler

$$\text{tr.deg}\, A \leqslant \frac{1}{2}(\dim\mathfrak{q} - \dim\mathfrak{q}_\gamma) + \operatorname{ind}\mathfrak{q} = \frac{1}{2}\dim(Q\gamma) + \operatorname{ind}\mathfrak{q}.$$

If $\operatorname{ind}\mathfrak{q}_\gamma \neq \operatorname{ind}\mathfrak{q}$, then $\operatorname{ind}\mathfrak{q}_\gamma > \operatorname{ind}\mathfrak{q}$ and the bound is larger than the sum above.

Checking the equality $\operatorname{ind}\mathfrak{q}_\gamma = \operatorname{ind}\mathfrak{q}$ is an intricate task. It does not hold for all Lie algebras, see e.g. [PY06, Ex. 1.1].

Remark 1.4. The symplectic linear algebra calculation in the proof of Proposition 1.1 brings up the following observation. Let $A \subset \mathcal{S}(\mathfrak{q})^{\mathfrak{l}}$ be a Poisson-commutative subalgebra with the maximal possible transcendence degree. Suppose that $x \in \mathfrak{q}^*$ is generic, in particular $\dim d_x A = \text{tr.deg}\, A$. Then the orthogonal complement of $d_x A$ w.r.t. $\hat{x}$ equals $\mathfrak{l} + d_x A + \ker\hat{x}$.

1.2. **Lie algebras of Kostant type.** Throughout the paper we will use the direction derivatives $\partial_\gamma H$ of elements $H \in \mathcal{S}(\mathfrak{q}) = \mathbb{C}[\mathfrak{q}^*]$ with respect to $\gamma \in \mathfrak{q}^*$ which are defined by

$$\partial_\gamma H(x) = \frac{d}{dt} H(x + t\gamma)\Big|_{t=0}.$$

Given a nonzero $\gamma \in \mathfrak{q}^*$ we fix a decomposition $\mathfrak{q} = \mathbb{C}y \oplus \operatorname{Ker}\gamma$, where $\gamma(y) = 1$. For each nonzero $H \in \mathcal{S}(\mathfrak{q})$, we have a decomposition $H = y^m H_{[m]} + y^{m-1}H_{[m-1]} + \dots + yH_{[1]} + H_{[0]}$, where $H_{[m]} \neq 0$ and $H_{[i]} \in \mathcal{S}(\operatorname{Ker}\gamma)$ for every i. Following [PPY] set ${}^\gamma\! H = H_{[m]}$. Note that ${}^\gamma\! H$ does not depend on the choice of y. Note also that $H_{[m]} \in \mathbb{C}$ if and only if $H(\gamma) \neq 0$.

We will denote by Q_γ the stabiliser of γ in Q with respect to the coadjoint action.

Lemma 1.5. *Suppose that $H \in \mathcal{S}(\mathfrak{q})^{\mathfrak{q}}$. Let m and $H_{[m]} = {}^\gamma\! H$ be as above. Then ${}^\gamma\! H \in \mathcal{S}(\mathfrak{q}_\gamma)^{Q_\gamma}$. Furthermore, $\partial_\gamma^m H = m! H_{[m]}$ and for all $k \geqslant 0$, we have $\partial_\gamma^k H = 0$ if and only if $k > m$.*

Proof. We repeat the argument of [PPY, Appendix]. Suppose that $H_{[m]} \notin \mathcal{S}(\mathfrak{q}_\gamma)$. Then there is $\xi \in \mathfrak{q}$ such that $\{\xi, H_{[m]}\} = y^{m+1}\tilde{H} + y^m \tilde{H}_0$ with $\tilde{H} \neq 0$. Since $\deg_y\{\xi, y^d H_{[d]}\} \leqslant d+1$, we see that $\{\xi, H\} \neq 0$, a contradiction. Observe that $\operatorname{Ker}\gamma$ is a Q_γ-stable subspace and that $Q_\gamma y \in y + \operatorname{Ker}\gamma$. Since H is a Q-invariant and hence also a Q_y-invariant, $H_{[m]}$ is a Q_γ-invariant as well. The statements concerning derivatives follow from the facts that $\partial_\gamma y = 1$ and that $\partial_\gamma H_{[d]} = 0$ for each d. □

Recall that $\xi_1, \dots, \xi_r$ is a basis of $\mathfrak{q}$ and let $n = \operatorname{ind}\mathfrak{q}$. Using notation (1·1), for any $k > 0$, set

$$\Lambda^k \pi = \underbrace{\pi \wedge \pi \wedge \ldots \wedge \pi}_{k \text{ factors}}$$

which is regarded as an element of $\mathcal{S}^k(\mathfrak{q}) \otimes \Lambda^{2k}\mathfrak{q}^*$. Then $\Lambda^{(r-n)/2}\pi \neq 0$ and all higher exterior powers of π are zero. We have $dF \in \mathcal{S}(\mathfrak{q}) \otimes \mathfrak{q}$ for each $F \in \mathcal{S}(\mathfrak{q})$. We will also regard dF as a differential 1-form on $\mathfrak{q}^*$. Take $H_1, \dots, H_n \in \mathcal{S}(\mathfrak{q})^{\mathfrak{q}}$. Then $dH_1 \wedge \ldots \wedge dH_n \in \mathcal{S}(\mathfrak{q}) \otimes \Lambda^n \mathfrak{q}$. At the same time, $\Lambda^{(r-n)/2}\pi \in \mathcal{S}(\mathfrak{q})\otimes\Lambda^{r-n}\mathfrak{q}^*$. The volume form $\omega = \xi_1 \wedge \ldots \wedge \xi_r$ defines a non-degenerate pairing between $\Lambda^n\mathfrak{q}$ and $\Lambda^{r-n}\mathfrak{q}$. If $u \in \Lambda^n\mathfrak{q}$ and $v \in \Lambda^{r-n}\mathfrak{q}$, then $u \wedge v = c\omega$ with $c \in \mathbb{C}$. We write this as $\dfrac{u \wedge v}{\omega} = c$ and let $\dfrac{u}{\omega}$ be an element of $(\Lambda^{r-n}\mathfrak{q})^*$ such that $\dfrac{u}{\omega}(v) = \dfrac{u \wedge v}{\omega}$. For any $\mathbf{u} \in \mathcal{S}(\mathfrak{q})\otimes\Lambda^n\mathfrak{q}$, we let $\dfrac{\mathbf{u}}{\omega}$ be the corresponding element of

$$\mathcal{S}(\mathfrak{q})\otimes(\Lambda^{r-n}\mathfrak{q})^* \cong \mathcal{S}(\mathfrak{q})\otimes\Lambda^{r-n}\mathfrak{q}^*.$$

One says that $H_1, \ldots, H_n$ *satisfy the Kostant identity* if

$$\frac{dH_1 \wedge \ldots \wedge dH_n}{\omega} = C\Lambda^{(r-n)/2}\pi \tag{1·2}$$

for some nonzero constant C. Identity (1·2) encodes the following equivalence

$$d_\gamma H_1 \wedge \ldots \wedge d_\gamma H_n \neq 0 \iff \gamma \in \mathfrak{q}^*_{\text{reg}}.$$

Definition 1.6. A Lie algebra $\mathfrak{q}$ is of *Kostant type* if $\mathcal{S}(\mathfrak{q})^Q$ is freely generated by homogenous polynomials $H_1, \ldots, H_n$ that satisfy the Kostant identity.

Any reductive Lie algebra is of Kostant type [K63, Thm 9]. Another easy observation is that

$$\sum_{i=1}^{n} \deg H_i = \frac{r-n}{2} + n = \mathbf{b}(\mathfrak{q})$$

if homogeneous invariants $H_1, \ldots, H_n$ satisfy the Kostant identity. If $\mathfrak{q}$ is of Kostant type, then any set of algebraically independent homogeneous generators of $\mathcal{S}(\mathfrak{q})^Q$ satisfies the Kostant identity.

Definition 1.7 (cf. [PPY, Sec. 2.7])**.** Let $H_1, \ldots, H_n \in \mathcal{S}(\mathfrak{q})^Q$ be algebraically independent homogeneous elements such that $\sum\limits_{i=1}^{n} \deg {}^\gamma\! H_i = \mathbf{b}(\mathfrak{q}_\gamma)$. If these polynomials satisfy the Kostant identity (1·2), then they form a *good system* (*g.s.*) for γ. If they satisfy the Kostant identity and generate $\mathcal{S}(\mathfrak{q})^Q$, then they form a *good generating system* (*g.g.s.*) for $\gamma \in \mathfrak{q}^*$.

Lemma 1.8. *Suppose that homogeneous elements $H_1, \ldots, H_n \in \mathcal{S}(\mathfrak{q})^{\mathfrak{q}}$ satisfy the Kostant identity and $\gamma \in \mathfrak{q}^*$ is such that $\operatorname{ind}\mathfrak{q}_\gamma = \operatorname{ind}\mathfrak{q}$. Then*

(i) $\sum\limits_{i=1}^{n} \deg {}^\gamma\! H_i \leqslant \mathbf{b}(\mathfrak{q}_\gamma)$;

(ii) *$\{H_i\}$ is a g.s. (for γ) if and only if the polynomials ${}^\gamma\! H_i$ are algebraically independent;*

(iii) *if $\{H_i\}$ is a g.s., then the invariants ${}^\gamma\! H_i$ satisfy the Kostant identity related to $\mathfrak{q}_\gamma$;*

(iv) *if $\{H_i\}$ is a g.s. and $\mathfrak{q}_\gamma$ has the codim-2 property, then $\mathcal{S}(\mathfrak{q}_\gamma)^{Q_\gamma} = \mathcal{S}(\mathfrak{q}_\gamma)^{\mathfrak{q}_\gamma} = \mathbb{C}[\{{}^\gamma\! H_i\}]$.*

Proof. We may suppose that the basis elements $\xi_1, \ldots, \xi_r$ of $\mathfrak{q}$ are chosen in such a way that the last $\dim\mathfrak{q}_\gamma$ elements ξ_i form a basis of $\mathfrak{q}_\gamma$. Whenever $\xi \in \mathfrak{q}_\gamma$, we have $\gamma\big([\xi, \eta]\big) = 0$ for each $\eta \in \mathfrak{q}$ and hence $\deg_y[\xi,\eta] \leqslant 0$. The y-degree of $\Lambda^{(r-n)/2}\pi$ is at most $\frac{1}{2}\dim(Q\gamma)$. We note that

$$r - n = r - \operatorname{ind}\mathfrak{q}_\gamma = (r - \dim\mathfrak{q}_\gamma) + (\dim\mathfrak{q}_\gamma - \operatorname{ind}\mathfrak{q}_\gamma).$$

Choosing two complementary subspaces of $\mathfrak{q}/\mathfrak{q}_\gamma$ that are Lagrangian w.r.t. $\hat{\gamma}$, we derive that the highest y-component of $\Lambda^{(r-n)/2}\pi$ is equal (up to a nonzero scalar) to

$$y^{(r-\dim\mathfrak{q}_\gamma)/2}(x_1 \wedge \ldots \wedge x_{r-\dim\mathfrak{q}_\gamma}) \wedge \Lambda^{(\dim\mathfrak{q}_\gamma - n)/2}\pi_\gamma \neq 0,$$

where π_γ is the Poisson tensor of $\mathfrak{q}_\gamma$. The y-degree of the expression on the left hand side of (1·2) is at most $\sum\limits_{i=1}^{n}(\deg H_i - \deg {}^\gamma\! H_i)$. This leads to the inequality

$$\sum_{i=1}^{n}(\deg H_i - \deg {}^\gamma\! H_i) \geqslant \frac{1}{2}\dim(Q\gamma)$$

which is equivalent to

$$\sum_{i=1}^{n}\deg {}^\gamma\! H_i \leqslant \sum_{i=1}^{n}\deg H_i - \frac{1}{2}\dim(Q\gamma) = \frac{1}{2}(r + n - r + \dim\mathfrak{q}_\gamma) = \mathbf{b}(\mathfrak{q}_\gamma).$$

We have the equality here if and only if

$$y^{(r-\dim\mathfrak{q}_\gamma)/2}\,\frac{d\,{}^\gamma\! H_1 \wedge \ldots \wedge d\,{}^\gamma\! H_n}{\omega}$$

is the highest y-component of the left hand side in (1·2). Moreover, this is the case if and only if the polynomials ${}^\gamma\! H_i$ are algebraically independent. Therefore (i) and (ii) follow.

If $\sum\limits_{i=1}^{n}\deg {}^\gamma\! H_i = \mathbf{b}(\mathfrak{q}_\gamma)$, then

$$y^{(r-\dim\mathfrak{q}_\gamma)/2}\,\frac{d\,{}^\gamma\! H_1 \wedge \ldots \wedge d\,{}^\gamma\! H_n}{\omega} = \tilde{C} y^{(r-\dim\mathfrak{q}_\gamma)/2}(x_1 \wedge \ldots \wedge x_{r-\dim\mathfrak{q}_\gamma}) \wedge \Lambda^{(\dim\mathfrak{q}_\gamma - n)/2}\pi_\gamma$$

for some nonzero $\tilde{C} \in \mathbb{C}$. Writing $\omega = (\xi_1 \wedge \ldots \wedge \xi_{r-\dim\mathfrak{q}_\gamma}) \wedge \omega_\gamma$, one proves that

$$\frac{d\,{}^\gamma\! H_1 \wedge \ldots \wedge d\,{}^\gamma\! H_n}{\omega_\gamma} = \tilde{C}\Lambda^{(\dim\mathfrak{q}_\gamma - n)/2}\pi_\gamma.$$

Thus (iii) is proved as well.

The Kostant identity implies that the differentials $d\,{}^\gamma\! H_i$ are linearly independent on $(\mathfrak{q}_\gamma^*)_{\rm reg}$. If $\mathfrak{q}_\gamma$ has the codim-2 property, then $\mathfrak{q}^*_{\rm reg}$ is a big open subset. Thereby the homogeneous invariants ${}^\gamma\! H_i$ generate an algebraically closed subalgebra of $\mathcal{S}(\mathfrak{q}_\gamma)$, see [PPY, Thm 1.1]. Since $\operatorname{tr.deg}\mathcal{S}(\mathfrak{q}_\gamma)^{Q_\gamma} = \operatorname{tr.deg}\mathcal{S}(\mathfrak{q}_\gamma)^{\mathfrak{q}_\gamma} = n$, the assertion (iv) follows. ☐

The statements of Lemma 1.8 generalise analogous assertions proven in [PPY] for $\mathfrak{q} = \mathfrak{g}$ reductive and γ nilpotent. Parts (i), (ii), and (iii) of Lemma 1.8 constitute [PPY, Thm 2.1], which is proved via finite $\mathcal{W}$-algebras. Our current approach is more direct and more general.

Suppose that $\operatorname{ind}\mathfrak{q}_\gamma = \operatorname{ind}\mathfrak{q}$ and $H_1, \ldots, H_n$ is a g.s. for γ. Then no ${}^\gamma H_i$ can be a constant, see Lemma 1.8(ii). Therefore we must have $H_i(\gamma) = 0$ for each i.

1.3. **Sheets and limits in reductive Lie algebras.** Suppose now that $\mathfrak{q} = \mathfrak{g}$ is reductive and $Q = G$. Choose a G-isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$. Recall that the *sheets* in $\mathfrak{g}$ are the irreducible components of the locally closed subsets $X^{(d)} = \{\xi \in \mathfrak{g} \mid \dim(G\xi) = d\}$. At a certain further point in this paper, we will have to pass from nilpotent to arbitrary elements. To this end, sheets in $\mathfrak{g}$ and the method of associated cones developed in [BK79, § 3] will be used. The *associated cone* of $\mu \in \mathfrak{g}^*$ is the intersection $\overline{\mathbb{C}^\times(G\mu)} \cap \mathfrak{N}$, where $\mathfrak{N}$ is the nilpotent cone. Each irreducible component of $\overline{\mathbb{C}^\times(G\mu)} \cap \mathfrak{N}$ is of dimension $\dim(G\mu)$. Let $G\gamma$ be the dense orbit in an irreducible component of the associated cone. The set $\mathbb{C}^\times(G\mu)$ is irreducible, hence is contained in a sheet. The orbit $G\gamma$ is also contained in the same sheet. Therefore this orbit is unique and the associated cone is irreducible.

The following statement should be well-known in algebraic geometry. We still give a short proof.

Lemma 1.9. *There is a morphism of algebraic varieties $\tau : \mathbb{C} \to \overline{\mathbb{C}^\times(G\mu)}$ with the properties $\tau(\mathbb{C} \setminus \{0\}) \subset \mathbb{C}^\times(G\mu)$ and $\tau(0) = \gamma$. Moreover, τ is given by a 1-parameter subgroup of $\mathrm{GL}(\mathfrak{g}^*)$.*

Proof. Let $f \in \mathfrak{g}$ be the image of γ under the G-isomorphism $\mathfrak{g}^* \cong \mathfrak{g}$. We keep the same letter for the image of μ. Then f can be included into an $\mathfrak{sl}_2$-triple $\{e, h, f\} \subset \mathfrak{g}$. The element $\operatorname{ad}(h)$ induces a $\mathbb{Z}$-grading on $\mathfrak{g}$, where the components are

$$\mathfrak{g}_m = \{\xi \in \mathfrak{g} \mid [h, \xi] = m\xi\}.$$

The centraliser G_h of h acts on $\mathfrak{g}_{-2}$ and $G_h f$ is a dense open subset of $\mathfrak{g}_{-2}$. Since $\mathfrak{g}_{\geqslant 0}$ is a parabolic subalgebra of $\mathfrak{g}$, we have $G(\mathfrak{g}_{\geqslant -2}) = \mathfrak{g}$ and $G(G_0 f + \mathfrak{g}_{\geqslant -1})$ is a dense open subset of $\mathfrak{g}$, which meets $\overline{\mathbb{C}^\times(G\mu)}$. Hence

$$G(G_0 f + \mathfrak{g}_{\geqslant -1}) \cap \overline{\mathbb{C}^\times(G\mu)}$$

is a nonempty open subset of $\overline{\mathbb{C}^\times(G\mu)}$ and there is $cg\mu \in G(G_0 f + \mathfrak{g}_{\geqslant -1})$ with $g \in G$, $c \in \mathbb{C}^\times$. We may assume that $\mu \in f + \mathfrak{g}_{\geqslant -1}$.

Let $\{\chi(t) \mid t \in \mathbb{C}^\times\} \subset \mathrm{GL}(\mathfrak{g})$ be the 1-parameter subgroup defined by

$$\chi(t) = t^2 \mathrm{id}_{\mathfrak{g}} \exp(\operatorname{ad}(th)).$$

Then $\lim_{t \to 0} \chi(t)\mu = f$ and τ is given by $\tau(t) = \chi(t)\mu$ for $t \neq 0$ and $\tau(0) = f$. □

The existence of τ and the fact that $\dim(G\gamma) = \dim(G\mu)$ imply that $\lim_{t \to 0} \mathfrak{g}_{\tau(t)} = \mathfrak{g}_\gamma$, where the limit is taken in a suitable Grassmannian.

Let $\mu = x + y$ be the Jordan decomposition of μ in $\mathfrak{g}$, where x is semisimple and y is nilpotent. Then $\mathfrak{l} = \mathfrak{g}_x$ is a Levi subalgebra of $\mathfrak{g}$. Set $L = \exp(\mathfrak{l})$. By [Bor, Sec. 3], the nilpotent orbit $G\gamma$ in a sheet of μ is *induced* from $Ly \subset \mathfrak{l}$. In standard notation, $G\gamma = \mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(Ly)$. For the classical Lie algebras, the description of this induction is given in [K83].

Type A. Let $\mathfrak{g} = \mathfrak{gl}_N$. Suppose that the distinct eigenvalues of $\mu \in \mathfrak{gl}_N$ are $\lambda_1, \dots, \lambda_r$ and that the Jordan canonical form of μ is the direct sum of the respective Jordan blocks

$$J_{\alpha_1^{(1)}}(\lambda_1), \dots, J_{\alpha_{m(1)}^{(1)}}(\lambda_1), \dots, J_{\alpha_1^{(r)}}(\lambda_r), \dots, J_{\alpha_{m(r)}^{(r)}}(\lambda_r), \tag{1·3}$$

of sizes $\alpha_1^{(i)} \geqslant \alpha_2^{(i)} \geqslant \dots \geqslant \alpha_{m(i)}^{(i)} \geqslant 1$ for $i = 1, \dots, r$. We let $\alpha^{(i)}$ denote the corresponding Young diagram whose j-th row is $\alpha_j^{(i)}$. The nilpotent orbit $G\gamma$ corresponds to the partition Π such that the row k of its Young diagram is the sum of the k-th rows of all diagrams $\alpha^{(i)}$. That is,

$$\Pi = \alpha^{(1)} + \dots + \alpha^{(r)} = \Big(\sum_{i=1}^{r} \alpha_1^{(i)}, \sum_{i=1}^{r} \alpha_2^{(i)}, \dots \Big), \tag{1·4}$$

where we assume that $\alpha_j^{(i)} = 0$ for $j > m(i)$; see [K83, §1] and in particular Cor. 2 there.

Type C. Suppose now that $\mathfrak{g} = \mathfrak{sp}_{2n}$. Keep notation (1·3) for the Jordan blocks of $\mu \in \mathfrak{sp}_{2n}$. To separate the zero eigenvalue we will assume that $\lambda_1 = 0$. The case where zero is not an eigenvalue of μ will be taken care of by the zero multiplicity $m(1) = 0$. In type C, the following additional conditions on the parameters of the Jordan canonical form must be satisfied, cf. [K83, §2] and [J04, Sec. 1],

- ⋄ any row of odd length in $\alpha^{(1)}$ must occur an even number of times;
- ⋄ for each $k > 1$ there is k' such that $\lambda_k = -\lambda_{k'}$ and $\alpha^{(k)} = \alpha^{(k')}$.

Define the partition Π by (1·4). It may happen that some rows of odd length in Π occur odd number of times. We will modify Π in order to produce a new partition Π_γ, which corresponds to a nilpotent orbit in $\mathfrak{sp}_{2n}$, by the sequence of steps described in the proof of [K83, Lemma 2.2]. Working from the top of the current Young diagram, consider the first row β of odd length which occurs an odd number of times. Remove one box from the last occurrence of β and add this box to the next row. This operation is possible because the length of the next row is necessarily odd due to the conditions on the Jordan canonical form in type C. Repeating the procedure will yield a diagram Π_γ with the property that any row of odd length occurs an even number of times. The orbit $G\gamma$ is given by the partition Π_γ [K83, Prop. 3.2].

2. MISHCHENKO–FOMENKO SUBALGEBRAS AND THEIR LIMITS

Take $\gamma \in \mathfrak{q}^*$ and let $\overline{\mathcal{A}}_\gamma$ denote the corresponding *Mishchenko–Fomenko subalgebra* of $\mathcal{S}(\mathfrak{q})$ which is generated by all γ*-shifts* $\partial_\gamma^k H$ with $k \geqslant 0$ of all elements $H \in \mathcal{S}(\mathfrak{q})^{\mathfrak{q}}$. Note that $\partial_\gamma^k H$ is a constant for $k = \deg H$. The generators $\partial_\gamma^k H$ of $\overline{\mathcal{A}}_\gamma$ can be equivalently defined by shifting the argument of the invariants H. Set $H_{\gamma,t}(x) = H(x + t\gamma)$. Suppose that $\deg H = m$. Then $H_{\gamma,t}(x)$ expands as a polynomial in t as

$$H(x + t\gamma) = H_{(0)} + H_{(1)}\,t + \cdots + H_{(m)}\,t^m, \tag{2·1}$$

where $H_{(k)} = \frac{1}{k!}\partial_\gamma^k H$. The subalgebra $\overline{\mathcal{A}}_\gamma$ is generated by all elements $H_{(k)}$ associated with all $\mathfrak{q}$-invariants $H \in \mathcal{S}(\mathfrak{q})^{\mathfrak{q}}$. When H is a homogeneous polynomial, we will also use an equivalent form of (2·1), where we make the substitution $x \mapsto \gamma + z^{-1}x$ for a variable z, and expand as a polynomial in z^{-1},

$$H\big(\gamma + z^{-1}x\big) = H_{(0)}\,z^{-m} + \cdots + H_{(m-1)}\,z^{-1} + H_{(m)}.$$

Take now $\lambda \in \mathbb{C}$ and consider $H_{\gamma,\lambda} \in \mathcal{S}(\mathfrak{q})$ with $H_{\gamma,\lambda}(x) = H(x + \lambda\gamma)$. A standard argument with the Vandermonde determinant shows that $\overline{\mathcal{A}}_\gamma$ is generated by $H_{\gamma,\lambda}$ with $H \in \mathcal{S}(\mathfrak{q})^{\mathfrak{q}}$ and $\lambda \in \mathbb{C}$. One readily sees that

$$\overline{\mathcal{A}}_\gamma \subset \mathcal{S}(\mathfrak{q})^{Q_\gamma}.$$

Clearly it suffices to take only homogeneous H and only nonzero λ in order to generate $\overline{\mathcal{A}}_\gamma$. Under these assumptions on H and λ, we have

$$d_\mu H_{\gamma,\lambda} = d_{\mu+\lambda\gamma} H = \lambda^{m-1} d_{\lambda^{-1}\mu+\gamma} H = \lambda^{m-1} d_\gamma H_{\mu,\lambda^{-1}}, \tag{2·2}$$

where $m = \deg H$. Hence

$$d_\mu \overline{\mathcal{A}}_\gamma = d_\gamma \overline{\mathcal{A}}_\mu \tag{2·3}$$

for any $\gamma, \mu \in \mathfrak{q}^*$, where both sides are regarded as subspaces of $\mathfrak{q}$.

2.1. **Maximal Poisson-commutative subalgebras.** Our next goal is to formulate certain conditions that assure that $\overline{\mathcal{A}}_\gamma$ is a maximal Poisson-commutative subalgebra of $\mathcal{S}(\mathfrak{q})^{\mathfrak{q}_\gamma}$. From now on assume that $\mathfrak{q}$ is of Kostant type with $\operatorname{ind}\mathfrak{q} = n$ and that $H_1, \ldots, H_n$ are homogeneous generators of $\mathcal{S}(\mathfrak{q})^{\mathfrak{q}}$. Then the Mishchenko–Fomenko subalgebra $\overline{\mathcal{A}}_\gamma$ is generated by the γ-shifts $\partial_\gamma^k H_i$ with $1 \leqslant i \leqslant n$ and $0 \leqslant k \leqslant \deg H_i - 1$.

Lemma 2.1. *Suppose that $\mathfrak{q}$ is of Kostant type and has the codim-2 property. Suppose further that $\operatorname{ind}\mathfrak{q}_\gamma = n$. Then*

(i) $\operatorname{tr.deg}\overline{\mathcal{A}}_\gamma = \frac{1}{2}\dim(Q\gamma) + n$;
(ii) $\dim d_x\overline{\mathcal{A}}_\gamma = \frac{1}{2}\dim(Q\gamma) + n$ *if* $x + \mathbb{C}\gamma \subset \mathfrak{q}^*_{\rm reg}$ *and* $x|_{\mathfrak{q}_\gamma} \in (\mathfrak{q}_\gamma^*)_{\rm reg}$.

Proof. By (2·3), $d_x\overline{\mathcal{A}}_\gamma = d_\gamma\overline{\mathcal{A}}_x$. According to [B91, Thm 3.2 and its proof], see also [B91, Thm 2.1], if $x+\mathbb{C}\gamma \subset \mathfrak{q}^*_{\rm reg}$ and $x|_{\mathfrak{q}_\gamma} \in (\mathfrak{q}_\gamma^*)_{\rm reg}$, then $d_\gamma\overline{\mathcal{A}}_x$ contains a subspace U of dimension $\frac{1}{2}\dim(Q\gamma)$ such that $U \cap \mathfrak{q}_\gamma = 0$. A part of [B91, Thm 3.2] asserts that such an element x exists. For each $H \in \mathcal{S}(\mathfrak{q})^{\mathfrak{q}}$ and each $\mu \in \mathfrak{q}^*$, we have $d_\mu H \in \operatorname{Ker}\hat{\mu} = \mathfrak{q}_\mu$. Since $\mathfrak{q}$ is of Kostant type, the differentials $d_{ux+\gamma}H_i$ are linearly independent for each $u \in \mathbb{C}^\times$ and therefore their linear span is equal to $\mathfrak{q}_{ux+\gamma}$. In view of (2·2), $\mathfrak{q}_{ux+\gamma} \subset d_\gamma\overline{\mathcal{A}}_x$ and hence

$U_\gamma = \lim\limits_{u\to 0} \mathfrak{q}_{ux+\gamma}$ is a subspace of $d_\gamma\overline{\mathcal{A}}_x$ as well. Clearly $\dim U_\gamma = n$ and $U_\gamma \subset \mathfrak{q}_\gamma$. It follows that

$$\dim d_\gamma\overline{\mathcal{A}}_x \geqslant \dim U + \dim U_\gamma = \frac{1}{2}\dim(Q\gamma) + n.$$

Since $d_x\overline{\mathcal{A}}_\gamma \leqslant \frac{1}{2}\dim(Q\gamma)+n$ by Proposition 1.1, part (ii) follows. To prove (i) recall that $\mathrm{tr.deg}\,\overline{\mathcal{A}}_\gamma = \max_{x\in\mathfrak{q}^*}\dim d_x\overline{\mathcal{A}}_\gamma = \frac{1}{2}\dim(Q\gamma) + \mathrm{ind}\,\mathfrak{q}$. □

Proposition 2.2. *Suppose that $\mathfrak{q}$ satisfies the codim-2 condition and that $H_1,\dots,H_n$ is a g.g.s. for $\gamma\in\mathfrak{q}^*$. If in addition $\mathrm{ind}\,\mathfrak{q}_\gamma = n$, then $\overline{\mathcal{A}}_\gamma$ is freely generated by $\partial_\gamma^k H_i$ with $1\leqslant i\leqslant n$ and $0\leqslant k\leqslant \deg H_i - \deg{}^\gamma H_i$.*

Proof. Since $\sum\limits_{i=1}^n \deg{}^\gamma H_i = \mathbf{b}(\mathfrak{q}_\gamma)$, there are only

$$\sum_{i=1}^n(\deg H_i - \deg{}^\gamma H_i + 1) = \frac{\dim\mathfrak{q}+n}{2} - \frac{\dim\mathfrak{q}_\gamma+n}{2} + n = \frac{1}{2}\dim(Q\gamma)+n$$

nonzero elements $\partial_\gamma^k H_i$ and they have to be algebraically independent by Lemma 2.1(i). □

Theorem 2.3. *Suppose that $\mathfrak{q}$ satisfies the codim-3 condition and that $H_1,\dots,H_n$ is a g.g.s. for γ. Suppose further that $\mathfrak{q}_\gamma$ satisfies the codim-2 condition and $\mathrm{ind}\,\mathfrak{q}_\gamma = n$. Let $F_1,\dots,F_s$ with $s=\frac{1}{2}\dim(Q\gamma)+n$ be algebraically independent homogeneous generators of $\overline{\mathcal{A}}_\gamma$ such that each F_j is a γ-shift $\partial_\gamma^k H_i$ as in Proposition 2.2. Then*

(i) *the differentials of F_j are linearly independent on a big open subset;*
(ii) *$\overline{\mathcal{A}}_\gamma$ is a maximal (w.r.t. inclusion) Poisson-commutative subalgebra of $\mathcal{S}(\mathfrak{q})^{\mathfrak{q}_\gamma}$.*

Proof. (i) We have $d_x\overline{\mathcal{A}}_\gamma = \langle d_xF_j \mid 1\leqslant j\leqslant s\rangle_{\mathbb{C}}$. Whenever $\dim d_x\overline{\mathcal{A}}_\gamma = s$, the differentials d_xF_j are linearly independent. According to Lemma 2.1, the equality $\dim d_x\overline{\mathcal{A}}_\gamma = s$ holds if $x+\mathbb{C}\gamma\subset\mathfrak{q}^*_{\rm reg}$ and $x|_{\mathfrak{q}_\gamma}\in(\mathfrak{q}^*_\gamma)_{\rm reg}$. Choosing any complement of $\mathfrak{q}_\gamma$ in $\mathfrak{q}$, we can embed $\mathfrak{q}^*_\gamma$ into $\mathfrak{q}^*$. If $x|_{\mathfrak{q}_\gamma}$ is non-regular, then x belongs to $(\mathfrak{q}^*_\gamma)_{\rm sing} + \mathrm{Ann}(\mathfrak{q}_\gamma)$. This is a closed subset of codimension at least 2, since $\mathfrak{q}_\gamma$ satisfies the codim-2 condition.

Let us examine the property $x+\mathbb{C}\gamma\subset\mathfrak{q}^*_{\rm reg}$. The desired condition on x holds if $x\in\mathfrak{q}^*_{\rm reg}$ and $\gamma+cx\in\mathfrak{q}^*_{\rm reg}$ for all $c\in\mathbb{C}^\times$. The first restriction is inessential. In order to deal with the second one, $\gamma+\mathbb{C}^\times x\subset\mathfrak{q}^*_{\rm reg}$, we choose γ as the origin in $\mathfrak{q}^*$ and consider the corresponding map

$$\psi\colon \mathfrak{q}^*\setminus\{\gamma\}\to\mathbb{P}\mathfrak{q}^*$$

with $\psi(x)=\mathbb{C}(x-\gamma)$. We have

$$\dim\overline{\psi(\mathfrak{q}^*_{\rm sing})}\leqslant\dim\mathfrak{q}^*_{\rm sing}\leqslant\dim\mathfrak{q}-3.$$

Hence the preimage $\psi^{-1}(\overline{\psi(\mathfrak{q}^*_{\rm sing})})$ is a closed subset of $\mathfrak{q}^* \setminus \{\gamma\}$ of codimension at least 2. Assume that $x \neq 0$. Note that

$$\gamma + \mathbb{C}^{\times} x = \psi^{-1}\big(\psi(x+\gamma)\big).$$

If $\gamma + \mathbb{C}^{\times} x \cap \mathfrak{q}^*_{\rm sing} \neq \varnothing$, then $\psi(x+\gamma) \in \psi(\mathfrak{q}^*_{\rm sing})$ and $x+\gamma \in \psi^{-1}\big(\psi(\mathfrak{q}^*_{\rm sing})\big)$.

Since

$$\psi^{-1}\big(\overline{\psi(\mathfrak{q}^*_{\rm sing})}\big) \cup \{\gamma\}$$

is a closed subset of $\mathfrak{q}^*$ of dimension at most $\dim\mathfrak{q}-2$, part (i) follows.

(ii) We have $\mathrm{tr.deg}\,\overline{\mathcal{A}}_\gamma = \frac{1}{2}\dim(Q\gamma) + n$. Assume on the contrary that $\overline{\mathcal{A}}_\gamma$ is not maximal. Then $\overline{\mathcal{A}}_\gamma \subsetneq A \subset \mathcal{S}(\mathfrak{q})^{\mathfrak{q}_\gamma}$, where A is a Poisson-commutative subalgebra. In view of Proposition 1.1, $\mathrm{tr.deg}\,A \leqslant \mathrm{tr.deg}\,\overline{\mathcal{A}}_\gamma$ and hence A is a non-trivial algebraic extension of $\overline{\mathcal{A}}_\gamma$. Since (i) holds, $\overline{\mathcal{A}}_\gamma$ is an algebraically closed subalgebra of $\mathcal{S}(\mathfrak{q})$ by [PPY, Thm. 1.1]. This contradiction completes the proof. □

Now suppose that $\mathfrak{g} = \mathrm{Lie}\,G$ is reductive. Then $\mathfrak{g}$ has the codim-3 property. It will be convenient to consider elements of $\mathfrak{g}$ as linear functions on $\mathfrak{g}$. We have $\mathrm{ind}\,\mathfrak{g}_\gamma = \mathrm{rk}\,\mathfrak{g} = n$ for each $\gamma \in \mathfrak{g}^*$ [Y06, dG08, CM10]. A nilpotent element γ may or may not posses a good generating system [PPY]. But if an element is not nilpotent, then there is no g.g.s. for it. Our next step will be to develop a transition from nilpotent to arbitrary elements of $\mathfrak{g}$.

We point out a few properties of directional derivatives to be used below. We have

$$\partial_{g\mu}(gF) = g(\partial_\mu F)$$

for all $g \in G$, $\mu \in \mathfrak{g}^*$, and $F \in \mathcal{S}(\mathfrak{g})$. Hence, $\partial^k_{g\mu} H = g(\partial^k_\mu H)$ for $H \in \mathcal{S}(\mathfrak{g})^G$. Moreover, $\partial_{t\mu} F = t\partial_\mu F$ for each $t \in \mathbb{C}^{\times}$.

Consider an arbitrary element $\mu \in \mathfrak{g}^*$ and the associated nilpotent orbit $G\gamma \subset \mathfrak{g}^*$ as defined in the first paragraph of Section 1.3.

Proposition 2.4. *Suppose that γ has a g.g.s. and that $\mathfrak{g}_\gamma$ satisfies the codim-2 condition. Then $\overline{\mathcal{A}}_\mu$ is a maximal Poisson-commutative subalgebra of $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}_\mu}$.*

Proof. Let $F_1, \ldots, F_s$ with $s = \frac{1}{2}\dim(G\gamma) + \mathrm{rk}\,\mathfrak{g} = \frac{1}{2}\dim(G\mu) + \mathrm{rk}\,\mathfrak{g}$ be algebraically independent homogeneous generators of $\overline{\mathcal{A}}_\gamma$. As in Theorem 2.3, we have $F_j = \partial^k_\gamma H_i$, where $0 \leqslant k \leqslant \deg H_i - \deg{}^\gamma H_i$. Set accordingly $\hat{F}_j = \partial^k_\mu H_i$. By Theorem 2.3(i), the differentials dF_j are linearly independent on a big open subset. Assume on the contrary that this is not the case for the differentials of $\hat{F}_j$. Then

$$\bigwedge_{j=1}^{s} d\hat{F}_j = \mathbf{F}R,$$

where $\mathbf{F}$ is a non-constant homogenous polynomial and $R \in \mathcal{S}(\mathfrak{g}) \otimes \Lambda^s\mathfrak{q}$ is a regular differential s-form that is nonzero on a big open subset of $\mathfrak{q}^*$.

Let τ be the map of Lemma 1.9, which is constructed as an orbit of

$$\{\chi(t) = t^2 \mathrm{id}_{\mathfrak{g}^*} \exp(\mathrm{ad}^*(th)) \mid t \in \mathbb{C}^\times\} \subset \mathrm{GL}(\mathfrak{g}^*).$$

Then $\lim\limits_{t\to 0} \partial^k_{\tau(t)} H_i = \partial^k_\gamma H_i$ for all i and all k. The appearing partial derivatives can be expressed via the group action:

$$\partial^k_{\tau(t)} H_i = \partial^k_{\chi(t)\mu} H_i = t^{2k} \exp(\mathrm{ad}^*(th))(\partial^k_\mu H_i).$$

Letting G act on the differential forms as well, we obtain that

$$\bigwedge_{j=1}^{s} dF_j = \lim_{t\to 0} t^K (\exp(\mathrm{ad}^*(th))\mathbf{F})(\exp(\mathrm{ad}^*(th))R),$$

where $K \in \mathbb{Z}_{\geqslant 0}$. Since $\mathbf{F} \notin \mathbb{C}$ is a homogeneous polynomial, the lowest t-component of $\exp(\mathrm{ad}^*(th))\mathbf{F}$ is a non-constant homogeneous polynomial as well. This component divides $dF_1 \wedge \ldots \wedge dF_s$, a contradiction. Thus, the differentials $d\hat{F}_j$ are linearly independent on a big open subset.

By [PPY, Thm 1.1], the homogeneous elements $\hat{F}_1, \ldots, \hat{F}_s$ generate an algebraically closed subalgebra of $\mathcal{S}(\mathfrak{g})$, which is contained in $\overline{\mathcal{A}}_\mu$ and is of transcendence degree s. Since $\mathrm{tr.deg}\,\overline{\mathcal{A}}_\mu = s$, these elements actually generate $\overline{\mathcal{A}}_\mu$. By Proposition 1.1, a Poisson-commutative extension of $\overline{\mathcal{A}}_\mu$ in $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}_\mu}$ must be algebraic and is therefore trivial. □

If $\mathfrak{q} = \mathfrak{g}$ is of type A or C and γ is nilpotent, then $\mathfrak{g}_\gamma$ has the codim-2 property and there is a g.g.s. for γ, see [PPY]. Therefore, the assumptions of Theorem 2.3 are satisfied and we have the following.

Corollary 2.5. *Suppose that $\mathfrak{g}$ is of type* A *or* C*. Then $\overline{\mathcal{A}}_\mu$ is a maximal Poisson-commutative subalgebra of $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}_\mu}$ for each $\mu \in \mathfrak{g}^*$.* □

If $\gamma \in \mathfrak{q}^*_{\mathrm{sing}}$, then $\mathrm{tr.deg}\,\overline{\mathcal{A}}_\gamma < \boldsymbol{b}(\mathfrak{q})$ [B91, Thm 2.1], i.e., this algebra does not have the maximal possible transcendence degree. On the one hand, this property can be a disadvantage for some applications, while on the other, $\overline{\mathcal{A}}_\gamma$ Poisson-commutes with $\mathfrak{q}_\gamma$ and can be extended, in many ways, to a Poisson-commutative subalgebra of the maximal possible transcendence degree. Below we present a construction of such an extension.

2.2. **Vinberg's limits in the nilpotent case.** Take $\gamma \in \mathfrak{q}^*_{\mathrm{sing}}$, $\mu \in \mathfrak{q}^*_{\mathrm{reg}}$, and $u \in \mathbb{C}$ and consider the Mishchenko–Fomenko subalgebra $\overline{\mathcal{A}}_{\gamma+u\mu}$ of $\mathcal{S}(\mathfrak{q})$. For $F \in \mathcal{S}^N(\mathfrak{q})$, we have $dF \in \mathcal{S}^{N-1}(\mathfrak{q}) \otimes \mathfrak{q}$ and $\partial_x F = dF(\,.\,, x)$ for any $x \in \mathfrak{q}^*$. Hence $\partial_{\gamma+u\mu} F = \partial_\gamma F + u\partial_\mu F$. More generally, $\partial^k_{\gamma+u\mu} F \in \mathcal{S}(\mathfrak{q})[u]$. We have

$$\partial^k_{\gamma+u\mu} = \partial^k_\gamma + u\partial_\mu\partial^{k-1}_\gamma + \cdots + \binom{k}{s} u^s \partial^s_\mu \partial^{k-s}_\gamma + \cdots + u^k \partial^k_\mu. \tag{2·4}$$

Obviously $\lim_{u\to 0} \partial^k_{\gamma+u\mu} F = \partial^k_\gamma F$. In the case $\partial^k_\gamma F = 0$, the limit $\lim_{u\to 0} \mathbb{C}\partial^k_{\gamma+u\mu} F$ still makes sense as an element of the projective space $\mathbb{P}\mathcal{S}(\mathfrak{q})$. This limit line is spanned by the lowest u-component of $\partial^k_{\gamma+u\mu} F$. In the same projective sense set

$$\overline{\mathcal{C}}_{\gamma,\mu} = \lim_{u\to 0} \overline{\mathcal{A}}_{\gamma+u\mu}.$$

Formally speaking, $\overline{\mathcal{C}}_{\gamma,\mu}$ is a subspace of $\mathcal{S}(\mathfrak{q})$ such that

$$\overline{\mathcal{C}}_{\gamma,\mu} \cap \mathcal{S}^m(\mathfrak{q}) = \lim_{u\to 0} \left(\overline{\mathcal{A}}_{\gamma+u\mu} \cap \mathcal{S}^m(\mathfrak{q})\right)$$

for each $m \geqslant 0$. In other words, this subspace is the linear span

$$(2{\cdot}5) \qquad \overline{\mathcal{C}}_{\gamma,\mu} = \left\langle \text{lowest } u\text{-component of } F \mid F \in \overline{\mathcal{A}}_{\gamma+u\mu} \right\rangle_{\mathbb{C}}.$$

We call $\overline{\mathcal{C}}_{\gamma,\mu}$ *Vinberg's limit* at γ along μ, see [V14]. Note that $\overline{\mathcal{C}}_{\gamma,\mu}$ is a subalgebra of $\mathcal{S}(\mathfrak{q})$ and that it does depend on μ.

Clearly $\overline{\mathcal{A}}_\gamma \subset \overline{\mathcal{C}}_{\gamma,\mu}$. By [BK76, Satz 4.5], we have $\mathrm{tr.deg}\, \overline{\mathcal{C}}_{\gamma,\mu} = \mathrm{tr.deg}\, \overline{\mathcal{A}}_{\gamma+u\mu}$ with a generic u. Therefore $\mathrm{tr.deg}\, \overline{\mathcal{C}}_{\gamma,\mu} = \boldsymbol{b}(\mathfrak{q})$ assuming that $\mathfrak{q}$ satisfies the codim-2 condition and has enough symmetric invariants [B91]. Set $\bar\mu = \mu|_{\mathfrak{q}_\gamma}$.

Theorem 2.6. *Suppose that $\mathfrak{q}$ satisfies the codim-2 condition, $\mathrm{ind}\,\mathfrak{q}_\gamma = n$, there is a g.g.s. for γ, $\mathfrak{q}_\gamma$ has the codim-2 property, and $\bar\mu \in (\mathfrak{q}^*_\gamma)_{\rm reg}$. Then $\overline{\mathcal{C}}_{\gamma,\mu}$ is a free algebra generated by $\overline{\mathcal{A}}_\gamma$ and the Mishchenko–Fomenko subalgebra $\overline{\mathcal{A}}_{\bar\mu} \subset \mathcal{S}(\mathfrak{q}_\gamma)$.*

Proof. Let $H_1, \ldots, H_n$ be a g.g.s. for γ. Set $F_{i,k} = \partial^k_\gamma H_i$. For $k = \deg H_i - \deg {}^\gamma H_i$, we have $\partial^k_\gamma H_i = k!\,{}^\gamma H_i \in \mathcal{S}(\mathfrak{q}_\gamma)$ and $\partial^P_\gamma H_i = 0$ if $P > k$, see Lemma 1.5. Together with (2·4) this gives

$$\lim_{u\to 0} \mathbb{C}\partial^k_{\gamma+u\mu} H_i = \begin{cases} \mathbb{C}F_{i,k} & \text{if} \quad k \leqslant \deg H_i - \deg {}^\gamma H_i, \\ \mathbb{C}\partial^{\bar k}_{\bar\mu}({}^\gamma H_i) & \text{if} \quad k > \deg H_i - \deg {}^\gamma H_i, \end{cases}$$

where $\bar k = k - (\deg H_i - \deg {}^\gamma H_i)$. According to Proposition 2.2, the elements $F_{i,k}$ with the conditions $k \leqslant \deg H_i - \deg {}^\gamma H_i$ freely generate $\overline{\mathcal{A}}_\gamma$.

Since $H_1, \ldots, H_n$ is a g.g.s. for γ, the elements ${}^\gamma H_1, \ldots, {}^\gamma H_n$ are algebraically independent, they satisfy the Kostant identity (1·2) and freely generate $\mathcal{S}(\mathfrak{q}_\gamma)^{\mathfrak{q}_\gamma}$, see Lemma 1.8. In particular, $\mathfrak{q}_\gamma$ is of Kostant type. Applying Proposition 2.2 to $\mathcal{S}(\mathfrak{q}_\gamma)$, we see that $\overline{\mathcal{A}}_{\bar\mu}$ is freely generated by the elements $\partial^{\bar k}_{\bar\mu}({}^\gamma H_i)$ with $0 \leqslant \bar k < \deg {}^\gamma H_i$.

It remains to show that there are no algebraic relations among $F_{i,k}$ with $k \leqslant \deg H_i - \deg {}^\gamma H_i$ and $\partial^{\bar k}_{\bar\mu}({}^\gamma H_i)$ with $1 \leqslant \bar k < \deg {}^\gamma H_i$. Once this is done, we will know that the lowest u-components of the generators $\partial^k_{\gamma+u\mu} H_i$ are algebraically independent and therefore the lowest u-component of a polynomial in $\partial^k_{\gamma+u\mu} H_i$ is the same polynomial in their lowest u-components.

Take $x \in \mathfrak{q}^*$. Then $U_1 = \langle d_x F_{i,k} \rangle_{\mathbb{C}}$ is a subspace of $\mathfrak{q}$ and $\hat x(U_1, \mathfrak{q}_\gamma) = 0$. At the same time

$$U_2 = \left\langle d_x \partial^{\bar k}_{\bar\mu}({}^\gamma H_i) \mid 0 \leqslant \bar k < \deg {}^\gamma H_i \right\rangle_{\mathbb{C}}$$

is a subspace of $\mathfrak{q}_\gamma$. Therefore $\hat{x}(U_1 \cap U_2, \mathfrak{q}_\gamma) = 0$ and $U_1 \cap U_2 \subset (\mathfrak{q}_\gamma)_{\bar{x}}$ for $\bar{x} = x|_{\mathfrak{q}_\gamma}$. Suppose that x is a generic point, $x \in \mathfrak{q}^*_{\rm reg}$ and $\bar{x} \in (\mathfrak{q}^*_\gamma)_{\rm reg}$. Since $\operatorname{ind} \mathfrak{q}_\gamma = n$, we have $\dim(\mathfrak{q}_\gamma)_{\bar{x}} = n$. Hence $\dim(U_1 \cap U_2) = n$. It follows that $U_1 + U_2 = U_1 \oplus \tilde{U}_2$, where

$$\tilde{U}_2 = \left\langle d_x \partial_{\bar{\mu}}^{\bar{k}}({}^\gamma H_i) \mid 1 \leqslant \bar{k} < \deg {}^\gamma H_i \right\rangle_{\mathbb{C}}.$$

This completes the proof. □

The assumptions of Theorem 2.6 are satisfied in types A and C for all nilpotent elements γ and for generic elements μ.

Remark 2.7. Suppose that $\mathfrak{g}$ is of type A. Then $\mathfrak{g}_\gamma$ has the codim-3 property [Y09] and thereby $\overline{\mathcal{A}}_\nu$ is a maximal Poisson-commutative subalgebra of $\mathcal{S}(\mathfrak{q}_\gamma)$ for each $\nu \in (\mathfrak{q}^*_\gamma)_{\rm reg}$, see [PY08] and also [AP17]. Hence $\overline{\mathcal{C}}_{\gamma,\mu}$ is a maximal Poisson-commutative subalgbera of $\mathcal{S}(\mathfrak{g})$ for any nilpotent γ and any generic μ.

3. Quantisation and symmetrisation

As we recalled in the Introduction, Vinberg's quantisation problem [V91] concerns the existence and construction of a commutative subalgebra $\mathcal{A}_\mu$ of $\mathcal{U}(\mathfrak{g})$ with the property $\operatorname{gr} \mathcal{A}_\mu = \overline{\mathcal{A}}_\mu$. In the case where $\mu \in \mathfrak{g}^*$ is regular semisimple, explicit constructions of the subalgebras $\mathcal{A}_\mu$ in the classical types were given by Nazarov and Olshanski [NO96] with the use of the Yangian for $\mathfrak{gl}_N$ and the twisted Yangians associated with the orthogonal and symplectic Lie algebras. Positive solutions in the general case were given by Rybnikov [R06] for regular semisimple μ and Feigin, Frenkel and Toledano Laredo [FFTL] for any regular μ. The solutions rely on the properties of a commutative subalgebra $\mathfrak{z}(\widehat{\mathfrak{g}})$ of the universal enveloping algebra $\mathcal{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)$. This subalgebra is known as the *Feigin–Frenkel centre* and is defined as the centre of the universal affine vertex algebra associated with the affine Kac–Moody algebra $\widehat{\mathfrak{g}}$ at the critical level. In particular, each element of $\mathfrak{z}(\widehat{\mathfrak{g}})$ is annihilated by the adjoint action of $\mathfrak{g}$. Furthermore, the subalgebra $\mathfrak{z}(\widehat{\mathfrak{g}})$ is invariant with respect to the derivation $T = -\partial_t$ of the algebra $\mathcal{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)$. By a theorem of Feigin and Frenkel [FF92] (see also [F07]), there exists a family of elements $S_1, \dots, S_n \in \mathfrak{z}(\widehat{\mathfrak{g}})$ (a *complete set of Segal–Sugawara vectors*), where $n = \operatorname{rk} \mathfrak{g}$, such that

$$\mathfrak{z}(\widehat{\mathfrak{g}}) = \mathbb{C}[T^r S_l \mid l = 1, \dots, n, \ \ r \geqslant 0].$$

One can assume that each S_l is homogeneous with respect to the gradation on $\mathcal{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)$ defined by $\deg X[r] = -r$, where we set $X[r] = X t^r$.

For any $\mu \in \mathfrak{g}^*$ and a nonzero $z \in \mathbb{C}$, the mapping

$$(3{\cdot}1) \qquad \varrho_{\mu,z} \colon \mathcal{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big) \to \mathcal{U}(\mathfrak{g}), \qquad X[r] \mapsto X z^r + \delta_{r,-1}\, \mu(X), \quad X \in \mathfrak{g},$$

defines a G_μ-equivariant algebra homomorphism. The image of $\mathfrak{z}(\widehat{\mathfrak{g}})$ under $\varrho_{\mu,z}$ is a commutative subalgebra $\mathcal{A}_\mu$ of $\mathcal{U}(\mathfrak{g})$ which does not depend on z. If $S \in \mathcal{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)$ is a homogeneous element of degree d, then regarding $\varrho_{\mu,z}(S)$ as a polynomial in z^{-1}, define the elements $S^{(i)} \in \mathcal{U}(\mathfrak{g})$ by the expansion

$$\varrho_{\mu,z}(S) = S^{(0)} z^{-d} + \dots + S^{(d-1)} z^{-1} + S^{(d)}.$$

Suppose that $\mu \in \mathfrak{g}^*$ is regular and that $S_1, \dots, S_n \in \mathfrak{z}(\widehat{\mathfrak{g}})$ is a complete set of homogeneous Segal–Sugawara vectors of the respective degrees $d_1, \dots, d_n$. Then

- ⋄ the elements $S_k^{(i)}$ with $k = 1, \dots, n$ and $i = 0, 1, \dots, d_k - 1$ are algebraically independent generators of $\mathcal{A}_\mu$ and $\operatorname{gr} \mathcal{A}_\mu = \overline{\mathcal{A}}_\mu$;
- ⋄ the subalgebra $\mathcal{A}_\mu$ of $\mathcal{U}(\mathfrak{g})$ is maximal commutative.

The first of these statements is due to [FFTL] and the second is implied by the results of [PY08]. The elements $S_k^{(i)}$ generate $\mathcal{A}_\mu$ for any $\mu \in \mathfrak{g}^*$ and the inclusion $\operatorname{gr} \mathcal{A}_\mu \supset \overline{\mathcal{A}}_\mu$ holds. It was conjectured in [FFTL, Conjecture 1], that the property $\operatorname{gr} \mathcal{A}_\mu = \overline{\mathcal{A}}_\mu$ extends to all μ. Its proof in type A was given in [FM15]. In what follows we give a more general argument which will imply the conjecture in types A and C thus providing another proof in type A. We will rely on the properties of the canonical symmetrisation map

$$\varpi : \mathcal{S}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g}). \tag{3·2}$$

This map was already used by Tarasov [T00] in type A to show that if $\mu \in \mathfrak{gl}_N^*$ is semisimple, then the images of the μ-shifts of certain generators of $\mathcal{S}(\mathfrak{gl}_N)^{\mathfrak{gl}_N}$ under the map (3·2) generate a commutative subalgebra of $\mathcal{U}(\mathfrak{gl}_N)$. By another result of Tarasov [T03] this subalgebra coincides with $\mathcal{A}_\mu$ if μ is regular semisimple. Below we prove a similar statement for all classical types B, C and D and all $\mu \in \mathfrak{g}^*$, which allows us to suggest that it holds for all simple Lie algebras.

Conjecture 3.1. *For any simple Lie algebra $\mathfrak{g}$ there exist generators $H_1, \dots, H_n$ of the algebra $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ such that for any element $\mu \in \mathfrak{g}^*$ the images $\varpi(\partial_\mu^k H_i)$ of their μ-shifts with respect to the symmetrisation map ϖ generate the algebra $\mathcal{A}_\mu$.*

In proving this conjecture for the classical types we will take advantage of the explicit constructions of generators of $\mathcal{A}_\mu$ obtained from the complete sets of Segal–Sugawara vectors found in [CT06], [CM09] and [M13]; see also [M]. It turns out that such generators can be obtained from the μ-shifts of certain elements of $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ via the symmetrisation map (3·2). This will show how to choose the generators of $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ to satisfy the condition of the conjecture.

In what follows we will identify any element $X \in \mathfrak{g}$ with its images under the canonical embeddings $\mathfrak{g} \hookrightarrow \mathcal{S}(\mathfrak{g})$ and $\mathfrak{g} \hookrightarrow \mathcal{U}(\mathfrak{g})$. It should always be clear from the context whether X is regarded as an element of $\mathcal{S}(\mathfrak{g})$ or $\mathcal{U}(\mathfrak{g})$.

3.1. **Symmetrised determinants and permanents.** Consider the symmetriser $h^{(m)}$ and anti-symmetriser $a^{(m)}$ in the group algebra $\mathbb{C}[\mathfrak{S}_m]$ of the symmetric group $\mathfrak{S}_m$. These are the idempotents defined by

$$h^{(m)} = \frac{1}{m!} \sum_{\sigma \in \mathfrak{S}_m} \sigma \qquad \text{and} \qquad a^{(m)} = \frac{1}{m!} \sum_{\sigma \in \mathfrak{S}_m} \operatorname{sgn} \sigma \cdot \sigma.$$

We let $H^{(m)}$ and $A^{(m)}$ denote their respective images under the natural action of $\mathfrak{S}_m$ on the tensor product space $(\mathbb{C}^N)^{\otimes m}$. We will denote by P_σ the image of $\sigma \in \mathfrak{S}_m$ with respect to this action.

For an arbitrary associative algebra $\mathcal{M}$ we will identify $H^{(m)}$ and $A^{(m)}$ with the respective elements $H^{(m)} \otimes 1$ and $A^{(m)} \otimes 1$ of the algebra

$$\underbrace{\operatorname{End} \mathbb{C}^N \otimes \ldots \otimes \operatorname{End} \mathbb{C}^N}_{m} \otimes \mathcal{M}. \tag{3·3}$$

Any $N \times N$ matrix $M = [M_{ij}]$ with entries in $\mathcal{M}$ will be represented as the element

$$M = \sum_{i,j=1}^{N} e_{ij} \otimes M_{ij} \in \operatorname{End} \mathbb{C}^N \otimes \mathcal{M}. \tag{3·4}$$

For each $a \in \{1, \ldots, m\}$ introduce the element M_a of the algebra (3·3) by

$$M_a = \sum_{i,j=1}^{N} 1^{\otimes(a-1)} \otimes e_{ij} \otimes 1^{\otimes(m-a)} \otimes M_{ij}. \tag{3·5}$$

The partial trace tr_a will be understood as the linear map

$$\operatorname{tr}_a \colon \operatorname{End} (\mathbb{C}^N)^{\otimes m} \to \operatorname{End} (\mathbb{C}^N)^{\otimes(m-1)}$$

which acts as the trace map on the a-th copy of $\operatorname{End} \mathbb{C}^N$ and is the identity map on all the remaining copies.

For any $m = 1, \ldots, N$ define the *m-th symmetrised minor* of the matrix M by

$$\operatorname{Det}_m(M) = \operatorname{tr}_{1,\ldots,m} A^{(m)} M_1 \ldots M_m,$$

or explicitly,

$$\operatorname{Det}_m(M) = \frac{1}{m!} \sum_{1 \leqslant a_1 < \cdots < a_m \leqslant N} \; \sum_{\sigma, \tau \in \mathfrak{S}_m} \operatorname{sgn} \sigma\tau \cdot M_{a_{\sigma(1)} a_{\tau(1)}} \ldots M_{a_{\sigma(m)} a_{\tau(m)}}.$$

The *symmetrised determinant* of M is $\operatorname{Det}(M) = \operatorname{Det}_N(M)$. It coincides with the usual determinant $\det(M)$ in the case of a commutative algebra $\mathcal{M}$. For any $m \geqslant 1$ the *m-th symmetrised permanent* is defined by

$$\operatorname{Per}_m(M) = \operatorname{tr}_{1,\ldots,m} H^{(m)} M_1 \ldots M_m,$$

which is written explicitly as

$$\mathrm{Per}_m(M) = \frac{1}{m!} \sum_{1\leqslant a_1\leqslant\cdots\leqslant a_m\leqslant N} \frac{1}{\alpha_1!\dots\alpha_N!} \sum_{\sigma,\tau\in\mathfrak{S}_m} M_{a_{\sigma(1)}\, a_{\tau(1)}} \dots M_{a_{\sigma(m)}\, a_{\tau(m)}},$$

where α_k denotes the multiplicity of $k \in \{1, \dots, N\}$ in the multiset $\{a_1, \dots, a_m\}$.

Both symmetrised determinants and permanents were considered by Itoh [I07, I09] in relation to Casimir elements for classical Lie algebras.

3.2. **Generators of $\mathcal{A}_\mu$ in type** A. We will work with the reductive Lie algebra $\mathfrak{gl}_N$ rather than the simple Lie algebra $\mathfrak{sl}_N$. We will denote the standard basis elements of $\mathfrak{gl}_N$ by E_{ij} for $i, j = 1, \dots, N$ and combine them into the $N \times N$ matrix $E = [E_{ij}]$. Regarding the entries of E as elements of the symmetric algebra $\mathcal{S}(\mathfrak{gl}_N)$, consider the characteristic polynomial

$$\det(u + E) = u^N + \Phi_1 u^{N-1} + \dots + \Phi_N.$$

Its coefficients $\Phi_1, \dots, \Phi_N$ are algebraically independent generators of the algebra of $\mathfrak{gl}_N$-invariants $\mathcal{S}(\mathfrak{gl}_N)^{\mathfrak{gl}_N}$. All coefficients Ψ_i of the series

$$\det(1 - qE)^{-1} = 1 + \Psi_1 q + \Psi_2 q^2 + \dots$$

belong to the algebra $\mathcal{S}(\mathfrak{gl}_N)^{\mathfrak{gl}_N}$ and $\Psi_1, \dots, \Psi_N$ are its algebraically independent generators. Writing the matrix E in the form (3·4) with $\mathcal{M} = \mathcal{S}(\mathfrak{gl}_N)$ we can represent the generators Φ_m and Ψ_m in the symmetrised form

$$\Phi_m = \mathrm{Det}_m(E) \qquad \text{and} \qquad \Psi_m = \mathrm{Per}_m(E).$$

The respective μ-shifts are found as the coefficients of the polynomials in z^{-1} so that

$$\mathrm{Det}_m(\mu + Ez^{-1}) = \Phi_m z^{-m} + \frac{1}{1!}\, \partial_\mu \Phi_m z^{-m+1} + \dots + \frac{1}{m!}\, \partial_\mu^m \Phi_m \tag{3·6}$$

and

$$\mathrm{Per}_m(\mu + Ez^{-1}) = \Psi_m z^{-m} + \frac{1}{1!}\, \partial_\mu \Psi_m z^{-m+1} + \dots + \frac{1}{m!}\, \partial_\mu^m \Psi_m \tag{3·7}$$

where $\mu \in \mathfrak{gl}_N^*$ is regarded as the $N \times N$ matrix $\mu = [\mu_{ij}]$ with $\mu_{ij} = \mu(E_{ij})$.

The next theorem shows that Conjecture 3.1 holds for each of the families $\Phi_1, \dots, \Phi_N$ and $\Psi_1, \dots, \Psi_N$. It is the first family which was considered by Tarasov [T00], [T03].

Theorem 3.2. *Suppose that $\mu \in \mathfrak{gl}_N^*$ is arbitrary. The algebra $\mathcal{A}_\mu$ is generated by each family of elements $\varpi(\partial_\mu^k \Phi_m)$ and $\varpi(\partial_\mu^k \Psi_m)$ with $m = 1, \dots, N$ and $k = 0, 1, \dots, m-1$.*

Proof. Since the coefficients of the polynomials in (3·6) and (3·7) are already written in a symmetrised form, their images under the symmetrisation map (3·2) are given by the same expressions $\mathrm{Det}_m(\mu+Ez^{-1})$ and $\mathrm{Per}_m(\mu+Ez^{-1})$, where the matrix E is now regarded as the element

$$E = \sum_{i,j=1}^{N} e_{ij} \otimes E_{ij} \in \operatorname{End} \mathbb{C}^N \otimes \mathcal{U}(\mathfrak{gl}_N). \tag{3·8}$$

This follows from the easily verified property of the symmetrisation map (3·2):

$$\varpi\colon (c_1 + Y_1) \dots (c_k + Y_k) \mapsto \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} (c_{\sigma(1)} + Y_{\sigma(1)}) \dots (c_{\sigma(k)} + Y_{\sigma(k)}) \tag{3·9}$$

for any constants c_i and any elements $Y_i \in \mathfrak{g}$. Therefore, we only need to show that $\mathcal{A}_\mu$ is generated by the coefficients of each family of polynomials $\mathrm{Det}_m(\mu + Ez^{-1})$ and $\mathrm{Per}_m(\mu+Ez^{-1})$ with $m = 1, \dots, N$, where the matrix E is defined in (3·8). However, these properties of the coefficients were already established in [FM15, Sec. 4]. □

3.3. **Generators of $\mathcal{A}_\mu$ in types** B, C **and** D**.** Define the orthogonal Lie algebras $\mathfrak{o}_N$ with $N = 2n + 1$ and $N = 2n$ and symplectic Lie algebra $\mathfrak{sp}_N$ with $N = 2n$, as subalgebras of $\mathfrak{gl}_N$ spanned by the elements F_{ij} with $i, j \in \{1, \dots, N\}$,

$$F_{ij} = E_{ij} - E_{j'i'} \qquad \text{and} \qquad F_{ij} = E_{ij} - \varepsilon_i \, \varepsilon_j \, E_{j'i'},$$

respectively, for $\mathfrak{o}_N$ and $\mathfrak{sp}_N$. We use the notation $i' = N - i + 1$, and in the symplectic case set $\varepsilon_i = 1$ for $i = 1, \dots, n$ and $\varepsilon_i = -1$ for $i = n + 1, \dots, 2n$. We will denote by $\mathfrak{g}_N$ any of the Lie algebras $\mathfrak{o}_N$ or $\mathfrak{sp}_N$. Introduce the $N \times N$ matrix $F = [F_{ij}]$. Regarding its entries as elements of the symmetric algebra $\mathcal{S}(\mathfrak{g}_N)$, in the symplectic case consider the characteristic polynomial

$$\det(u + F) = u^{2n} + \Phi_2 \, u^{2n-2} + \dots + \Phi_{2n}. \tag{3·10}$$

The coefficients $\Phi_2, \dots, \Phi_{2n}$ are algebraically independent generators of the algebra of $\mathfrak{sp}_{2n}$-invariants $\mathcal{S}(\mathfrak{sp}_{2n})^{\mathfrak{sp}_{2n}}$. In the orthogonal case, all coefficients Ψ_{2k} of the series

$$\det(1 - q\, F)^{-1} = 1 + \Psi_2 \, q^2 + \Psi_4 \, q^4 + \dots$$

belong to the algebra $\mathcal{S}(\mathfrak{o}_N)^{\mathfrak{o}_N}$. In the case of even $N = 2n$ we also define the *Pfaffian* by

$$\operatorname{Pf} F = \frac{1}{2^n n!} \sum_{\sigma \in \mathfrak{S}_{2n}} \operatorname{sgn} \sigma \cdot F_{\sigma(1)\, \sigma(2)'} \dots F_{\sigma(2n-1)\, \sigma(2n)'}.$$

The coefficients $\Psi_2, \dots, \Psi_{2n}$ are algebraically independent generators of the algebra $\mathcal{S}(\mathfrak{o}_{2n+1})^{\mathfrak{o}_{2n+1}}$, while the elements $\Psi_2, \dots, \Psi_{2n-2}, \operatorname{Pf} F$ are algebraically independent generators of $\mathcal{S}(\mathfrak{o}_{2n})^{\mathfrak{o}_{2n}}$.

Write the matrix F in the form (3·4) with $\mathcal{M} = \mathcal{S}(\mathfrak{g}_N)$ and represent the generators Φ_m and Ψ_m for even values of m in the symmetrised form

$$\Phi_m = \mathrm{Det}_m(F) \qquad \text{and} \qquad \Psi_m = \mathrm{Per}_m(F).$$

Then the respective μ-shifts are found as the coefficients of the polynomials in z^{-1},

$$\mathrm{Det}_m(\mu + F z^{-1}) = \Phi_m z^{-m} + \frac{1}{1!}\,\partial_\mu \Phi_m z^{-m+1} + \dots + \frac{1}{m!}\,\partial_\mu^m \Phi_m \tag{3·11}$$

and

$$\mathrm{Per}_m(\mu + F z^{-1}) = \Psi_m z^{-m} + \frac{1}{1!}\,\partial_\mu \Psi_m z^{-m+1} + \dots + \frac{1}{m!}\,\partial_\mu^m \Psi_m, \tag{3·12}$$

where $\mu \in \mathfrak{g}_N^*$ is regarded as the $N \times N$ matrix $\mu = [\mu_{ij}]$ with $\mu_{ij} = \mu(F_{ij})$. The μ-shifts of the Pfaffian $\mathrm{Pf}\, F$ are the coefficients of the polynomial

$$\mathrm{Pf}\,\big(\mu + F\, z^{-1}\big) = \pi_{(0)} z^{-n} + \dots + \pi_{(n-1)} z^{-1} + \pi_{(n)}, \qquad \pi_{(k)} \in \mathcal{S}(\mathfrak{o}_{2n}),$$

where

$$\mathrm{Pf}\,\big(\mu + F\, z^{-1}\big) = \frac{1}{2^n n!} \sum_{\sigma \in \mathfrak{S}_{2n}} \mathrm{sgn}\,\sigma \cdot \big(\mu + F\, z^{-1}\big)_{\sigma(1)\,\sigma(2)'} \dots \big(\mu + F\, z^{-1}\big)_{\sigma(2n-1)\,\sigma(2n)'}. \tag{3·13}$$

The following theorem implies Conjecture 3.1 for the orthogonal and symplectic Lie algebras for the families of generators of $\mathcal{S}(\mathfrak{g}_N)^{\mathfrak{g}_N}$ described above.

Theorem 3.3. *Suppose that $\mu \in \mathfrak{g}_N^*$ is arbitrary.*

- *The family $\varpi(\partial_\mu^p \Phi_m)$ with $m = 2, 4, \dots, 2n$ and $p = 0, 1, \dots, m-1$ generates the algebra $\mathcal{A}_\mu$ in type* C.
- *The family $\varpi(\partial_\mu^p \Psi_m)$ with $m = 2, 4, \dots, 2n$ and $p = 0, 1, \dots, m-1$ generates the algebra $\mathcal{A}_\mu$ in type* B.
- *The family $\varpi(\partial_\mu^p \Psi_m)$ with $m = 2, 4, \dots, 2n-2$ and $p = 0, 1, \dots, m-1$ together with the elements $\varpi(\pi_{(k)})$ for $k = 0, \dots, n-1$ generate the algebra $\mathcal{A}_\mu$ in type* D.

Proof. It follows from the results of [M13] that the generators of $\mathcal{S}(\mathfrak{g}_N)^{\mathfrak{g}_N}$ introduced above admit the well-defined forms

$$\Psi_m = \gamma_m(N)\, \mathrm{tr}_{1,\dots,m}\, S^{(m)} F_1 \dots F_m \qquad \text{and} \qquad \Phi_m = \gamma_m(-2n)\, \mathrm{tr}_{1,\dots,m}\, S^{(m)} F_1 \dots F_m$$

in the orthogonal and symplectic case, respectively, where

$$\gamma_m(\omega) = \frac{\omega + m - 2}{\omega + 2m - 2},$$

and $S^{(m)} \in \mathrm{End}\,(\mathbb{C}^N)^{\otimes m}$ is the *Brauer algebra symmetriser* which admits a few equivalent expressions; see also [M]. The corresponding μ-shifts are found as the coefficients of the polynomial

$$\gamma_m(\omega)\, \mathrm{tr}_{1,\dots,m}\, S^{(m)} \big(\mu_1 + F_1 z^{-1}\big) \dots \big(\mu_m + F_m z^{-1}\big), \tag{3·14}$$

where we extend notation (3·5) to the matrix μ and assume the specialisations $\omega = N$ and $\omega = -2n$ in the orthogonal and symplectic case, respectively. Due to (3·9), the images of the polynomials (3·11) and (3·12) under the symmetrisation map (3·2) are given by the same expressions $\mathrm{Det}_m(\mu+Fz^{-1})$ and $\mathrm{Per}_m(\mu+Fz^{-1})$ where the matrix F is now regarded as the element

$$F=\sum_{i,j=1}^{N} e_{ij}\otimes E_{ij}\in \mathrm{End}\,\mathbb{C}^N\otimes \mathcal{U}(\mathfrak{g}_N). \tag{3·15}$$

The same observation shows that the image of the polynomial (3·14) under the symmetrisation map (3·2) is given by the expression (3·14) where the matrix F is given by (3·15). This allows us to conclude that the following identities hold for polynomials with coefficients in $\mathcal{U}(\mathfrak{g}_N)$:

$$\mathrm{Det}_m(\mu+Fz^{-1})=\gamma_m(-2n)\,\mathrm{tr}_{1,\dots,m}\,S^{(m)}\big(\mu_1+F_1z^{-1}\big)\dots\big(\mu_m+F_mz^{-1}\big) \tag{3·16}$$

in the symplectic case, and

$$\mathrm{Per}_m(\mu+Fz^{-1})=\gamma_m(N)\,\mathrm{tr}_{1,\dots,m}\,S^{(m)}\big(\mu_1+F_1z^{-1}\big)\dots\big(\mu_m+F_mz^{-1}\big) \tag{3·17}$$

in the orthogonal case.

Now consider the orthogonal and symplectic Lie algebras simultaneously and define the polynomials $\phi_m(z)$ in z^{-1} by

$$\phi_m(z)=\gamma_m(\omega)\,\mathrm{tr}_{1,\dots,m}\,S^{(m)}\big(-\partial_z+\mu_1+F_1z^{-1}\big)\dots\big(-\partial_z+\mu_m+F_mz^{-1}\big)\,1, \tag{3·18}$$

where ∂_z is understood as the differential operator with $\partial_z\,1=0$ so that

$$\phi_m(z)=\phi_{m\,(0)}z^{-m}+\dots+\phi_{m\,(m-1)}z^{-1}+\phi_{m\,(m)},\qquad \phi_{m\,(k)}\in\mathcal{U}(\mathfrak{g}_N).$$

As for the expression (3·14), the right hand side of (3·18) in the symplectic case is assumed to be written in a certain equivalent form which is well-defined for all $1\leqslant m\leqslant 2n+1$; see [M13] and [M] for details. The same assumption will apply to all expressions of this kind throughout the rest of the proof.

In the case $\mathfrak{g}_N=\mathfrak{o}_{2n}$ the ϖ-image of $\mathrm{Pf}\,\big(\mu+Fz^{-1}\big)$ coincides with the expression defined by (3·13), where F is now defined by (3·15). With this interpretation of F we can write

$$\mathrm{Pf}\,\big(\mu+Fz^{-1}\big)=\varpi(\pi_{(0)})z^{-n}+\dots+\varpi(\pi_{(n-1)})z^{-1}+\varpi(\pi_{(n)}). \tag{3·19}$$

By the general properties of the algebra $\mathcal{A}_\mu$ from [FFTL] which we recalled in the beginning of this section and the results of [M13], given any $\mu\in\mathfrak{g}_N^*$, all coefficients of the polynomials $\phi_m(z)$ belong to the commutative subalgebra $\mathcal{A}_\mu$ of $\mathcal{U}(\mathfrak{g}_N)$. All coefficients of the polynomial (3·19) belong to the subalgebra $\mathcal{A}_\mu$ of $\mathcal{U}(\mathfrak{o}_{2n})$. Moreover, the elements

$$\phi_{2k\,(p)}\qquad\text{with}\quad k=1,\dots,n\quad\text{and}\quad p=0,1,\dots,2k-1$$

generate the algebra $\mathcal{A}_\mu$ in the cases B and C, while the elements

$$\phi_{2k\,(p)}\qquad\text{with}\quad k=1,\dots,n-1\quad\text{and}\quad p=0,1,\dots,2k-1$$

together with $\varpi(\pi_{(0)}),\dots,\varpi(\pi_{(n-1)})$ generate the algebra $\mathcal{A}_\mu$ in the case D.

Due to (3·18), $\phi_{m\,(p)}$ is found as the coefficient of z^{-m+p} in the expression

$$\sum_{i_1<\dots<i_p}\ \sum_{j_1<\dots<j_{m-p}} \gamma_m(\omega)\,\mathrm{tr}_{1,\dots,m}\,S^{(m)}\mu_{i_1}\dots\mu_{i_p}\big(-\partial_z+F_{j_1}z^{-1}\big)\dots\big(-\partial_z+F_{j_{m-p}}z^{-1}\big)\,1,$$

summed over disjoint subsets of indices $\{i_1,\dots,i_p\}$ and $\{j_1,\dots,j_{m-p}\}$ of $\{1,\dots,m\}$. Note that for any $\sigma\in\mathfrak{S}_m$ we have $S^{(m)}P_\sigma=P_\sigma S^{(m)}=\pm S^{(m)}$. Hence, applying the cyclic property of trace together with the relations $P_\sigma\mu_i=\mu_{\sigma(i)}P_\sigma$ and $P_\sigma F_j=F_{\sigma(j)}P_\sigma$, we get

$$\phi_{m\,(p)}=z^{m-p}\binom{m}{p}\gamma_m(\omega)\,\mathrm{tr}_{1,\dots,m}\,S^{(m)}\mu_1\dots\mu_p\big(-\partial_z+F_{p+1}z^{-1}\big)\dots\big(-\partial_z+F_mz^{-1}\big)\,1.$$

Furthermore,

$$z^{m-p}\big(-\partial_z+F_{p+1}z^{-1}\big)\dots\big(-\partial_z+F_mz^{-1}\big)\,1$$
$$=F_{p+1}\dots F_m+\text{a linear combination of }\ \{F_{a_1}\dots F_{a_s}\}$$

with $p+1\leqslant a_1<\dots<a_s\leqslant m$. Applying again the cyclic property of trace and appropriate conjugations by the elements P_σ, we bring the expression for $\phi_{m\,(p)}$ to the form

$$\phi_{m\,(p)}=\binom{m}{p}\gamma_m(\omega)\,\mathrm{tr}_{1,\dots,m}\,S^{(m)}\mu_1\dots\mu_p\,F_{p+1}\dots F_m$$
$$+\sum_{r=p+1}^{m-1}c_r\,\gamma_m(\omega)\,\mathrm{tr}_{1,\dots,m}\,S^{(m)}\mu_1\dots\mu_p\,F_{p+1}\dots F_r$$

for certain constants c_r. For the partial trace of the symmetriser we have

$$\mathrm{tr}_m\,\gamma_m(\omega)\,S^{(m)}=\pm\frac{\omega+m-2}{m}\,\gamma_{m-1}(\omega)\,S^{(m-1)},$$

with the plus and minus sign taken in the orthogonal and symplectic case, respectively (assuming $m\leqslant n$ for the latter); see [M13]. This yields

$$\phi_{m\,(p)}=\binom{m}{p}\gamma_m(\omega)\,\mathrm{tr}_{1,\dots,m}\,S^{(m)}\mu_1\dots\mu_p\,F_{p+1}\dots F_m$$
$$+\sum_{r=p+1}^{m-1}d_r\,\gamma_r(\omega)\,\mathrm{tr}_{1,\dots,r}\,S^{(r)}\mu_1\dots\mu_p\,F_{p+1}\dots F_r$$

for certain constants d_r. Introduce the coefficients of the respective polynomials on the right hand sides of (3·16) and (3·17) by

$$\gamma_m(\omega)\,\mathrm{tr}_{1,\dots,m}\,S^{(m)}\big(\mu_1+F_1z^{-1}\big)\dots\big(\mu_m+F_mz^{-1}\big)=\varphi_{m\,(0)}z^{-m}+\dots+\varphi_{m\,(m-1)}z^{-1}+\varphi_{m\,(m)}$$

with $\varphi_{m\,(p)}\in\mathcal{U}(\mathfrak{g}_N)$. The same argument as for the coefficients $\phi_{m\,(p)}$ gives

$$\varphi_{m\,(p)}=\binom{m}{p}\gamma_m(\omega)\,\mathrm{tr}_{1,\dots,m}\,S^{(m)}\mu_1\dots\mu_p\,F_{p+1}\dots F_m.$$

This yields a triangular system of linear relations

$$\phi_{m\,(p)} = \varphi_{m\,(p)} + \sum_{r=p+1}^{m-1} d'_r\, \varphi_{r\,(p)}.$$

Therefore, we may now conclude that the elements $\varphi_{m\,(p)}$ with even $m = 2, 4, \dots, 2n$ and $p = 0, 1, \dots, m-1$ generate the algebra $\mathcal{A}_\mu$ in types B and C, while the elements $\varphi_{m\,(p)}$ with $m = 2, 4, \dots, 2n-2$ and $p = 0, 1, \dots, m-1$ together with $\varpi(\pi_{(0)}), \dots, \varpi(\pi_{(n-1)})$ generate the algebra $\mathcal{A}_\mu$ in type D. The proof is completed by taking into account the relations (3·16) and (3·17). □

4. Quantisations of MF subalgebras with a non-regular μ

Take any $\mu \in \mathfrak{g}^*$. As we already know, $\operatorname{gr} \mathcal{A}_\mu$ is a Poisson-commutative subalgebra of $\mathcal{S}(\mathfrak{g})$ and $\overline{\mathcal{A}}_\mu \subset \operatorname{gr} \mathcal{A}_\mu$ by [FFTL, Prop. 3.12]. As can be seen from the construction (3·1), $\mathcal{A}_\mu \subset \mathcal{U}(\mathfrak{g})^{G_\mu}$ and thereby $\operatorname{gr} \mathcal{A}_\mu \subset \mathcal{S}(\mathfrak{g})^{G_\mu}$. According to Proposition 1.1, $\operatorname{tr.deg}(\operatorname{gr} \mathcal{A}_\mu) \leqslant \frac{1}{2}\dim(G\mu) + \operatorname{rk} \mathfrak{g}$. At the same time, $\operatorname{tr.deg} \overline{\mathcal{A}}_\mu = \frac{1}{2}\dim(G\mu) + \operatorname{rk} \mathfrak{g}$ by Lemma 2.1. Hence we have the following general result.

Proposition 4.1. *For any reductive $\mathfrak{g}$ and any $\mu \in \mathfrak{g}^*$, $\operatorname{gr} \mathcal{A}_\mu$ is an algebraic extension of $\overline{\mathcal{A}}_\mu$.* □

Suppose that $\overline{\mathcal{A}}_\mu$ is a maximal Poisson-commutative subalgebra of $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}_\mu}$. Then necessary $\operatorname{gr} \mathcal{A}_\mu = \overline{\mathcal{A}}_\mu$. In view of Corollary 2.5, the FFTL-conjecture in types A and C follows.

Theorem 4.2. *Suppose that $\mathfrak{g}$ is of type* A *or* C*. Then $\operatorname{gr} \mathcal{A}_\mu = \overline{\mathcal{A}}_\mu$ for each $\mu \in \mathfrak{g}^*$.* □

We can rely on Proposition 2.4 to conclude that $\operatorname{gr} \mathcal{A}_\mu = \overline{\mathcal{A}}_\mu$ for some $\mu \in \mathfrak{g}^* \cong \mathfrak{g}$ in the other types.

Example 4.3 (The minimal nilpotent orbit). Let $\gamma \in \mathfrak{g}$ be a minimal nilpotent element in a simple Lie algebra $\mathfrak{g}$. Suppose that $\mathfrak{g}$ is not of type E_8. Then there is a g.g.s. for γ and $\mathfrak{g}_\gamma$ has the codim-2 property, see [PPY]. Hence $\operatorname{gr} \mathcal{A}_\gamma = \overline{\mathcal{A}}_\gamma$.

Example 4.4 (The subregular case). Keep the assumption that $\mathfrak{g}$ is simple and assume that $\dim(G\mu) = \dim \mathfrak{g} - \operatorname{rk} \mathfrak{g} - 2$. Let γ be as in Lemma 1.9. Then $G\gamma$ is the subregular nilpotent orbit. There is a g.g.s. for γ and $\mathfrak{g}_\gamma$ has the codim-2 property if $\mathfrak{g}$ is not of type G_2, see [PY13, Sec. 6]. In view of Proposition 2.4, we have $\operatorname{gr} \mathcal{A}_\mu = \overline{\mathcal{A}}_\mu$ outside of type G_2.

In types A and C, it is possible to describe the generators of $\mathcal{A}_\mu$ explicitly. If $\mathfrak{g} = \mathfrak{gl}_N$, then $\Phi_1, \dots, \Phi_N$ is a g.g.s. for any nilpotent $\gamma \in \mathfrak{g}$; if $\mathfrak{g} = \mathfrak{sp}_{2n}$, then $\Phi_2, \dots, \Phi_{2n}$ is a g.g.s. for any nilpotent $\gamma \in \mathfrak{g}$, see [PPY]. The degrees of ${}^\gamma\Phi_i$ can be found in [PPY, Sec. 4]. We give more details below.

Type A. Suppose that $\mathfrak{g} = \mathfrak{gl}_N$ and that $\mu \in \mathfrak{gl}_N$ has the Jordan blocks as in (1·3). Then the nilpotent element of Lemma 1.9 is given by the partition Π defined in (1·4). Let $\Pi = (\beta_1, \dots, \beta_M)$. Using the results of Sections 2 and 3.2 we find that the algebra $\mathcal{A}_\mu$ is freely generated by the elements $\varpi(\partial_\mu^k \Phi_m)$ with $1 \leqslant m \leqslant N$ and $0 \leqslant k \leqslant m - \mathbf{r}(m)$, where $\mathbf{r}(m)$ is uniquely determined by the conditions

$$\sum_{i=1}^{\mathbf{r}(m)-1} \beta_i < m \leqslant \sum_{i=1}^{\mathbf{r}(m)} \beta_i.$$

To give an equivalent definition of $\mathbf{r}(m)$, consider the row-tableau of shape Π which is obtained by writing the numbers $1, 2, \dots, N$ consecutively from left to right in the boxes of each row of the Young diagram Π beginning from the top row. Then $\mathbf{r}(m)$ equals the row number of m in the tableau.

Note that in the case where Π corresponds to a nilpotent element μ, the elements $\varpi(\partial_\mu^k \Phi_m)$ with $k > m - \mathbf{r}(m)$ are equal to zero. In the general case, associate the elements of the family $\Phi_{m\,k} = \varpi(\partial_\mu^k \Phi_m)$ with the boxes of the diagram $\Gamma = (N, N-1, \dots, 1)$, as illustrated:

$$\Gamma \quad = \quad \begin{array}{ccccc} \Phi_{N\,N-1} & \Phi_{N\,N-2} & \dots & \Phi_{N\,1} & \Phi_{N\,0} \\ \Phi_{N-1\,N-2} & \Phi_{N-1\,N-3} & \dots & \Phi_{N-1\,0} & \\ \dots & \dots & \dots & & \\ \Phi_{2\,1} & \Phi_{2\,0} & & & \\ \Phi_{1\,0} & & & & \end{array}$$

Then the free generators of $\mathcal{A}_\mu$ correspond to the skew diagram Γ/σ, where

$$\sigma = \big(\mathbf{r}(N) - 1, \dots, \mathbf{r}(1) - 1\big).$$

This agrees with the results of [FM15] which were applied to a few different families of generators, where the transpose of Γ/σ was used instead. The transpose of σ is found by

$$\sigma' = \big(\beta_2 + \dots + \beta_M, \beta_3 + \dots + \beta_M, \dots, \beta_M\big)$$

which coincides with the diagram γ in the notation of that paper. It was shown there that any generator $\Phi_{m\,k}$ associated with a box of the diagram σ is represented by a *linear combination* of the generators corresponding to the boxes of Γ/σ which occur in the column containing $\Phi_{m\,k}$.

Type C. Now suppose that $\mathfrak{g} = \mathfrak{sp}_{2n}$. Keep the notation of (1·3) for the Jordan blocks of $\mu \in \mathfrak{sp}_{2n}$. The nilpotent element of Lemma 1.9 is associated with the partition $\Pi_\gamma = (\beta_1, \dots, \beta_M)$ produced in Section 1.3. Similar to type A, for each $m = 1, \dots, n$ define positive integers $\mathbf{r}(2m)$ by the conditions

$$\sum_{i=1}^{\mathbf{r}(2m)-1} \beta_i < 2m \leqslant \sum_{i=1}^{\mathbf{r}(2m)} \beta_i.$$

Equivalently, they can be defined in terms of the initial Young diagram Π given in (1·4) by the following rule. Consider the row-tableau of shape Π which is obtained by writing the numbers $1, 2, \dots, 2n$ consecutively from left to right in the boxes of each row of Π beginning from the top row. Then $\mathbf{r}(2m)$ equals the row number of $2m$ in the tableau.

The algebra $\mathcal{A}_\mu$ is freely generated by the elements $\Phi_{2m\,k} = \varpi(\partial_\mu^k \Phi_{2m})$ with $1 \leqslant m \leqslant n$ and $0 \leqslant k \leqslant 2m - \mathbf{r}(2m)$. To illustrate this result in terms of diagrams, associate the elements of the family $\Phi_{2m\,k}$ with the boxes of the diagram $\Gamma = (2n, 2n-2, \dots, 2)$, as follows:

$$\Gamma \quad = \quad \begin{array}{llllll} \Phi_{2n\,2n-1} & \Phi_{2n\,2n-2} & \dots & \Phi_{2n\,2} & \Phi_{2n\,1} & \Phi_{2n\,0} \\ \Phi_{2n-2\,2n-3} & \Phi_{2n-2\,2n-4} & \dots & \Phi_{2n-2\,0} & & \\ \dots & \dots & \dots & & & \\ \Phi_{2\,1} & \Phi_{2\,0} & & & & \end{array}$$

Then the free generators of $\mathcal{A}_\mu$ correspond to the skew diagram Γ/σ, where

$$\sigma = \big(\mathbf{r}(2n) - 1, \mathbf{r}(2n-2) - 1, \dots, \mathbf{r}(2) - 1\big).$$

If μ is regular then σ is empty and all generators in Γ are algebraically independent. In the other extreme case where $\mu = 0$ the diagram σ is $(2n-1, 2n-3, \dots, 1)$ and $\mathcal{A}_\mu$ coincides with the center of the universal enveloping algebra $\mathcal{U}(\mathfrak{sp}_{2n})$. It is freely generated by the elements $\Phi_{2\,0}, \Phi_{4\,0}, \dots, \Phi_{2n\,0}$.

For another illustration consider $\mathfrak{sp}_{10}$ and suppose that $\mu \in \mathfrak{sp}_{10}$ has the zero eigenvalue with the corresponding Young diagram $(1,1)$ and two opposite sign eigenvalues, each corresponding to the diagram $(2,1,1)$. Then $\Pi = (5,3,2)$ with

$$\mathbf{r}(2) = \mathbf{r}(4) = 1, \qquad \mathbf{r}(6) = \mathbf{r}(8) = 2 \qquad \text{and} \qquad \mathbf{r}(10) = 3.$$

Hence $\sigma = (2,1,1,0,0)$ and the algebra $\mathcal{A}_\mu$ is freely generated by all elements in Γ except for $\Phi_{10\,9}, \Phi_{10\,8}, \Phi_{8\,7}$ and $\Phi_{6\,5}$.

Remark 4.5. Let $\gamma \in \mathfrak{g}^*$ be a nilpotent element and assume that there is a g.g.s. $H_1, \dots, H_n$ for γ. Then $\overline{\mathcal{A}}_\gamma$ is freely generated by $F_1, \dots, F_s$, where each F_j is a γ-shift $\partial_\gamma^k H_i$, see Section 2. Take elements $\mathcal{F}_j \in \mathcal{A}_\gamma$ such that the symbol of $\mathcal{F}_j$ is F_j. Then the commutative subalgebra $\widetilde{\mathcal{A}}_\gamma \subset \mathcal{U}(\mathfrak{g})$ generated by $\mathcal{F}_1, \dots, \mathcal{F}_s$ is a quantisation of $\overline{\mathcal{A}}_\gamma$, i.e., $\operatorname{gr} \widetilde{\mathcal{A}}_\gamma = \overline{\mathcal{A}}_\gamma$, and therefore solves Vinberg's problem. However we cannot claim that $\widetilde{\mathcal{A}}_\gamma = \mathcal{A}_\gamma$.

Suppose now that $\mathfrak{g} = \mathfrak{o}_N$. There are nilpotent elements $\gamma \in \mathfrak{g}$ that have a g.g.s. and the codim-2 property [PPY, Thm 4.7]. There are some other elements that have only a g.g.s. [PPY, Lemmas 4.5, 4.6]. We postpone the detailed exploration of subalgebras $\mathcal{A}_\mu \subset \mathcal{U}(\mathfrak{g})$ until a forthcoming paper.

5. Quantisations of Vinberg's limits

Take $\gamma \in \mathfrak{g}^*$, $\mu \in \mathfrak{g}^*_{\rm reg}$ and let $\overline{\mathcal{C}}_{\gamma,\mu}$ be Vinberg's limit at γ along μ as defined in Section 2.2. By the construction (3·1), we have $\mathcal{A}_{\gamma+u\mu} \subset \mathcal{U}(\mathfrak{g})[u]$. Let

$$\mathcal{C}_{\gamma,\mu} = \lim_{u\to 0} \mathcal{A}_{\gamma+u\mu}$$

be the limit taken in the same sense as in (2·5). Clearly $\mathcal{C}_{\gamma,\mu}$ is a commutative subalgebra. One could expect that this subalgebra is a quantisation of $\overline{\mathcal{C}}_{\gamma,\mu}$ subject to some reasonable conditions. However, this is not necessarily the case because the operations of taking the limit and the symbol may not commute.

Lemma 5.1. *Suppose that $\overline{\mathcal{A}}_{\gamma+u\mu}$ is freely generated by some elements $F_1(u), \ldots, F_s(u)$, depending on u, such that the nonzero vectors $F_j \in \lim_{u\to 0} \mathbb{C}F_j(u)$ are algebraically independent. Suppose further that $\mathcal{A}_{\gamma+u\mu}$ is generated by $\mathcal{F}_j(u) = \varpi\big((F_j(u)\big)$. Then $\operatorname{gr} \mathcal{C}_{\gamma,\mu} = \overline{\mathcal{C}}_{\gamma,\mu}$.*

Proof. Since $\mathcal{F}_j(u)$ is the symmetrisation of $F_j(u)$, we have

$$\lim_{u\to 0} \mathbb{C}\mathcal{F}_j(u) = \varpi\big(\lim_{u\to 0} \mathbb{C}F_j(u)\big) = \varpi(\mathbb{C}F_j) = \mathbb{C}\varpi(F_j).$$

Furthermore, the elements $\varpi(F_j)$ are algebraically independent. Hence $\mathcal{C}_{\gamma,\mu}$ is freely generated by $\varpi(F_j)$. Because $\overline{\mathcal{C}}_{\gamma,\mu}$ is (freely) generated by F_j, the result follows. ☐

Proposition 5.2. *Let $\mathfrak{g}$ be of type A or C. Suppose that the element $\gamma \in \mathfrak{g}^* \cong \mathfrak{g}$ is nilpotent, $\mu \in \mathfrak{g}^*_{\rm reg}$ and $\bar\mu := \mu|_{\mathfrak{g}_\gamma} \in (\mathfrak{g}^*_\gamma)_{\rm reg}$. Then $\operatorname{gr} \mathcal{C}_{\gamma,\mu} = \overline{\mathcal{C}}_{\gamma,\mu}$.*

Proof. Under the assumptions, $\mathfrak{g}_\gamma$ has the codim-2 property [PPY]. Furthermore, let $\{H_1, \ldots, H_n\} \subset \mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ be the set of generators, where $H_i = \Phi_i$ in type A and $H_i = \Phi_{2i}$ in type C. Then $H_1, \ldots, H_n$ is a g.g.s. for any nilpotent $\gamma \in \mathfrak{g}$ [PPY]. Since $\bar\mu \in (\mathfrak{g}_\gamma)^*$, Theorem 2.6 applies and asserts that $\overline{\mathcal{C}}_{\gamma,\mu}$ has a set of algebraically independent generators, say $F_1, \ldots, F_{\mathbf{b}(\mathfrak{g})}$. Here $F_j \in \lim_{u\to 0} \mathbb{C}F_j(u)$, where $F_j(u) = \partial^k_{\gamma+u\mu} H_i$. By Theorems 3.2 and 3.3, each subalgebra $\mathcal{A}_{\gamma+u\mu}$ is generated by $\varpi\big(F_j(u)\big)$. Hence Lemma 5.1 applies thus completing the proof. ☐

5.1. **Limits along regular series and symmetrisation.** Limits of Mishchenko–Fomenko subalgebras have been studied since [V91]. A rather general definition and a detailed discussion can be found in [V14]. The following construction will be sufficient for our purposes.

Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra. Consider a sequence of elements $h(0), \ldots, h(\ell) \in \mathfrak{h}$ such that $\mathfrak{g}_{h(0)} \cap \mathfrak{g}_{h(1)} \cap \ldots \cap \mathfrak{g}_{h(\ell)} = \mathfrak{h}$ for the centralisers of $h(m)$. Set

$$\mu(u) = h(0) + uh(1) + u^2h(2) + \cdots + u^\ell h(\ell).$$

Then $\overline{\mathcal{A}}_{\mu(u)} \subset \mathcal{S}(\mathfrak{g})[u]$. Further, set

$$\overline{\mathcal{C}} = \lim_{u\to 0} \overline{\mathcal{A}}_{\mu(u)}$$

in the same sense as in (2·5). In our previous considerations ℓ was equal to 1, $h(1)$ was regular, but neither of $h(0), h(1)$ had to be semisimple.

The first interesting property is that $\mathrm{tr.deg}\,\overline{\mathcal{C}} = \boldsymbol{b}(\mathfrak{g})$ [V91, BK76, V14]. Another one is that $\overline{\mathcal{C}}$ is a free algebra [Sh02]. Moreover, each $\overline{\mathcal{C}}$ is a maximal Poisson-commutative subalgebra of $\mathcal{S}(\mathfrak{g})$ [T02]. In type A, the symmetrisation map ϖ provides a quantisation of $\overline{\mathcal{C}}$ and a quantisation of this subalgebra is unique [T00, T03].

Set $\mathfrak{g}_0 = \mathfrak{g}_{h(0)}$ and $\mathfrak{g}_i = \mathfrak{g}_{i-1} \cap \mathfrak{g}_{h(i)}$ for each $i \geqslant 1$. Regard $h(i)$ as a linear function on $\mathfrak{g}$ and on each $\mathfrak{g}_j$. Denote by $\overline{\mathcal{A}}^{(i)}_{h(i)}$ the Mishchenko–Fomenko subalgebra of $\mathcal{S}(\mathfrak{g}_{i-1})$ associated with $h(i)$.

Theorem 5.3 ([Sh02, Thm 1]). *The algebra $\overline{\mathcal{C}}$ is generated by the Mishchenko–Fomenko subalgebras $\overline{\mathcal{A}}_{h(0)}$ and $\overline{\mathcal{A}}^{(i)}_{h(i)}$ with $1 \leqslant i \leqslant \ell$, and by $\mathfrak{h} = \mathfrak{g}_\ell$.*

Recall that the subalgebra $\mathcal{A}_{h(0)}$ of $\mathcal{U}(\mathfrak{g})$ is defined as the image of the Feigin–Frenkel centre under the homomorphism (3·1) with $\mu = h(0)$. Consider the subalgebras $\mathcal{A}^{(i)}_{h(i)} \subset \mathcal{U}(\mathfrak{g}_{i-1})$ for $i \geqslant 1$ defined in the same way. Let $\widetilde{\mathcal{C}} \subset \mathcal{U}(\mathfrak{g})$ be the subalgebra generated by $\mathcal{A}_{h(0)}$ and $\mathcal{A}^{(i)}_{h(i)} \subset \mathcal{U}(\mathfrak{g}_{i-1})$ with $1 \leqslant i \leqslant \ell$, and by $\mathfrak{h}$. Since $\mathcal{A}^{(i)}_{h(i)}$ commutes with $\mathfrak{g}_i$, the subalgebra $\widetilde{\mathcal{C}}$ is commutative.

Theorem 5.4. *Let $\mathfrak{g}$ be a reductive Lie algebra. Then $\mathrm{gr}\,\widetilde{\mathcal{C}} = \overline{\mathcal{C}}$.*

Proof. By the construction $\overline{\mathcal{C}} \subset \mathrm{gr}\,\widetilde{\mathcal{C}}$. Since $\overline{\mathcal{C}}$ is a maximal Poisson-commutative subalgebra [T02], we have the equality here. □

Suppose now that $\mathfrak{g}$ is a classical Lie algebra. Then each $\mathfrak{g}_i$ is also a classical Lie algebra and every simple factor of $\mathfrak{g}_i$ is either of type A or of the same type as $\mathfrak{g}$. In this case, $\overline{\mathcal{C}}$ has a set of generators $F_1, \ldots, F_{\boldsymbol{b}(\mathfrak{g})}$ such that $\widetilde{\mathcal{C}}$ is generated by $\varpi(F_i)$, see Section 3.

One can also consider the commutative subalgebra $\mathcal{C} = \lim_{u\to 0} \mathcal{A}_{\mu(u)} \subset \mathcal{U}(\mathfrak{g})$ and ask whether $\mathrm{gr}\,\mathcal{C} = \overline{\mathcal{C}}$.

Example 5.5 (cf. [R06, Lemma 4]). Take $\mu(u) = E_{11} + uE_{22} + \cdots + u^{N-1}E_{NN} \in \mathfrak{gl}_N$. Then $\overline{\mathcal{C}} = \lim_{u\to 0} \overline{\mathcal{A}}_{\mu(u)}$ is generated by $\mathfrak{h}$, $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$, and $\mathcal{S}(\mathfrak{g}_i)^{\mathfrak{g}_i}$ with $0 \leqslant i \leqslant N-2$ [V91]. It is the associated graded algebra of the Gelfand–Tsetlin subalgebra $\mathcal{GT}(\mathfrak{gl}_N) \subset \mathcal{U}(\mathfrak{gl}_N)$. For $\gamma = E_{11}$, we have $\deg^{\gamma} \Phi_m = m-1$ and $\partial_\gamma \Phi_1 = 1$ and so

$$\overline{\mathcal{A}}_\gamma = \mathbb{C}[\Phi_1, \ldots, \Phi_N, \partial_\gamma\Phi_2, \ldots, \partial_\gamma\Phi_N].$$

Arguing as in the proof of Theorem 2.6, we conclude that $\overline{\mathcal{C}}$ is freely generated by the lowest u-components of the elements $\partial^k_{\mu(u)}\Phi_m$ with $1 \leqslant m \leqslant N$ and $0 \leqslant k \leqslant m-1$. At the same time, $\mathcal{A}_{\mu(u)}$ is generated by $\varpi(\partial^k_{\mu(u)}\Phi_m)$. Therefore, applying Lemma 5.1 we get $\operatorname{gr}\mathcal{C} = \overline{\mathcal{C}}$. It is clear that $\mathcal{C} = \mathcal{GT}(\mathfrak{gl}_N)$. □

The main property of $\mathcal{GT}(\mathfrak{gl}_N)$ is that this subalgebra has a simple spectrum in any irreducible finite-dimensional GL_N-module V. It was noticed in [R06] that $\mathcal{A}_\mu \subset \mathcal{U}(\mathfrak{gl}_N)$ has the same property if $\mu \in \mathfrak{h}$ is sufficiently generic. The observation was extended to all reductive Lie algebras $\mathfrak{g}$ in [FFR].

Example 5.6. Suppose that $\mathfrak{g} = \mathfrak{sp}_{2n}$. The subalgebras $\widetilde{\mathcal{C}}$ admit an explicit description. For a particular choice of the parameters $h(i)$, we obtain a symplectic analogue $\mathcal{GT}(\mathfrak{sp}_{2n})$ of the Gelfand–Tsetlin subalgebra. In the notation of Section 3.3 take

$$h(i-1) = F_{ii} \qquad \text{for} \quad i = 1, \dots, n.$$

Then $\mathfrak{g}_{m-1} = \mathbb{C}^m \oplus \mathfrak{sp}_{2n-2m}$, where we identify $\mathfrak{sp}_{2n-2m}$ with the Lie subalgebra of $\mathfrak{sp}_{2n}$ spanned by the elements F_{ij} with $m+1 \leqslant i, j \leqslant (m+1)'$. The arising algebra $\mathcal{GT}(\mathfrak{sp}_{2n}) = \widetilde{\mathcal{C}}$ is freely generated by the centres $\mathcal{U}(\mathfrak{sp}_{2k})^{\mathfrak{sp}_{2k}}$ with $1 \leqslant k \leqslant n$ and by the symmetrisations

$$\varpi\big(\partial_{h(m)}\Phi^{(m)}_{2i}\big) \in \mathcal{U}(\mathfrak{sp}_{2n-2m}) \qquad \text{for} \quad m = 0, \dots, n-1 \quad \text{and} \quad i = 1, \dots, n-m,$$

where $\Phi^{(0)}_{2i} = \Phi_{2i}$ and $\Phi^{(m)}_{2i} \in \mathcal{S}(\mathfrak{sp}_{2n-2m}) \subset \mathcal{S}(\mathfrak{g}_{m-1})$ with $m \geqslant 1$ denotes the $2i$-th coefficient of the characteristic polynomial associated with $\mathfrak{sp}_{2n-2m}$ as defined in (3·10). The subalgebra $\mathcal{GT}(\mathfrak{sp}_{2n})$ is maximal commutative. If it has a simple spectrum in a finite-dimensional irreducible representation V of $\mathfrak{sp}_{2n}$, then V thus gets a basis of Gelfand–Tsetlin-type; cf. [M99]. Making use of the anti-involution on $\mathcal{U}\big(t^{-1}\mathfrak{g}[t^{-1}]\big)$ defined in the proof of [FFR, Lemma 2], one can show that the action of $\mathcal{GT}(\mathfrak{sp}_{2n})$ on V is diagonalisable.

In the first nontrivial example with $n = 2$ the subalgebra $\mathcal{GT}(\mathfrak{sp}_4) \subset \mathcal{U}(\mathfrak{sp}_4)$ is generated by the centres of $\mathcal{U}(\mathfrak{sp}_2)$ and $\mathcal{U}(\mathfrak{sp}_4)$, the Cartan elements F_{11}, F_{22} and one more element

$$\operatorname{Det}\begin{bmatrix} F_{22} & F_{23} & F_{24} \\ F_{32} & F_{33} & F_{34} \\ F_{42} & F_{43} & F_{44} \end{bmatrix} - \operatorname{Det}\begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix},$$

where we used the symmetrised determinants introduced in Section 3.1.

5.2. **Vinberg's problem for centralisers.** Let $\gamma \in \mathfrak{g} \cong \mathfrak{g}^*$ be a nilpotent element. Set $\mathfrak{q} = \mathfrak{g}_\gamma$, take $\nu \in \mathfrak{q}^*$ and consider the Mishchenko–Fomenko subalgebra $\overline{\mathcal{A}}_\nu \subset \mathcal{S}(\mathfrak{q})$. In this setting, Vinberg's problem was recently solved by Arakawa and Premet [AP17] under the assumptions that $\mathfrak{q}$ has the codim-2 property, there is a g.g.s. for γ, and $\nu \in \mathfrak{q}^*_{\mathrm{reg}}$. We will restrict ourselves to types A and C, where the first two assumptions are satisfied.

Let $\{H_1, \ldots, H_n\} \subset \mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ be the set of generators, where $H_i = \Phi_i$ in type A and $H_i = \Phi_{2i}$ in type C. Recall that $H_1, \ldots, H_n$ is a g.g.s. for any nilpotent $\gamma \in \mathfrak{g}$ [PPY]. Set $P_i := {}^{\gamma}H_i$. Then $\mathcal{S}(\mathfrak{q})^{\mathfrak{q}} = \mathbb{C}[P_1, \ldots, P_n]$ [PPY].

Proposition 5.7. *Suppose that $\mathfrak{g}$ is of type A or C. Take any $\nu \in \mathfrak{q}^*$. Then the elements $\varpi(\partial_\nu^k P_i)$ generate a commutative subalgebra $\widetilde{\mathcal{A}}_\nu$ of $\mathcal{U}(\mathfrak{q})$. If $\nu \in \mathfrak{q}^*_{\rm reg}$, then $\operatorname{gr}\widetilde{\mathcal{A}}_\nu = \overline{\mathcal{A}}_\nu$.*

Proof. Suppose first that $\nu \in \mathfrak{q}^*_{\rm reg}$ and that $\nu = \bar{\mu} = \mu|_{\mathfrak{q}}$ with $\mu \in \mathfrak{g}^*_{\rm reg}$. Then

$$\varpi(\partial^k_{\gamma+u\mu} H_i) \in \mathcal{A}_{\gamma+u\mu}$$

as in the proof of Proposition 5.2. Recall that if $k \geqslant \deg P_i$ and $\bar{k} = k - \deg P_i$, then

$$\lim_{u\to 0} \mathbb{C}\partial^k_{\gamma+u\mu} H_i = \mathbb{C}\partial^{\bar{k}}_\nu P_i.$$

Hence $\varpi(\partial^k_\nu P_i) \in \mathcal{C}_{\gamma,\mu}$ for all $k \geqslant 0$ and therefore any two such elements commute. The statement holds for all ν in a dense open subset implying that

$$\left[\varpi(\partial^k_{\nu'} P_i), \varpi(\partial^{k'}_{\nu'} P_{i'})\right] = 0$$

for all $\nu' \in \mathfrak{q}^*$, all $k, k' \in \mathbb{Z}_{\geqslant 0}$ and all $i, i' \in \{1, \ldots, n\}$.

The Lie algebra $\mathfrak{q}$ satisfies the codim-2 condition and has n, where $n = \operatorname{ind}\mathfrak{q}$, algebraically independent symmetric invariants [PPY]. Hence $\operatorname{tr.deg}\overline{\mathcal{A}}_\nu = \mathbf{b}(\mathfrak{q})$ if $\nu \in \mathfrak{q}^*_{\rm reg}$ [B91, Thm 3.1], see also [PY08, Sec. 2.3]. At the same time $\sum\limits_{i=1}^{n} \deg P_i = \mathbf{b}(\mathfrak{q})$ [PPY]. Therefore the elements $\partial^k_\nu P_i$ with $0 \leqslant k < \deg P_i$ have to be algebraically independent for each $\nu \in \mathfrak{q}^*_{\rm reg}$; see also Proposition 2.2. By a standard argument, the symbol of $\mathbf{Q}\big(\varpi(\partial^k_\nu P_i)\big)$ is equal to $\mathbf{Q}(\partial^k_\nu P_i)$ for every polynomial $\mathbf{Q}$ in $\mathbf{b}(\mathfrak{q})$ variables. Hence $\operatorname{gr}\widetilde{\mathcal{A}}_\nu = \overline{\mathcal{A}}_\nu$. □

Conjecture 5.8. *If $\nu \in \mathfrak{q}^*_{\rm reg}$, then $\widetilde{\mathcal{A}}_\nu$ coincides with the quantisation of $\overline{\mathcal{A}}_\nu$ constructed in* [AP17].

For a reductive $\mathfrak{g}$, the uniqueness of the quantisation in the case of a generic semisimple element $\mu \in \mathfrak{g} \cong \mathfrak{g}^*$ is proved by Rybnikov [R05]. However, it is not known whether this uniqueness property extends to the quantisation of $\overline{\mathcal{A}}_\nu$ and we cannot conclude that the symmetrisation in the sense of Proposition 5.7 coincides with the quantisation of Arakawa and Premet.

References

[AP17] T. Arakawa and A. Premet, Quantizing Mishchenko–Fomenko subalgebras for centralizers via affine W-algebras, *Trans. Mosc. Math. Soc.*, **78** (2017), to appear.

[B91] A. V. Bolsinov, Commutative families of functions related to consistent Poisson brackets, *Acta Appl. Math.*, **24**, no. 3 (1991), 253–274.

[Bor] W. Borho, Über Schichten halbeinfacher Lie-Algebren, *Invent. Math.* **65** (1981), 283–317.

[BK76] W. Borho and H. Kraft, Über die Gelfand-Kirillov-Dimension, *Math. Ann.*, **220** (1976), no. 1, 1–24.

[BK79] W. BORHO and H. KRAFT, Über Bahnen und deren Deformationen bei linearen Aktionen reduktiver Gruppen, *Comment. Math. Helv.* **54** (1979), no. 1, 61–104.

[CM10] J.-Y. CHARBONNEL and A. MOREAU, The index of centralizers of elements of reductive Lie algebras, *Doc. Math.*, **15** (2010), 387–421.

[CM09] A. V. CHERVOV and A. I. MOLEV, On higher order Sugawara operators, *Int. Math. Res. Not.* (2009), 1612–1635.

[CT06] A. CHERVOV and D. TALALAEV, Quantum spectral curves, quantum integrable systems and the geometric Langlands correspondence, `arXiv:hep-th/0604128`.

[FF92] B. FEIGIN and E. FRENKEL, Affine Kac–Moody algebras at the critical level and Gelfand–Dikii algebras, *Int. J. Mod. Phys.* A**7**, *Suppl*. 1A (1992), 197–215.

[FFR] B. FEIGIN, E. FRENKEL, and L. RYBNIKOV, Opers with irregular singularity and spectra of the shift of argument subalgebra, *Duke Math. J.* **155** (2010), no. 2, 337–363.

[FFTL] B. FEIGIN, E. FRENKEL, and V. TOLEDANO LAREDO, Gaudin models with irregular singularities, *Adv. Math.* **223** (2010), 873–948.

[F07] E. FRENKEL, Langlands correspondence for loop groups, Cambridge Studies in Advanced Mathematics, 103. Cambridge University Press, Cambridge, 2007.

[FM15] V. FUTORNY and A. MOLEV, Quantization of the shift of argument subalgebras in type A, *Adv. Math.* **285** (2015), 1358–1375.

[dG08] W. DE GRAAF, Computing with nilpotent orbits in simple Lie algebras of exceptional type, *LMS J. Comput. Math.*, **11** (2008), 280–297.

[I07] M. ITOH, Two determinants in the universal enveloping algebras of the orthogonal Lie algebras, *J. Algebra* **314** (2007), 479–506.

[I09] M. ITOH, Two permanents in the universal enveloping algebras of the symplectic Lie algebras, *Internat. J. Math.* **20** (2009), 339–368.

[J04] J. C. JANTZEN, Nilpotent orbits in representation theory, in: B. Orsted (Ed.), Representation and Lie Theory, in: Progr. Math., vol. **228**, Birkhäuser, Boston, MA, 2004, 1–211.

[K83] G. KEMPKEN. Induced conjugacy classes in classical Lie-algebras, *Abh. Math. Sem. Univ. Hamburg*, **53** (1983), 53–83.

[K63] B. KOSTANT, Lie group representations on polynomial rings, *Amer. J. Math.*, **85** (1963), 327–404.

[MF78] A. S. MISHCHENKO and A. T. FOMENKO, Euler equation on finite-dimensional Lie groups (in Russian), *Izv. AN SSSR. Ser. matem.* **42** (1978), no. 2, 396–415; English translation in *Math. USSR-Izv.* **12** (1978), 371–389.

[M99] A. I. MOLEV, A basis for representations of symplectic Lie algebras, *Comm. Math. Phys.* **201** (1999), 591–618.

[M13] A. I. MOLEV, Feigin–Frenkel center in types B, C and D, *Invent. Math.* **191** (2013), 1–34.

[M] A. MOLEV, Sugawara operators for classical Lie algebras. Mathematical Surveys and Monographs, AMS, Providence, RI (in press).

[NO96] M. NAZAROV and G. OLSHANSKI, Bethe subalgebras in twisted Yangians, *Comm. Math. Phys.* **178** (1996), 483–506.

[P03] D. PANYUSHEV, The index of a Lie algebra, the centralizer of a nilpotent element, and the normalizer of the centralizer, *Math. Proc. Cambr. Phil. Soc.* **134** (2003), no. 1, 41–59.

[PPY] D. PANYUSHEV, A. PREMET, and O. YAKIMOVA, On symmetric invariants of centralisers in reductive Lie algebras, *J. Algebra* **313** (2007), 343–391.

[PY06] D. PANYUSHEV and O. YAKIMOVA, The index of representations associated with stabilisers, *J. Algebra* **302** (2006), 280–304.

[PY08] D. PANYUSHEV and O. YAKIMOVA, The argument shift method and maximal commutative subalgebras of Poisson algebras, *Math. Res. Letters* **15** (2008), no. 2, 239–249.

[PY13] D. PANYUSHEV and O. YAKIMOVA, Parabolic contractions of semisimple Lie algebras and their invariants, *Selecta Math.* (New Series) **19** (2013), 699–717.

[R05] L. G. RYBNIKOV, Centralizers of certain quadratic elements in Poisson-Lie algebras and the method of translation of invariants, *Russian Math. Surveys* **60** (2005), no. 2, 367–369.

[R06] L. G. RYBNIKOV, The shift of invariants method and the Gaudin model, *Funct. Anal. Appl.* **40** (2006), 188–199.

[S04] S. T. SADETOV, A proof of the Mishchenko-Fomenko conjecture (in Russian), *Dokl. Akad. Nauk*, **397** (2004), no. 6, 751–754; English translation in *Doklady Math.* **70** (2004), no. 1, 634–638.

[Sh02] V. V. SHUVALOV, On the limits of Mishchenko-Fomenko subalgebras in Poisson algebras of semisimple Lie algebras (in Russian), *Funktsional. Anal. i Prilozhen.* **36** (2002), no. 4, 55–64; English translation in *Funct. Anal. Appl.* **36** (2002), no. 4, 298–305.

[T00] A. A. TARASOV, On some commutative subalgebras in the universal enveloping algebra of the Lie algebra $\mathfrak{gl}(n, C)$, *Sb. Math.* **191** (2000), 1375–1382.

[T02] A. A. TARASOV, The maximality of certain commutative subalgebras in the Poisson algebra of a semisimple Lie algebra, *Russian Math. Surveys* **57** (2002), no. 5, 1013–1014.

[T03] A. A. TARASOV, Uniqueness of liftings of maximal commutative subalgebras of the Poisson–Lie algebra to the enveloping algebra, *Sb. Math.* **194** (2003), 1105–1111.

[V91] E. B. VINBERG, Some commutative subalgebras of a universal enveloping algebra, *Math. USSR-Izv.* **36** (1991), 1–22.

[V14] E. B. VINBERG, Limits of integrable Hamiltonians on semisimple Lie algebras, *Funct. Anal. Appl.* **48** (2014), 107–115.

[Y06] O. YAKIMOVA, The centralisers of nilpotent elements in classical Lie algebras, *Funct. Anal. Appl.* **40**, no. 1 (2006), 42–51.

[Y09] O. YAKIMOVA, Surprising properties of centralisers in classical Lie algebras, *Ann. Inst. Fourier (Grenoble)* **59** no. 3 (2009), 903–935.

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