EXISTENCE OF SHARP ASYMPTOTIC PROFILES OF SINGULAR SOLUTIONS TO AN ELLIPTIC EQUATION WITH A SIGN-CHANGING NON-LINEARITY

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ABSTRACT. The first two authors [Proc. Lond. Math. Soc. (3) **114**(1):1–34, 2017] classified the behaviour near zero for all positive solutions of the perturbed elliptic equation with a critical Hardy–Sobolev growth

$$-\Delta u = |x|^{-s} u^{2^\star(s)-1} - \mu u^q \text{ in } B \setminus \{0\},$$

where B denotes the open unit ball centred at 0 in $\mathbb{R}^n$ for $n \geq 3$, $s \in (0,2)$, $2^\star(s) := 2(n-s)/(n-2)$, $\mu > 0$ and $q > 1$. For $q \in (1, 2^\star - 1)$ with $2^\star = 2n/(n-2)$, it was shown in the op. cit. that the positive solutions with a non-removable singularity at 0 could exhibit up to three different singular profiles, although their existence was left open. In the present paper, we settle this question for all three singular profiles in the maximal possible range. As an important novelty for $\mu > 0$, we prove that for every $q \in (2^\star(s) - 1, 2^\star - 1)$ there exist infinitely many positive solutions satisfying $|x|^{s/(q-2^\star(s)+1)} u(x) \to \mu^{-1/(q-2^\star(s)+1)}$ as $|x| \to 0$, using a dynamical system approach. Moreover, we show that there exists a positive singular solution with $\liminf_{|x|\to 0} |x|^{(n-2)/2} u(x) = 0$ and $\limsup_{|x|\to 0} |x|^{(n-2)/2} u(x) \in (0, \infty)$ if (and only if) $q \in (2^\star - 2, 2^\star - 1)$.

1. INTRODUCTION AND MAIN RESULTS

The Hardy–Sobolev inequality is obtained by interpolating between the Sobolev inequality ($s = 0$) and the Hardy inequality ($s = 2$): For every $s \in (0,2)$ and $n \geq 3$, there exists a positive constant $K_{s,n}$ such that

$$\int_{\mathbb{R}^n} |\nabla u|^2 \, dx \geq K_{s,n} \left(\int_{\mathbb{R}^n} |x|^{-s} |u|^{2^\star(s)} \, dx \right)^{\frac{2}{2^\star(s)}} \quad \text{for all } u \in C_c^\infty(\mathbb{R}^n),$$

where $2^\star(s) := 2(n-s)/(n-2)$ denotes the critical Hardy–Sobolev exponent. The critical Sobolev exponent $2^\star$ corresponds to $2^\star(s)$ with $s = 0$. Recent results and challenges on the Hardy–Sobolev inequalities are surveyed by Ghoussoub–Robert in [12], see also [13]. For $s \in (0,2)$, the best Hardy–Sobolev constant $K_{s,n}$ is attained by a one-parameter family $(U_\eta)_{\eta>0}$ of functions

$$U_\eta(x) := c_{n,s} \, \eta^{\frac{n-2}{2}} \left(\eta^{2-s} + |x|^{2-s} \right)^{-\frac{n-2}{2-s}} \quad \text{for } x \in \mathbb{R}^n, \tag{1.1}$$

where $c_{n,s} := ((n-s)(n-2))^{1/(2^\star(s)-2)}$ is a positive normalising constant. The functions U_η are the only positive *non-singular* solutions of the equation (see Chen–Lin [8] and Chou–Chu [9])

$$-\Delta U = |x|^{-s} U^{2^\star(s)-1} \quad \text{in } \mathbb{R}^n \setminus \{0\}. \tag{1.2}$$

Moreover, any positive $C^2(\mathbb{R}^n \setminus \{0\})$ *singular* solution U of (1.2) is radially symmetric around 0 and $v(t) = e^{-(n-2)t/2} U(e^{-t})$ is a positive periodic function of t in $\mathbb{R}$ (see Hsia–Lin–Wang [14]).

The isolated singularity problem has been studied extensively, see Véron's monograph [21]. Recent works of the first author and her collaborators such as [4, 10, 11] give a full classification of the isolated singularities for various classes of elliptic equations.

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In this paper, we settle an open question arising from [11] with regard to the *existence* of all the singular profiles at zero for the positive solutions of the perturbed non-linear elliptic equation

$$-\Delta u = |x|^{-s} u^{2^\star(s)-1} - \mu u^q \qquad \text{for } x \in B(0,R) \setminus \{0\}, \tag{1.3}$$

where μ is a *positive* parameter, $q > 1$ and $s \in (0,2)$. By $B(0,R)$ we denote the open ball in $\mathbb{R}^n$ $(n \geq 3)$ centred at 0 with radius $R > 0$. The first two authors have proved in [11] that the positive singular solutions of (1.3) can exhibit up to *three* types of singular profiles at zero in a suitable range for q:

• A **(ND) type** profile (for "Non Differential") if

$$\lim_{|x|\to 0} |x|^{\frac{s}{q-(2^\star(s)-1)}} u(x) = \mu^{-\frac{1}{q-(2^\star(s)-1)}}. \tag{ND}$$

• A profile of **(MB) type** (for "Multi-Bump") in the sense that there exists a sequence $(r_k)_{k\geq 0}$ of positive numbers decreasing to 0 such that $r_{k+1} = o(r_k)$ as $k \to +\infty$ and

$$u(x) = (1 + o(1)) \sum_{k=0}^{\infty} U_{r_k}(x) \text{ as } |x| \to 0, \text{ where } U_\eta \text{ is as in (1.1)}. \tag{MB}$$

• A profile of **(CGS) type** (for "Caffarelli–Gidas–Spruck") if there exists a positive periodic function $v \in C^\infty(\mathbb{R})$ such that

$$\lim_{|x|\to 0} \left(|x|^{\frac{n-2}{2}} u(x) - v(-\log|x|) \right) = 0. \tag{CGS}$$

The case $q = 2^\star - 1$ in (1.3) was fully dealt with in [11]. Hence, in the sequel we assume that $q \neq 2^\star - 1$. We recall the relevant classification result from [11]:

Theorem 1.1 ([11])**.** *Let $u \in C^\infty(B(0,R) \setminus \{0\})$ be an arbitrary positive solution to (1.3).*

- *If $q > 2^\star - 1$, then 0 is a removable singularity;*
- *If $2^\star(s) - 1 < q < 2^\star - 1$, then either 0 is a removable singularity, or u develops a profile of type (CGS), (MB) or (ND);*
- *If $1 < q \leq 2^\star(s) - 1$, then either 0 is a removable singularity, or u has a profile of type (CGS) or (MB).*

Moreover, if u develops a profile of (MB) type, then $2^\star - 2 < q < 2^\star - 1$.

However, no examples of the three singular profiles of Theorem 1.1 were given in [11], leaving open the question of their existence. In the present paper, we fill this gap by proving the following:

Theorem 1.2. *The three singular profiles of Theorem 1.1 actually do exist.*

The existence assertion of Theorem 1.2 is a corollary of the following precise result:

Theorem 1.3. *Equation (1.3) admits positive radially symmetric solutions developing (CGS), (MB) and (ND) profiles in the exact range of parameters given by Theorem 1.1. More precisely, when $q \in (1, 2^\star - 1)$, there exists $R_0 > 0$ such that for every $R \in (0, R_0)$, the following hold:*

(i) *For every $\gamma > 0$, there exists a unique positive radial solution u_γ of (1.3) with a removable singularity at 0 and $\lim_{|x|\to 0} u_\gamma(x) = \gamma$.*
(ii) *If $q > 2^\star - 2$, then (1.3) has at least a positive (MB) solution.*
(iii) *For every positive singular solution U of (1.2), there exists a unique positive radial (CGS) solution u of (1.3) with asymptotic profile U near zero.*
(iv) *If $q > 2^\star(s) - 1$, then (1.3) admits infinitely many positive (ND) solutions.*

Remark 1.4. *If $q \in (1, 2^\star(s)-1)$, then all positive radial solutions of* (1.3) *extend as positive radial solutions in $\mathbb{R}^n \setminus \{0\}$. For $q \in [2^\star(s)-1, 2^\star-1)$, any positive radial non-(ND) solution u of* (1.3) *extends as a positive radial solution at least in $B(0,R^*) \setminus \{0\}$ with R^* independent of u (see Lemma 3.2).*

From the three singular profiles of (1.3), only the (CGS) type is reminiscent of the asymptotics of the local singular solutions for the Yamabe problem in the case of a flat background metric ($\mu = s = 0$) studied in Caffarelli–Gidas–Spruck [3] (see also Korevaar–Mazzeo–Pacard–Schoen [16] for a refined asymptotics and Marques [19] for the case of a general background metric). But for $\mu > 0$, the introduction of the perturbation term in (1.3) yields two new singular profiles: the (ND) and (MB) types.

An important novelty in this paper is the *existence of infinitely many* positive radial (ND) solutions for (1.3) when $q \in (2^\star(s)-1, 2^\star-1)$. To our best knowledge, there are no previous existence results known for this type of singularities, which arise as a consequence of studying (1.3) with a critical Hardy–Sobolev growth (i.e., $s \in (0,2)$) rather than with a critical Sobolev growth ($s = 0$). Since (1.4) fails for the (ND) solutions, neither Pohozaev-type arguments nor Fowler-type transformations relevant for (CGS) or (MB) profiles can be used. Specific to the (ND) solutions, the *first term* in their asymptotics arises from the competition generated in the right-hand side of (1.3) and not directly from the differential structure. To overcome this obstacle, we rewrite the radial form of (1.3) as a dynamical system using an original transformation involving *three* variables, see (2.2). The variable X_1 in (2.2) is suggestive of a second order term in the asymptotics of the (ND) solutions, which will make apparent the differential structure of our equation in a dynamical systems setting. Nevertheless, by linearising the flow around the critical point, we find a positive eigenvalue, a null one and a negative eigenvalue so that we cannot apply the classical Hartman–Grobman theorem. Instead, we shall use Theorem 7.1 in the Appendix, which invokes the notion of center-stable manifold and ideas of Kelley [15].

For $1 < q < 2^\star - 1$, Theorem 1.1 yields that every positive non-(ND) solution of (1.3) satisfies

$$\limsup_{|x|\to 0} |x|^{\frac{n-2}{2}} u(x) < \infty. \tag{1.4}$$

Moreover, (1.4) holds for every positive solution of (1.3) when $q \in (1, 2^\star(s)-1]$. Note that (1.4) is crucial for Pohozaev type arguments [11], on the basis of which we prove in Sect. 3 the non-existence of smooth positive solutions for (1.3), subject to $u = 0$ on $\partial B(0,R)$.

Theorem 1.5. *Let $\mu > 0$ and $s \in (0,2)$ be arbitrary. Let Ω be a smooth bounded domain in $\mathbb{R}^n$ ($n \geq 3$) such that $0 \in \Omega$. Assume that Ω is star-shaped with respect to 0. Then, for every $q \in (1, 2^\star(s)-1]$, there are no positive smooth solutions for the problem*

$$\begin{cases} -\Delta u = |x|^{-s} u^{2^\star(s)-1} - \mu u^q & \text{in } \Omega \setminus \{0\}, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

If $q \in (2^\star(s)-1, 2^\star-1)$, then (1.5) *admits no positive smooth solutions of non-(ND) type.*

Motivated by the problem of finding a metric conformal to the flat metric of $\mathbb{R}^n$ such that $K(x)$ is the scalar curvature of the new metric, Chen–Lin [5, 6, 8] and Lin [17] analysed the local behaviour of the positive singular solutions $u \in C^2(B(0,1) \setminus \{0\})$ to

$$-\Delta u = K(x)\, u^{2^\star-1} \quad \text{in } B(0,1) \setminus \{0\}, \tag{1.6}$$

where K is a positive continuous function on $B(0,1)$ in $\mathbb{R}^n$ ($n \geq 3$) with $K(0) = 1$. Moreover, K was always assumed to be a C^1 function on $B(0,1) \setminus \{0\}$ such that

$$0 < \underline{L} := \liminf_{|x|\to 0} |x|^{1-\ell} |\nabla K(x)| \leq \overline{L} := \limsup_{|x|\to 0} |x|^{1-\ell} |\nabla K(x)| < \infty \ \text{ for some } \ell > 0. \tag{1.7}$$

In the above-mentioned works (see also Lin–Prajapat [18] and Taliaferro–Zhang [20]), the following question was investigated: *Under what conditions on K, the positive singular solutions of* (1.2) *with* $s=0$ *are asymptotic models at zero for the positive singular solutions of* (1.6)*?*

This question was settled positively in any of the following situations:

(a) Assumption (1.7) holds for $\ell \geq (n-2)/2$ (see [8, Theorems 1.1 and 1.2]);
(b) If (1.7) holds with $\ell \in (0,(n-2)/2)$, together with *extra* conditions, see [17, Theorem 1.2].

Extra conditions in situation (b) are needed to guarantee a positive answer to the above question. Otherwise, for every $0<\ell<(n-2)/2$, Chen–Lin [8, Theorem 1.6] provided general positive radial functions $K(r)$ non-increasing in $r=|x|\in[0,1]$ with $K(0)=1$ such that (1.7) holds and (1.6) has a positive singular solution with $\liminf_{|x|\to 0}|x|^{(n-2)/2}u(x)=0$.

The importance of condition (1.7) in settling the above question can be inferred from our next result as a by-product of Theorem 1.3(ii): For every $0<\ell<\min\{(n-2)/2,2\}$ and $s\in(0,2)\setminus\{\ell\}$, we construct a positive continuous function K on $B(0,R)$ for some $R>0$ with $K(0)=1$ such that *exactly one inequality in* (1.7) *fails*, yet generating for (1.8) a positive singular solution, the asymptotics of which at zero *cannot* be modelled by any positive singular solution of (1.2).

Corollary 1.6. *For every* $0<\ell<\min\{(n-2)/2,2\}$ *and* $s\in(0,2)\setminus\{\ell\}$*, there exist* $R>0$ *and a positive* C^1*-function* K *on* $B(0,R)\setminus\{0\}$ *in* $\mathbb{R}^n$ $(n\geq 3)$ *with* $K<\lim_{|x|\to 0}K(x)=1$ *on* $B(0,R)\setminus\{0\}$ *such that* $0=\underline{L}<\overline{L}<\infty$ *if* $\ell<s$ *and* $0<\underline{L}<\overline{L}=\infty$ *if* $\ell>s$*, yet*

$$-\Delta u = K(x)|x|^{-s}u^{2^\star(s)-1} \quad \text{in } B(0,R)\setminus\{0\} \tag{1.8}$$

admits a positive singular solution with $\liminf_{|x|\to 0}|x|^{(n-2)/2}u(x)=0$.

Structure of the paper. In Sect. 2, we prove Theorem 1.3(iv) on the existence of infinitely many positive (ND) solutions for (1.3). In Sect. 3, we establish Theorem 1.5, together with uniform *a priori* estimates for the positive radial solutions of (1.3) satisfying (1.4) (see Proposition 3.1). In Sect. 4, by setting $u(r)=y(\xi)$ with $\xi=r^{(2-s)/2}$, we reduce the assertion of Theorem 1.3(i) on removable singularities to the existence and uniqueness of the solution for (4.1) on an interval $[0,T]$. The latter follows from Biles–Robinson–Spraker [2, Theorems 1 and 2]. In Sect. 5, after giving the proof of Corollary 1.6, we use an argument influenced by Chen–Lin [8] to prove the existence of (MB) solutions for (1.3) in the whole possible range $q\in(2^\star-2,2^\star-1)$. In Sect. 6, with a dynamical system approach, we prove Theorem 1.3(iii): the positive singular solutions of (1.2) serve as asymptotic models for the positive radial (CGS) solutions of (1.3). For a dynamical approach to Emden–Fowler equations and systems, see Bidaut-Véron–Giacomini [1].

The results in this paper give the existence and profile at infinity for the positive solutions to

$$-\Delta\tilde{u} = |x|^{-s}\tilde{u}^{2^\star(s)-1} - \mu|x|^{(n-2)q-(n+2)}\tilde{u}^q \quad \text{for } |x|>1/R$$

by using the Kelvin transform $\tilde{u}(x)=|x|^{2-n}u(x/|x|^2)$, where u is a positive solution of (1.3).

2. (ND) SOLUTIONS

In this section, we let $q\in(2^\star(s)-1,2^\star-1)$ and prove Theorem 1.3(iv), restated below.

Proposition 2.1. *Assume that* $q\in(2^\star(s)-1,2^\star-1)$*. Then, there exists* $R_0>0$ *such that for every* $R\in(0,R_0)$*, equation* (1.3) *admits infinitely many positive (ND) solutions.*

The proof of Proposition 2.1 takes place in several steps. First, we reformulate the radial form of (1.3) as a first order autonomous differential system using a new transformation, see (2.2).

2.1. **Formulation of our problem as a dynamical system.** We first assume that u is a positive radial (ND) solution of (1.3). We define

$$\vartheta := \frac{s}{q-2^\star(s)+1}, \quad \beta := \frac{(q-1)\,\vartheta}{2}-1, \quad \zeta := \frac{2^\star(s)-2}{q-2^\star(s)+1}. \tag{2.1}$$

We introduce a new transformation involving three functions X_1, X_2 and X_3 as follows

$$X_1(t) = t\left(1-\mu r^s u^{q-2^\star(s)+1}\right), \qquad X_2(t) = \frac{1}{t}, \quad X_3(t) = \frac{ru'(r)}{u(r)}+\vartheta, \tag{2.2}$$

where $t := r^{-\beta}$ and β,ϑ are given by (2.1). Since u is a positive radial (ND) solution of (1.3), that is, $\lim_{r\to 0^+} r^\vartheta u(r) = \mu^{-1/(q-2^\star(s)+1)}$, it follows that

$$\begin{cases} 1-X_1(t)X_2(t) = \mu r^s u(r)^{q-2^\star(s)+1} > 0 \quad \text{for all } t\in[2R^{-\beta},\infty), \\ X_1(t)X_2(t)\to 0 \text{ as } t\to\infty. \end{cases} \tag{2.3}$$

If we set $\vec{X} = (X_1,X_2,X_3)$, then, as one easily checks, we have that

$$\vec{X}'(t) = (H_1(\vec{X}(t)),H_2(\vec{X}(t)),H_3(\vec{X}(t))) \tag{2.4}$$

for all $t\in[2R^{-\beta},\infty)$, where H_1, H_2 and H_3 are real-valued functions defined on $\mathbb{R}^3$ by

$$\begin{cases} H_1(\xi_1,\xi_2,\xi_3) := \xi_1\xi_2+\beta^{-1}(q-2^\star(s)+1)(1-\xi_1\xi_2)\xi_3, \\ H_2(\xi_1,\xi_2,\xi_3) := -\xi_2^2, \\ H_3(\xi_1,\xi_2,\xi_3) := \beta^{-1}\mu^{-\zeta}\xi_1(1-\xi_1\xi_2)_+^\zeta+\beta^{-1}\xi_2(\xi_3-\vartheta)(\xi_3-\vartheta+n-2). \end{cases} \tag{2.5}$$

By ξ_+ we mean the positive part of ξ. We define $\vec{Y} := (Y_1,Y_2,Y_3)$, where $\vec{Y}(t) = \vec{X}(t+2R^{-\beta})$ for all $t\ge 0$. Then, (2.4) gives that $\vec{Y}'(t) = (H_1(\vec{Y}(t)),H_2(\vec{Y}(t)),H_3(\vec{Y}(t)))$ for all $t\in[0,\infty)$. To get more regularity, for any $\varepsilon\in(0,1)$, we choose $\Psi_\varepsilon\in C^1(\mathbb{R})$ such that $\Psi_\varepsilon(t) = t^\zeta$ for all $t\ge\varepsilon$. By choosing $\varepsilon_0\in(0,1)$ small enough and using (2.3), we find that

$$\vec{Y}'(t) = (H_1(\vec{Y}(t)),H_2(\vec{Y}(t)),H_{3,\Psi_\varepsilon}(\vec{Y}(t))) \quad \text{for all } t\in[0,\infty) \tag{2.6}$$

for every $\varepsilon\in(0,\varepsilon_0)$, where the function $H_{3,\Psi_\varepsilon}:\mathbb{R}^3\to\mathbb{R}$ is defined by

$$H_{3,\Psi_\varepsilon}(\xi_1,\xi_2,\xi_3) := \beta^{-1}\mu^{-\zeta}\xi_1\Psi_\varepsilon(1-\xi_1\xi_2)+\beta^{-1}\xi_2(\xi_3-\vartheta)(\xi_3-\vartheta+n-2).$$

2.2. **Existence of solutions for (2.6).** Using ϑ, β and ζ in (2.1), we define Υ and Γ by

$$\Upsilon := \mu^{\zeta/2}\sqrt{q-2^\star(s)+1} \quad \text{and} \quad \Gamma := \vartheta\,(n-2-\vartheta)\,\mu^\zeta. \tag{2.7}$$

Lemma 2.2. *Let $q\in(2^\star(s)-1,2^\star-1)$ and $\varepsilon\in(0,1)$. Fix $\Psi_\varepsilon\in C^1(\mathbb{R})$ such that $\Psi_\varepsilon(t)=t^\zeta$ for all $t\ge\varepsilon$. For every $\delta>0$ small, there exist $r_0\in(0,\delta/2)$ and a Lipschitz function $w:[0,r_0]\times[-r_0,r_0]\to[-r_0,r_0]$ such that for any $(Y_{2,0},Z_{3,0})\in(0,r_0]\times[-r_0,r_0]$, the system (2.6) subject to the initial condition*

$$\vec{Y}(0) = (\Upsilon(Z_{3,0}-w(Y_{2,0},Z_{3,0}))+\Gamma Y_{2,0},Y_{2,0},w(Y_{2,0},Z_{3,0})+Z_{3,0}) \tag{2.8}$$

has a solution $\vec{Y}(t) = (Y_1(t),Y_2(t),Y_3(t))$ for all $t\ge 0$ satisfying

$$\lim_{t\to+\infty}\vec{Y}(t) = (0,0,0). \tag{2.9}$$

Moreover, we have $Y_2(t) = 1/(t+Y_{2,0}^{-1})$ for all $t\ge 0$.

Proof. Since $\Psi_\varepsilon(1)=1$, we find one critical point $(0,0,0)$ for (2.6). Linearising the flow around $(0,0,0)$, we get one *unstable* eigenvalue $\lambda_1=\mu^{-\zeta/2}\beta^{-1}\sqrt{q-2^\star(s)+1}$ with associated eigenvector $(\Upsilon,0,1)$, one *null* eigenvalue with associated eigenvector $(\Gamma,1,0)$ and one *stable* eigenvalue $-\lambda_1$ with associated eigenvector $(\Upsilon,0,1)$. For $\vec{Z}=(Z_1,Z_2,Z_3)$, using a change of coordinates

$$\vec{Y}=(\Upsilon(Z_1-Z_3)+\Gamma Z_2,Z_2,Z_1+Z_3),\ \text{i.e.,}\ \vec{Z}=\left(\frac{Y_1-\Gamma Y_2+\Upsilon Y_3}{2\Upsilon},Y_2,\frac{\Gamma Y_2+\Upsilon Y_3-Y_1}{2\Upsilon}\right), \tag{2.10}$$

we bring the system (2.6) to a diagonal form, namely

$$\vec{Z}'(t)=(\lambda_1 Z_1(t)+h_1(\vec{Z}(t)),-Z_2^2(t),-\lambda_1 Z_3(t)+h_3(\vec{Z}(t)))\quad\text{for all } t\geq 0. \tag{2.11}$$

For any $\delta>0$ small, the functions h_1 and h_3 are C^1 on the ball $B_\delta(0)$ in $\mathbb{R}^3$ centred at 0 with radius δ. Moreover, for some constant $C_1>0$, the functions h_1 and h_3 satisfy

$$|h_1(\vec{\xi})|+|h_3(\vec{\xi})|\leq C_1\sum_{j=1}^3\xi_j^2 \ \text{and}\ |\nabla h_1(\vec{\xi})|+|\nabla h_3(\vec{\xi})|\leq C_1\sum_{j=1}^3|\xi_j| \tag{2.12}$$

for all $\vec{\xi}=(\xi_1,\xi_2,\xi_3)\in B_\delta(0)$. By (2.10), proving Lemma 2.2 is equivalent to showing that for every small $\delta>0$, there exist $r_0\in(0,\delta/2)$ and a Lipschitz map $w:[0,r_0]\times[-r_0,r_0]\to[-r_0,r_0]$ such that for all $(Y_{2,0},Z_{3,0})\in(0,r_0]\times[-r_0,r_0]$, the system (2.11) subject to

$$\vec{Z}(0)=(w(Y_{2,0},Z_{3,0}),Y_{2,0},Z_{3,0}) \tag{2.13}$$

has a solution $\vec{Z}(t)$ for all $t\geq 0$ with $\lim_{t\to+\infty}\vec{Z}(t)=(0,0,0)$. Linearising the flow for (2.11) around $(0,0,0)$ yields one null eigenvalue, and the classical Hartman–Grobman theorem does not apply to (2.11). In Appendix, using the notion of center-stable manifold and inspired by Kelley [15], we prove Theorem 7.1 that can be applied to (2.11) due to (2.12). This ends the proof. □

2.3. **Proof of Proposition 2.1.** For fixed $\varepsilon\in(0,1)$, we choose $\Psi_\varepsilon\in C^1(\mathbb{R})$ such that $\Psi_\varepsilon(t)=t^\zeta$ for all $t\geq\varepsilon$. Let $\delta\in(0,(1-\varepsilon)^{1/2})$. Let $r_0\in(0,\delta/2)$ and $w:[0,r_0]\times[-r_0,r_0]\to[-r_0,r_0]$ be given by Lemma 2.2. We fix $Y_{2,0}:=r_0/2$. Then for any fixed $Z_{3,0}\in[-r_0,r_0]$, the system (2.6), subject to the initial condition (2.8) has a solution $\vec{Y}(t)$ for all $t\geq 0$ such that (2.9) holds. Moreover, we find that $Y_2(t)=1/(t+Y_{2,0}^{-1})$ for all $t\geq 0$. Let $t_0>0$ be large such that $\vec{Y}(t)\in B_\delta(0)$ for all $t\geq t_0$. Using that $0<\varepsilon<1-\delta^2$, for all $t\geq t_0$, we get that $1-Y_1(t)Y_2(t)>\varepsilon$ so that $\Psi_\varepsilon(1-Y_1(t)Y_2(t))=(1-Y_1(t)Y_2(t))^\zeta$. Hence, we have $H_{3,\Psi_\varepsilon}(\vec{Y}(t))=H_3(\vec{Y}(t))$ for all $t\geq t_0$. For every $t\geq T:=t_0+Y_{2,0}^{-1}$, we define $\vec{X}(t)$ by $\vec{X}(t):=\vec{Y}(t-Y_{2,0}^{-1})$, which yields that $X_2(t)=1/t$. Then, $\vec{X}(t)$ is a solution of the system (2.4) for all $t\geq T$ such that $\lim_{t\to\infty}\vec{X}(t)=(0,0,0)$. With ϑ and β be given by (2.1) and $t:=r^{-\beta}$, we define $u(r)$ as in (2.2). Then u is a positive radial (ND) solution of (1.3) with $R:=T^{-1/\beta}$. The above construction leads to an infinite number of positive radial (ND) solutions for (1.3) by varying $Z_{3,0}$ in $[-r_0,r_0]$. This completes the proof. □

3. Consequences of Pohozaev's identity

In this section, using Pohozaev's identity, we prove Theorem 1.5, followed by uniform *a priori* estimates for the positive radial solutions of (1.3) satisfying (1.4) (see Proposition 3.1).

Let u be any positive solution of (1.3) with $q\in(1,2^\star-1)$ such that (1.4) holds. As in [11], for every $r\in(0,R)$, we denote by $P_r^{(q)}(u)$ the Pohazev-type integral associated to u, namely

$$P_r^{(q)}(u):=\int_{\partial B(0,r)}\left[(x,\nu)\left(\frac{|\nabla u|^2}{2}-\frac{u^{2^\star(s)}}{2^\star(s)|x|^s}+\mu\frac{u^{q+1}}{q+1}\right)-T(x,u)\,\partial_\nu u\right]d\sigma, \tag{3.1}$$

where $T(x,u)=(x,\nabla u(x))+(n-2)u(x)/2$. Here, ν denotes the unit outward normal at $\partial B(0,r)$. Assuming u satisfies (1.4), it was shown in [11] that there exists $\lim_{r\to 0^+} P_r^{(q)}(u):=P^{(q)}(u)$ and

$$P^{(q)}(u)\geq 0 \tag{3.2}$$

with strict inequality if and only if u is a (CGS) solution of (1.3). We refer to $P^{(q)}(u)$ as the *asymptotic Pohozaev integral*. We introduce the notation

$$\lambda:=(n-2)(2^\star-1-q)/2 \ \text{ and } \ c_{\mu,q,n}:=\lambda\mu/(q+1). \tag{3.3}$$

Both λ and $c_{\mu,q,n}$ are positive by the assumption $q\in(1,2^\star-1)$.

3.1. **Proof of Theorem 1.5.** Let $q\in(1,2^\star-1)$. Suppose that (1.5) admits a positive smooth solution u satisfying (1.4). From $u=0$ on $\partial\Omega$, we have $\nabla u=(\partial_\nu u)\,\nu$ for $x\in\partial\Omega$, where ν denotes the unit outward normal at $\partial\Omega$. For every $r>0$ small, by applying the Pohozaev identity as in [11, Proposition 6.1] for $\omega=\omega_r=\Omega\setminus\overline{B(0,r)}$, we get that

$$-\frac{1}{2}\int_{\partial\Omega}(x,\nu)|\nabla u|^2\,d\sigma=P_r^{(q)}(u)+c_{\mu,q,n}\int_{\omega_r}u^{q+1}\,dx. \tag{3.4}$$

By letting $r\to 0^+$ in (3.4) and using (3.2), we arrive at

$$-\frac{1}{2}\int_{\partial\Omega}(x,\nu)|\nabla u|^2\,d\sigma=P^{(q)}(u)+c_{\mu,q,n}\int_{\Omega}u^{q+1}\,dx\geq 0. \tag{3.5}$$

Since Ω is star-shaped with respect to the origin, we have $(x,\nu)>0$ on $\partial\Omega$. Then, (3.5) can only hold when $\nabla u\equiv 0$ on $\partial\Omega$ and $u\equiv 0$ in Ω. Hence, (1.5) has no positive smooth solutions satisfying (1.4). Using the comments before statement of Theorem 1.5, we finish the proof. □

3.2. **Uniform *a priori* estimates.** Let $q\in(1,2^\star-1)$. For the positive radial solutions u of (1.3) satisfying (1.4), we derive uniform *a priori* estimates. These are crucial for proving the existence of (MB) solutions in Proposition 5.1 and (CGS) solutions in Proposition 6.1. We define

$$\begin{cases}\bar{R}(u):=\sup\{R>0:\ u \text{ is a positive radial solution of (1.3)}\},\\[4pt] z(r):=r^{\frac{n-2}{2}}u(r) \text{ for } r\in(0,R),\ \ F_0(\xi):=\dfrac{(n-2)^2}{4}\xi^2-\dfrac{2}{2^\star(s)}\xi^{2^\star(s)} \ \text{ for } \xi\geq 0.\end{cases} \tag{3.6}$$

If u has a removable singularity at 0 or u is a solution of (MB) type, then $\liminf_{r\to 0^+} z(r)=0$. If u is a (CGS) solution, then from [11], we can derive that

$$0<\liminf_{r\to 0^+} z(r)\leq [(n-2)/2]^{2/(2^\star(s)-2)}:=M_0. \tag{3.7}$$

For $R>0$, we also define

$$F_R(\xi):=\frac{(n-2)^2}{4}-\frac{2}{2^\star(s)}\xi^{2^\star(s)-2}+\frac{2\mu R^\lambda\xi^{q-1}}{q+1} \ \text{ for } \xi\geq 0. \tag{3.8}$$

For F_0 given by (3.6), let Λ_0 denote the unique positive solution of $F_0(\xi)=0$, that is

$$\Lambda_0:=[(n-2)(n-s)/4]^{\frac{1}{2^\star(s)-2}}. \tag{3.9}$$

For any $\Lambda>\Lambda_0$, we have $F_0(\Lambda)<0$. Let R_Λ denote the unique $R>0$ for which $F_R(\Lambda)=0$:

$$R_\Lambda:=\left[-\frac{(q+1)F_0(\Lambda)}{2\mu\Lambda^{q+1}}\right]^{\frac{1}{\lambda}}>0. \tag{3.10}$$

Moreover, it holds

$$R_\Lambda = \sup\left\{\xi > 0 : F_{R_\Lambda}(t) > 0 \quad \text{for all } t \in (0,\xi)\right\}. \tag{3.11}$$

Proposition 3.1 (Uniform *a priori* estimates)**.** *Let $q \in (1, 2^\star - 1)$. Then for every $\Lambda > \Lambda_0$, there exists $R_\Lambda > 0$ as in (3.10) such that any positive radial solution of (1.3) with $R \in (0, R_\Lambda)$ satisfying (1.4) can be extended as a positive radial solution of (1.3) in $B(0,R_\Lambda)\setminus\{0\}$ and*

$$r^{\frac{n-2}{2}} u(r) < \Lambda \qquad \text{for all } r \in (0, R_\Lambda]. \tag{3.12}$$

Let ω_{n-1} denote the volume of the Euclidean $(n-1)$-sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$. Let λ and $c_{\mu,q,n}$ be given by (3.3). For $q > 2^\star(s) - 1$, we define ℓ_q as follows

$$\ell_q := \frac{(2-s)(q+1)}{(n-s)(q-1)} \left[\frac{(n-2)(n-s)(q-1)}{4(q-2^\star(s)+1)}\right]^{-\frac{q-2^\star(s)+1}{2^\star(s)-2}}. \tag{3.13}$$

A key tool in proving Proposition 3.1 is given by Lemma 3.2, which is of interest in its own.

Lemma 3.2. *Let $q \in (1, 2^\star - 1)$. Let u be a positive radial solution of (1.3) satisfying (1.4).*

(a) *For all $r \in (0, \bar{R})$, the functions z and $F_r(z)$ in (3.6) and (3.8), respectively satisfy*

$$z^2(r)\, F_r(z(r)) = \frac{2P^{(q)}(u)}{\omega_{n-1}} + [rz'(r)]^2 + 2c_{\mu,q,n} \int_0^r \xi^{n-1} u^{q+1}(\xi)\, d\xi. \tag{3.14}$$

(b) *If $\bar{R} < +\infty$, then $\liminf_{r\nearrow \bar{R}} u(r) > 0$ and $\limsup_{r\nearrow\bar{R}} u(r) = +\infty$.*

(c) *If $1 < q < 2^\star(s) - 1$, then $\bar{R} = +\infty$.*

(d) *If $q = 2^\star(s) - 1$, then $\bar{R} \geq (1/\mu)^{1/s}$.*

(e) *If $q \in (2^\star(s) - 1, 2^\star - 1)$, then $\bar{R} > (\ell_q/\mu)^{1/\lambda}$, where ℓ_q is given by (3.13).*

Remark 3.3. *We have $\ell_q \to 1$ as $q \searrow 2^\star(s) - 1$ and using F_0 in (3.6), we get*

$$\ell_q = \frac{q+1}{2} \sup_{\Lambda\in(\Lambda_0,\infty)} \frac{-F_0(\Lambda)}{\Lambda^{q+1}}. \tag{3.15}$$

Proof. From our assumptions, it follows that $\lim_{r\to 0^+} r^n u^{q+1}(r) = 0$.

Proof of (a). Since u is a radial solution of (1.3), the *Pohozaev-type integral* $P_r^{(q)}(u)$ satisfies

$$\frac{2P_r^{(q)}(u)}{\omega_{n-1}} = -[rz'(r)]^2 + z^2(r)\, F_r(z(r)) \quad \text{for all } r \in (0, \bar{R}). \tag{3.16}$$

By the Pohozaev identity, see [11, Proposition 6.1], for every $0 < r_1 < r < \bar{R}$, we find that

$$P_r^{(q)}(u) - P_{r_1}^{(q)}(u) = \omega_{n-1} c_{\mu,q,n} \int_{r_1}^r \xi^{n-1} u^{q+1}(\xi)\, d\xi. \tag{3.17}$$

Letting $r_1 \to 0^+$ in (3.17), for any $r \in (0, \bar{R})$, we find that

$$P_r^{(q)}(u) = P^{(q)}(u) + \omega_{n-1} c_{\mu,q,n} \int_0^r \xi^{n-1} u^{q+1}(\xi)\, d\xi. \tag{3.18}$$

Then we conclude (3.14) by using (3.16) and (3.18). The proof of (a) is now complete. $\square$

Proof of (b). Assume that $\bar{R} < +\infty$. To prove that $\liminf_{r\nearrow\bar{R}} u(r) > 0$, we proceed by contradiction. Assume that for a sequence $(r_k)_{k\geq 1}$ of positive numbers with $r_k \nearrow \bar{R}$ as $k\to\infty$, we have $\lim_{k\to\infty} u(r_k) = 0$, that is $\lim_{k\to\infty} z(r_k) = 0$. We let $r = r_k$ in (3.14), then pass to the limit $k\to\infty$ to obtain a contradiction. For the other claim in (b), assume that $\limsup_{r\nearrow\bar{R}} u(r) < +\infty$. Then $\limsup_{r\nearrow\bar{R}} z(r) < +\infty$ since $\bar{R} < +\infty$. By the classical ODE theory, it follows that $\limsup_{r\nearrow\bar{R}} |u'(r)| = \infty$. On the other hand, by (3.14), we get that $\limsup_{r\nearrow\bar{R}} |rz'(r)| < +\infty$, which shows that $\limsup_{r\nearrow\bar{R}} |u'(r)| < +\infty$. This contradiction completes the proof of (b). □

Proof of (c). Let $q < 2^\star(s)-1$. If $\bar{R} < \infty$, then there exists a sequence $(r_k)_{k\geq 1}$ in $(0,\bar{R})$ with $\lim_{k\to\infty} r_k = \bar{R}$ and $\lim_{k\to\infty} z(r_k) = +\infty$. By letting $r = r_k$ in (3.14) and $k\to\infty$, the left-hand side of (3.14) diverges to $-\infty$ as $k\to\infty$, which is a contradiction. This proves that $\bar{R} = +\infty$. □

Proof of (d). Let $q = 2^\star(s)-1$. We argue by contradiction. Assume that $\bar{R} < (1/\mu)^{1/s}$. Then, there exists $(r_k)_{k\geq 1}$ in $(0,\bar{R})$ with $\lim_{k\to\infty} r_k = \bar{R}$ and $\lim_{k\to\infty} z(r_k) = +\infty$. Since $r^\lambda = r^s < \bar{R}^s$ for all $r\in(0,\bar{R})$, from (3.14) and the definition of F_r in (3.8) (with $R = r$), we have

$$\frac{(n-2)^2 z^2(r_k)}{4} - \frac{2\,(1-\mu\bar{R}^s)\, z^{2^\star(s)}(r_k)}{2^\star(s)} > 0 \quad \text{for all } k\geq 1. \tag{3.19}$$

By letting $k\to\infty$ in (3.19) and using that $1-\mu\bar{R}^s > 0$, we get that the left-hand side of (3.19) tends to $-\infty$ as $k\to\infty$. This contradiction proves that $\bar{R} \geq (1/\mu)^{1/s}$. □

Proof of (e). Let $q\in(2^\star(s)-1, 2^\star-1)$. To prove $\bar{R} > (\ell_q/\mu)^{1/\lambda}$ with ℓ_q as in (3.13), it suffices to assume $\bar{R} < +\infty$. Let $F_{\bar{R}}$ be the function F_R in (3.8) with $R = \bar{R}$. We distinguish two cases:

CASE 1: If u has a removable singularity at 0, or u is a (MB) solution, then $\liminf_{r\to 0^+} z(r) = 0$ using that $z(r) = r^{\frac{n-2}{2}} u(r)$. Since $\limsup_{r\nearrow\bar{R}} z(r) = +\infty$, to ensure (3.14) for a positive radial solution u of (1.3) which is *not* (CGS) *nor* (ND), it is necessary to have

$$F_{\bar{R}}(\xi) > 0 \quad \text{for all } \xi\in[0,\infty). \tag{3.20}$$

We next study the monotonicity of $F_{\bar{R}}$. We see that $F_{\bar{R}}$ has only one positive critical point ξ_c defined by

$$\xi_c := \left(\frac{(2-s)(q+1)}{\mu(n-s)(q-1)\bar{R}^\lambda}\right)^{\frac{1}{q-2^\star(s)+1}}. \tag{3.21}$$

Moreover, ξ_c is a global minimum point for $\bar{F}$ on $[0,\infty)$. Thus, (3.20) holds if and only if $F_{\bar{R}}(\xi_c) > 0$, which corresponds to $\bar{R} > (\ell_q/\mu)^{1/\lambda}$.

CASE 2: If u is a radial (CGS) solution of (1.3) then we need $F_{\bar{R}}(\xi) > 0$ for every $\xi \geq \liminf_{r\to 0^+} z(r)$. If M_0 in (3.7) satisfies $M_0 \leq \xi_c$ then $\bar{R} > (\ell_q/\mu)^{1/\lambda}$ is necessary to have $F_{\bar{R}}(\xi) > 0$ for every $\xi\in[\liminf_{r\to 0^+} z(r), +\infty)$. If $M_0 > \xi_c$, then from (3.21) and (3.7), we get

$$\bar{R}^\lambda > \frac{(2-s)(q+1)}{\mu(n-s)(q-1)}\left(\frac{n-2}{2}\right)^{-\frac{2(q-2^\star(s)+1)}{2^\star(s)-2}}, \tag{3.22}$$

which again implies $\bar{R} > (\ell_q/\mu)^{1/\lambda}$.

We have established the assertion of (e) in both Cases 1 and 2. □

This completes the proof of Lemma 3.2. □

Proof of Proposition 3.1. For any $q \in [2^\star(s)-1, 2^\star-1)$, we denote $R^* = R^*(q)$ as follows

$$R^* := \begin{cases} (1/\mu)^{1/s} & \text{if } q = 2^\star(s)-1, \\ (\ell_q/\mu)^{1/\lambda} & \text{if } 2^\star(s)-1 < q < 2^\star-1. \end{cases}$$

Let $\Lambda > \Lambda_0$ be fixed. Let u be any positive radial solution of (1.3) with $R \in (0, R_\Lambda)$ such that (1.4) holds. From Lemma 3.2, the maximum radius of existence $\bar{R} = \bar{R}(u)$ for u satisfies $\bar{R} = +\infty$ if $1 < q < 2^\star(s)-1$, $\bar{R} \geq R^*$ for $q = 2^\star(s)-1$ and $\bar{R} > R^*$ for $2^\star(s)-1 < q < 2^\star-1$. From (3.15) and (3.10), we have $R_\Lambda \leq R^*$ for all $2^\star(s)-1 < q < 2^\star-1$. When $q = 2^\star(s)-1$, then using the definition of F_0 and R^*, we see easily that $R_\Lambda < R^*$. Hence, we can extend u as a positive radial solution of (1.3) in $B(0,R_\Lambda]\setminus\{0\}$ for all $1 < q < 2^\star-1$.

We now prove (3.12). Assume by contradiction that (3.12) fails, that is, $z(r_0) \geq \Lambda$ for some $r_0 \in (0, R_\Lambda]$, where $z(r) := r^{\frac{n-2}{2}} u(r)$ is defined as in (3.6). Since $z(r_0) \geq \Lambda > \Lambda_0 > M_0$, the Mean Value Theorem, together with (3.11) and (3.7), gives that there exists $r_1 \in (0, r_0)$ such that $z(r_1) = \Lambda$. Hence, using Lemma 3.2(a), we find that $0 = \Lambda^2 F_{R_\Lambda}(\Lambda) > 0$. This contradiction ends the proof of Proposition 3.1. ☐

4. Removable singularities

The assertion of Theorem 1.3(i) follows from Lemma 3.2 and Lemma 4.1 below.

Lemma 4.1. *For $q > 1$ and every $\gamma \in (0,\infty)$, there exists $R > 0$ such that (1.3) has a unique positive radial solution u_γ with a removable singularity at 0 and $\lim_{r\to 0^+} u_\gamma(r) = \gamma$.*

Proof. Fix $\gamma \in (0,\infty)$ arbitrarily. We consider the following initial value problem:

$$\begin{cases} y''(\xi) + a\, y'(\xi)/\xi + 4(y^{2^\star(s)-1} - \mu\, \xi^{\frac{2s}{2-s}}\, y^q)/(2-s)^2 = 0 \text{ for } \xi > 0, \\ y(0) = \gamma, \quad y'(0) = 0, \end{cases} \tag{4.1}$$

where we denote $a := (2n-s-2)/(2-s)$. By Biles–Robinson–Spraker [2, Theorems 1 and 2], for every $\gamma > 0$, there exists a unique positive solution y_γ of (4.1) on some interval $[0,T]$ with $T > 0$. A solution y of (4.1) is defined in [2] as follows:

(a) y and y' are absolutely continuous on $[0,T]$;
(b) y satisfies the ODE in (4.1) a.e. on $[0,T]$;
(c) y satisfies the initial conditions in (4.1).

Since $a > 1$, the function $\xi \longmapsto \xi^a y_\gamma'(\xi)$ is absolutely continuous on $[0,T]$. From (4.1), we have

$$\left(\xi^a y_\gamma'(\xi)\right)' = -\frac{4}{(2-s)^2}\, \xi^a \left(y_\gamma^{2^\star(s)-1} - \mu\, \xi^{\frac{2s}{2-s}}\, y_\gamma^q \right) \quad \text{a.e. in } [0,T].$$

Thus, for all $\xi \in [0,T]$, we find that

$$\xi^a y_\gamma'(\xi) = -\frac{4}{(2-s)^2} \int_0^\xi t^a \left(y_\gamma^{2^\star(s)-1}(t) - \mu\, t^{\frac{2s}{2-s}}\, y_\gamma^q(t) \right) dt.$$

By the property (a) for y_γ, we find that $y_\gamma \in C^2(0,T]$ satisfies the ODE in (4.1) on $(0,T]$. The change of variable $u_\gamma(r) = y_\gamma(\xi)$ with $\xi = r^{(2-s)/2}$ yields that u_γ is a positive radial $C^2(0,R]$-solution of (1.3) with $R = T^{2/(2-s)}$ and $\lim_{r\to 0^+} u_\gamma(r) = \gamma$. This proves the existence claim.

We now show the uniqueness claim: any positive radial $C^2(0,R]$-solution u of (1.3) for some $R > 0$ such that $\lim_{r\to 0^+} u(r) = \gamma$ must coincide with u_γ on their common domain of existence. Indeed, using the change of variable $u(r) = y(\xi)$ with $\xi = r^{(2-s)/2}$, we get that $y \in C^2(0, R^{(2-s)/2}]$ satisfies the differential equation in (4.1) for all $\xi \in (0, R^{(2-s)/2})$ and $\lim_{\xi\to 0^+} y(\xi) = \lim_{r\to 0^+} u(r) = \gamma$. Hence, y can be extended by continuity at 0 by defining $y(0)=\gamma$. To conclude that y is a solution of (4.1) on $[0,R^{(2-s)/2}]$ in the sense of [2], that is, y satisfies properties (a)–(c) stated above with $T=R^{(2-s)/2}$, it suffices to show that

$$y'(\xi)\to 0 \text{ and } y''(\xi)\to -2\gamma^{2^\star(s)-1}/[(n-s)(2-s)] \text{ as } \xi\to 0^+. \tag{4.2}$$

This would give that $y\in C^2[0,R^{(2-s)/2}]$, and then, by applying Theorem 2 in [2], we conclude that $y=y_\gamma$ on $[0,\min\{T,R^{(2-s)/2}\}]$, proving our uniqueness assertion.

We prove (4.2). Since u is a positive radial solution of (1.3) with $\lim_{r\to 0^+}u(r)=\gamma$, we have

$$r^{-(n-1-s)}\left(r^{n-1}u'(r)\right)'=-u^{2^\star(s)-1}+\mu r^s u^q\to -\gamma^{2^\star(s)-1}\quad \text{as } r\to 0^+. \tag{4.3}$$

Hence, the function $r\longmapsto r^{n-1}u'(r)$ is decreasing on some interval $(0,r_0)$ for small $r_0>0$. Thus, there exists $\lim_{r\to 0^+}r^{n-1}u'(r)=\theta\in(-\infty,\infty]$. We next show that $\theta=0$. Assume by contradiction that $\theta\neq 0$. Then choosing $\min\{\theta,0\}<c<\max\{\theta,0\}$, we find that $h(r)=u(r)+c(n-2)^{-1}r^{2-n}$ is decreasing (respectively, increasing) on $(0,r_1)$ for $r_1>0$ small when $\theta<0$ (respectively, when $\theta>0$). Since $\lim_{r\to 0^+}h(r)=-\infty$ if $\theta<0$ and $\lim_{r\to 0^+}h(r)=+\infty$ if $\theta>0$, we arrive at a contradiction. This proves that $\lim_{r\to 0^+}r^{n-1}u'(r)=0$. Hence, by (4.3), we get that $\lim_{r\to 0^+}r^{s-1}u'(r)=-\gamma^{2^\star(s)-1}/(n-s)$. Coming back to the ξ variable, we obtain (4.2). This ends the proof of Lemma 4.1. □

5. (MB) SOLUTIONS

In Sect. 5.1 we prove Corollary 1.6. In Sect. 5.2 we prove Theorem 1.3(ii) given as Proposition 5.1.

5.1. **Proof of Corollary 1.6.** For every $0<\ell<\min\{(n-2)/2,2\}$, we set $q:=2^\star-1-2\ell/(n-2)$ so that $q\in(2^\star-2,2^\star-1)$ with $q>1$. Then, for every $s\in(0,2)$, Theorem 1.3(ii) yields a positive radial (MB) solution u_{MB} of (1.3) for some $R>0$. We define $z(r)=r^{(n-2)/2}u_{MB}(r)$ for $r\in(0,R)$. Since $z^*:=\limsup_{r\to 0^+}z(r)\in(0,\infty)$ and $z_*=\liminf_{r\to 0^+}z(r)=0$, the asymptotics of u_{MB} at zero is different from that of any positive singular solution of (1.2). By defining

$$K(r)=1-\mu r^s u_{MB}(r)^{q-2^\star(s)+1} \text{ and } C_{s,\ell}:=2(s-\ell)/(n-2) \text{ for } r=|x|\in(0,R),$$

we see that $u=u_{MB}$ is a positive singular solution of (1.8). Moreover, we find that

$$|r^{1-\ell}K'(r)|=\mu[z(r)]^{C_{s,\ell}}\left|\ell+C_{s,\ell}\,rz'(r)/z(r)\right|\quad \text{for all } r\in(0,R). \tag{5.1}$$

We have $C_{s,\ell}>0$ when $\ell<s$ and $C_{s,\ell}<0$ when $\ell>s$. With $\underline{L}$ and $\overline{L}$ as in (1.7), we prove that

$$\underline{L}=0<\overline{L}<\infty \text{ if } \ell\in(0,s), \text{ whereas } 0<\underline{L}<\overline{L}=\infty \text{ if } \ell>s. \tag{5.2}$$

Indeed, since $P^{(q)}(u_{MB})=0$ and $z^*<\infty$, Lemma 3.2(a) yields that

$$\limsup_{r\to 0^+} r|z'(r)|/z(r)<\infty \text{ and } F_0(z(r))-[rz'(r)]^2\to 0 \text{ as } r\to 0^+, \tag{5.3}$$

where F_0 is given by (3.6). Hence, $\underline{L}=0$ and $\overline{L}<\infty$ if $\ell\in(0,s)$. Since $z_*=0$, for every $\rho\in(0,z^*)$, there exists a sequence $\{r_k\}$ of positive numbers decreasing to 0 as $k\to\infty$ such that $\lim_{k\to\infty}z(r_k)=\rho$. Then, by (5.3), we have $\lim_{k\to\infty}(r_kz'(r_k))^2=F_0(\rho)$. For suitable ρ, using r_k in (5.1), we get that $\overline{L}>0$ for $\ell\in(0,s)$, and correspondingly $0<\underline{L}<\infty$ for $\ell>s$. It remains to show that $\overline{L}=\infty$ if $\ell>s$. Assuming the contrary, $R_kz'(R_k)/z(R_k)\to -\ell/C_{s,\ell}$ for every sequence $\{R_k\}$ of positive numbers decreasing to 0 such that $\lim_{k\to\infty}z(R_k)=0$. Lemma 3.2(a) gives that $F_{R_k}(z(R_k))\geq[R_kz'(R_k)/z(R_k)]^2$. Letting $k\to\infty$, we would have $(n-2)^2/4\geq \ell^2/C_{s,\ell}^2$, which is a contradiction with $s>0$. Thus, (5.2) holds and K satisfies the properties in Corollary 1.6. □

5.2. **Existence of (MB) solutions.**

Proposition 5.1. *Let $q \in (1, 2^\star - 1)$. Assuming that $q > 2^\star - 2$, then for $R > 0$ small, (1.3) admits at least a positive radial (MB) solution u, that is,*

$$\liminf_{r \to 0^+} r^{\frac{n-2}{2}} u(r) = 0 \quad \textit{and} \quad \limsup_{r \to 0^+} r^{\frac{n-2}{2}} u(r) \in (0, \infty). \tag{5.4}$$

Proof. We use an argument inspired by Chen–Lin [8]. Let $(\gamma_i)_{i \geq 1}$ be an increasing sequence of positive numbers with $\lim_{i \to \infty} \gamma_i = \infty$. By Lemmas 4.1 and 3.2, for every $i \geq 1$, there exists $R_i > 0$ such that (1.3), subject to $\lim_{|x| \to 0^+} u(x) = \gamma_i$, admits a unique positive radial $C^2(0, R_i]$-solution u_{γ_i}. From now on, we use u_i instead of u_{γ_i}. Let $\Lambda > \Lambda_0$ be fixed, where Λ_0 is given by (3.9). By Proposition 3.1, there exists $R_\Lambda > 0$ such that u_i can be extended as a positive radial $C^2(0, R_\Lambda] \cap C[0, R_\Lambda]$-solution of (1.3) in $(0, R_\Lambda]$ satisfying

$$u_i(0) = \gamma_i, \quad r^{\frac{n-2}{2}} u_i(r) \leq \Lambda \text{ for all } r \in (0, R_\Lambda] \text{ and every } i \geq 1. \tag{5.5}$$

CLAIM: For any $u_0 > 0$, there exist $r_0 \in (0, R_\Lambda)$ and $i_0 \geq 1$ such that

$$u_i(r_0) \geq u_0 \quad \text{for all } i \geq i_0.$$

We now complete the proof of Proposition 5.1 assuming the Claim. From (5.5), there exists a subsequence of (u_i), relabelled (u_i), converging uniformly to u_∞ on any compact subset of $(0, R_\Lambda]$. Moreover, $u_i \to u_\infty$ in $C^2_{\text{loc}}(0, R_\Lambda]$ and u_∞ is a radial solution of (1.3). The above Claim yields $\limsup_{r \to 0^+} u_\infty(r) = \infty$, that is, u_∞ has a non-removable singularity at 0. By (5.5), we get $\limsup_{r \to 0^+} r^{\frac{n-2}{2}} u_\infty(r) \in (0, \infty)$. Since $q < 2^\star - 1$, we thus find that $u_\infty^{q+1} \in L^1(B(0, R_\Lambda))$. We have $P^{(q)}(u_i) = 0$ for all $i \geq 1$. By letting $u = u_i$ in (3.18) and (3.16), then passing to the limit $i \to +\infty$, we find that

$$P_r^{(q)}(u_\infty) = c_{\mu,q,n} \int_{B(0,r)} u_\infty^{q+1}(x)\, dx \quad \text{for all } r \in (0, R_\Lambda]. \tag{5.6}$$

By letting $r \to 0^+$ in (5.6), we find that $P^{(q)}(u_\infty) = 0$. Hence by (3.2), u_∞ is not a (CGS) solution of (1.3). As u_∞ does not have a removable singularity at 0, we conclude that u_∞ is a radial (MB) solution of (1.3), that is u_∞ satisfies (5.4). This ends the proof of Proposition 5.1. □

Proof of the Claim. Suppose the contrary. Then for some $u_0 > 0$ and any $r_0 \in (0, R_\Lambda)$, there exists a subsequence of u_i, relabeled (u_i), such that

$$u_i(r_0) < u_0 \quad \text{for all } i \geq 1. \tag{5.7}$$

We apply the following transformation

$$w_i(t) = r^{\frac{n-2}{2}} u_i(r) \quad \text{with } t = \log r. \tag{5.8}$$

By $w_i'(t)$ and $w_i''(t)$, we denote the first and second derivative of w_i with respect to t, respectively. Then w_i satisfies the equation

$$w_i''(t) - f(w_i(t)) = \mu e^{\lambda t} w_i^q(t) \quad \text{for } -\infty < t < \log R_\Lambda, \tag{5.9}$$

where $\lambda := (n-2)(2^\star - 1 - q)/2$ and $f : [0, \infty) \to \mathbb{R}$ is defined by

$$f(\xi) := (n-2)^2 \xi / 4 - \xi^{2^\star(s) - 1} \quad \text{for all } \xi \geq 0. \tag{5.10}$$

From (5.5), we have that

$$w_i(t) \in (0, \Lambda] \quad \text{for all } t \in (-\infty, \log R_\Lambda] \text{ and } i \geq 1. \tag{5.11}$$

The proof of the Claim is now divided into five steps:

Step 5.1. *The family $(w_i'(t))_{i\geq 1}$ is uniformly bounded on $(-\infty, \log R_\Lambda]$.*

Proof of Step 5.1. Using F_R in (3.8) with $R = e^t$, we define $E_i : (-\infty, \log R_\Lambda] \to \mathbb{R}$ by

$$E_i(t) := \left(w_i'(t)\right)^2 - w_i^2(t)\, F_{e^t}(w_i(t)). \tag{5.12}$$

We have $\lambda > 0$ (since $q < 2^\star - 1$) and $\lim_{t\to -\infty} w_i(t) = 0$. By Lemma 3.2(a), we find that

$$E_i(t) = -2c_{\mu,q,n} \int_{-\infty}^{t} e^{\lambda \xi} w_i^{q+1}(\xi)\, d\xi \quad \text{and} \quad E_i'(t) = -2c_{\mu,q,n}\, e^{\lambda t} w_i^{q+1}(t) < 0 \tag{5.13}$$

for all $t \in (-\infty, \log R_\Lambda)$. It follows that

$$\lim_{t\to -\infty} w_i'(t) = \lim_{t\to -\infty} E_i(t) = 0. \tag{5.14}$$

From (5.13), we have $E_i < 0$ on $(-\infty, \log R_\Lambda]$. Thus, by (5.11), we get that $(w_i'(t))_{i\geq 1}$ is uniformly bounded for $t \in (-\infty, \log R_\Lambda]$, completing Step 5.1. □

Step 5.2. *For $\varepsilon_0 > 0$ and $r_0 \in (0, R_\Lambda)$ small such that $r_0^{(n-2)/2} u_0 < \varepsilon_0/2$, we set*

$$\mathscr{F}_i := \{t \in (-\infty, \log r_0) : \ w_i(t) \geq \varepsilon_0\} \quad \textit{for all } i \geq 1.$$

Then there exists $i_0 \geq 1$ such that

$$w_i(\log r_0) < \varepsilon_0/2 \textit{ and } \mathscr{F}_i \neq \emptyset \quad \textit{for every } i \geq i_0.$$

Proof of Step 5.2. For $0 < \varepsilon_0 < [(n-2)/2]^{(n-2)/(2-s)}$, we define

$$\beta_0 := 2\varepsilon_0^{2^\star(s)-2} \left(n - 2 + \sqrt{(n-2)^2 - 4\varepsilon_0^{2^\star(s)-2}}\right)^{-1} \quad \text{so that } 0 < \beta_0 < \frac{n-2}{2} \text{ is small.} \tag{5.15}$$

Since $\beta_0 \to 0^+$ as $\varepsilon_0 \to 0^+$, we can take $\varepsilon_0 > 0$ small enough such that β_0 is smaller than $\max\{(n-2)/4, 2/q, (2-s)/(2^\star(s)-1)\}$. Our choice of r_0 and (5.7) yield that

$$w_i(\log r_0) = r_0^{\frac{n-2}{2}} u_i(r_0) < r_0^{\frac{n-2}{2}} u_0 < \varepsilon_0/2 \quad \text{for all } i \geq 1. \tag{5.16}$$

To end Step 5.2, we show by contradiction that there exists $i_0 \geq 1$ such that $\mathscr{F}_i \neq \emptyset$ for all $i \geq i_0$. Indeed, suppose that for a subsequence $(w_{i_k})_{k\geq 1}$ of $(w_i)_{i\geq 1}$, we have

$$w_{i_k}(t) < \varepsilon_0 \quad \text{for all } t \in (-\infty, \log r_0] \text{ and every } k \geq 1. \tag{5.17}$$

Let $k \geq 1$ be arbitrary. Using (5.17) and (5.15), we infer that

$$\beta_0\,(n - 2 - \beta_0) = \varepsilon_0^{2^\star(s)-2} > w_{i_k}^{2^\star(s)-2}(t) \tag{5.18}$$

for all $t \leq \log r_0$. From (5.9) and (5.18), we obtain that

$$w_{i_k}''(t) > [(n-2)/2 - \beta_0]^2\, w_{i_k}(t) \quad \text{for all } t \leq \log r_0. \tag{5.19}$$

In particular, $t \longmapsto w_{i_k}'(t)$ is increasing on $(-\infty, \log r_0]$. Since $\lim_{t\to -\infty} w_{i_k}'(t) = 0$, we find that $w_{i_k}'(t) > 0$ for all $t \leq \log r_0$. Set

$$\mathscr{G}_{i_k}(t) := (w_{i_k}'(t))^2 - [(n-2)/2 - \beta_0]^2\, w_{i_k}^2(t).$$

Using (5.19), we get that $\mathscr{G}_{i_k}$ is increasing on $(-\infty, \log r_0]$ and $\lim_{t\to -\infty} \mathscr{G}_{i_k}(t) = 0$. Thus, $\mathscr{G}_{i_k} > 0$ on $(-\infty, \log r_0]$, which implies that

$$w_{i_k}'(t) > [(n-2)/2 - \beta_0]\, w_{i_k}(t) \quad \text{for all } t \leq \log r_0.$$

Thus, $t \longmapsto e^{-(\frac{n-2}{2} - \beta_0)t} w_{i_k}(t)$ is increasing on $(-\infty, \log r_0]$. Using (5.8) and (5.16), we find that

$$u_{i_k}(r) \leq c_0\, r^{-\beta_0} \quad \text{for every } r \in (0, r_0] \text{ and all } k \geq 1, \tag{5.20}$$

where $c_0 := (\varepsilon_0/2)\, r_0^{-\frac{n-2}{2}+\beta_0}$. Since β_0 can be made arbitrarily small, it follows from (5.20) that the right-hand side of (1.3) with $u = u_{i_k}$ is uniformly bounded in $L^p(B(0,r_0))$ for some $p > n/2$. Then, u_{i_k} satisfies (1.3) in $\mathscr{D}'(B(0,r_0))$ (in the sense of distributions) and $(u_{i_k})_{k\geq 1}$ is uniformly bounded in $W^{2,p}(B(0,r_0))$ for some $p > n/2$. Hence, $(u_{i_k}(r))_{k\geq 1}$ is uniformly bounded in $r \in [0, r_0/2]$, which leads to a contradiction with $u_{i_k}(0) = \gamma_{i_k} \to \infty$ as $k \to \infty$. This ends the proof of Step 5.2. □

For $i \geq i_0$, we define

$$t_i := \sup\{t \in (-\infty, \log r_0) :\ w_i(t) \geq \varepsilon_0\}.$$

It follows from Step 5.2 that t_i is well-defined and that $t_i \in (-\infty, \log r_0)$ for all $i \geq i_0$.

Step 5.3. *We claim that for every $i \geq i_0$, the function w_i is decreasing on $[t_i, \bar{t}_i]$ for some $\bar{t}_i \in (t_i, \log r_0]$. Moreover, by diminishing $\varepsilon_0 > 0$ and $r_0 > 0$, there exist positive constants c_1, c_2 independent of ε_0 and i such that*

$$w_i(\bar{t}_i) \geq c_1 \varepsilon_0^{\frac{q+2}{2}} e^{\frac{\lambda t_i}{2}} \quad \text{and} \quad \bar{t}_i - t_i \leq \frac{2}{n-2} \log \frac{\varepsilon_0}{w_i(\bar{t}_i)} + c_2. \tag{5.21}$$

Moreover, if $\bar{t}_i < \log r_0$, then w_i is increasing on $[\bar{t}_i, \log r_0]$ and

$$\log r_0 - \bar{t}_i \leq \frac{2}{n-2} \log \frac{w_i(\log r_0)}{w_i(\bar{t}_i)} + c_3, \tag{5.22}$$

where $c_3 > 0$ is a constant independent of ε_0 and i.

Proof of Step 5.3. Let $i \geq i_0$ be arbitrary. By Step 5.2, we have $w_i(t) \leq \varepsilon_0$ for every $t \in [t_i, \log r_0]$. Since (5.18) holds for all $t \in [t_i, \log r_0]$, as in the proof of Step 5.2, we regain (5.19) replacing w_{i_k} by w_i for all $t \in [t_i, \log r_0]$. Hence, $t \longmapsto w_i'(t)$ is increasing on $[t_i, \log r_0]$ since $w_i''(t) > 0$ for all $t \in [t_i, \log r_0]$. We next distinguish two cases:

CASE 1: $w_i'(t) \neq 0$ for all $t \in [t_i, \log r_0)$. Hence, $w_i' < 0$ on $[t_i, \log r_0)$ using that $w_i < \varepsilon_0$ on $(t_i, \log r_0]$.

CASE 2: $w_i'(\bar{t}_i) = 0$ for some $\bar{t}_i \in [t_i, \log r_0)$. Then, $w_i' < 0$ on $[t_i, \bar{t}_i)$ and $w_i' > 0$ on $(\bar{t}_i, \log r_0]$.

In both cases, w_i is decreasing on $[t_i, \bar{t}_i]$ such that

(1) $\bar{t}_i = \log r_0$ in Case 1;
(2) $\bar{t}_i \in (t_i, \log r_0)$ and $w_i'(t) > 0$ for all $t \in (\bar{t}_i, \log r_0]$ in Case 2.

Unless explicitly mentioned, the argument below applies for both Case 1 (when $\bar{t}_i = \log r_0$) and Case 2 (when $\bar{t}_i \in (t_i, \log r_0)$).

From (5.9), we have that

$$w_i''(t) \geq f(w_i(t)) \quad \text{for all } t \in [t_i, \log r_0]. \tag{5.23}$$

Thus, using (5.23), we find that

$$t \longmapsto \left(w_i'(t)\right)^2 - F_0(w_i(t)) \tag{5.24}$$

(a) is non-increasing on $[t_i, \bar{t}_i]$ (in both Case 1 and Case 2);
(b) is non-decreasing on $[\bar{t}_i, \log r_0]$ in Case 2.

Proof of the first inequality in (5.21). By (5.16) and $w_i(t_i) = \varepsilon_0$, we infer that there exists $\widetilde{t}_i \in (t_i, \log r_0)$ such that $w_i(\widetilde{t}_i) = \varepsilon_0/2$ and, moreover, $\bar{t}_i \in (\widetilde{t}_i, \log r_0]$. Hence, there exists $\xi_i \in [t_i, \widetilde{t}_i]$ such that

$$-\varepsilon_0/2 = w_i(\widetilde{t}_i) - w_i(t_i) = w_i'(\xi_i)(\widetilde{t}_i - t_i).$$

By Step 5.1, $(|w_i'(t)|)_{i\geq 1}$ is uniformly bounded on $(-\infty, \log R_\Lambda)$ so that

$$\widetilde{t}_i - t_i \geq c\varepsilon_0 \quad \text{for some constant } c > 0. \tag{5.25}$$

From (5.11), (5.12) and (5.13), there exists $\widetilde{c} > 0$ such that

$$-\widetilde{c}\, w_i^2(t) \leq E_i(t) \leq E_i(\widetilde{t}_i) \quad \text{for every } \widetilde{t}_i < t \leq \log r_0. \tag{5.26}$$

Moreover, using (5.25), together with $E_i(t_i) < 0$ and $w_i \geq \varepsilon_0/2$ on $[t_i, \widetilde{t}_i]$, we obtain that

$$E_i(\widetilde{t}_i) = E_i(t_i) - \frac{2\lambda\mu}{q+1}\int_{t_i}^{\widetilde{t}_i} e^{\lambda t} w_i^{q+1}(t)\, dt \leq -\frac{\lambda\mu c\, \varepsilon_0^{q+2} e^{\lambda t_i}}{2^q (q+1)}. \tag{5.27}$$

Since $\bar{t}_i \in (\widetilde{t}_i, \log r_0]$, by combining (5.26) and (5.27), there exists $c_1 > 0$ such that

$$w_i(\bar{t}_i) \geq c_1 \varepsilon_0^{\frac{q+2}{2}}\, e^{\frac{\lambda t_i}{2}}, \tag{5.28}$$

where $c_1 > 0$ is independent of ε_0 and i. □

Proof of the second inequality in (5.21). From (5.24), for all $t \in [t_i, \bar{t}_i)$, we have

$$[w_i'(t)]^2 - F_0(w_i(t)) \geq -F_0(w_i(\bar{t}_i)), \tag{5.29}$$

which jointly with $w_i'(t) < 0$ and F_0 increasing on $[0, \varepsilon_0]$, yields that

$$-w_i'(t)\, [F_0(w_i(t)) - F_0(w_i(\bar{t}_i))]^{-1/2} \geq 1 \text{ for all } t \in [t_i, \bar{t}_i). \tag{5.30}$$

Hence, for all $t \in [t_i, \bar{t}_i)$, by integrating (5.30) over $[t, \bar{t}_i]$, we get that

$$\bar{t}_i - t \leq \int_{w_i(\bar{t}_i)}^{w_i(t)} \frac{d\eta}{[F_0(\eta) - F_0(w_i(\bar{t}_i))]^{1/2}} =: \mathscr{D}_i(t). \tag{5.31}$$

We shall prove below that

$$\mathscr{D}_i(t) \leq \frac{2}{n-2}\left(\log \frac{w_i(t)}{w_i(\bar{t}_i)} + \log 2\right) + \tilde{k} w_i^{2^\star(s)-2}(t) \tag{5.32}$$

for all $t \in [t_i, \bar{t}_i)$, where $\tilde{k} > 0$ is a constant independent of ε_0 and i. Then, since $w_i \leq \varepsilon_0$ on $[t_i, \bar{t}_i]$, from (5.31) an (5.32), we conclude the proof of the second inequality in (5.21).

Proof of (5.32). For every $\xi \geq 0$, we define

$$g_i(\xi) := \left(\frac{n-2}{2}\right)^2 \xi^2 - \frac{2}{2^\star(s)}\, \xi^{2^\star(s)}\, [w_i(\bar{t}_i)]^{2^\star(s)-2}. \tag{5.33}$$

By a change of variable, we find that

$$\mathscr{D}_i(t) = \int_1^{w_i(t)/w_i(\bar{t}_i)} \frac{d\xi}{[g_i(\xi) - g_i(1)]^{1/2}} \quad \text{for all } t \in [t_i, \bar{t}_i). \tag{5.34}$$

By the definition of g_i in (5.33), for each $\xi > 1$, we have

$$\frac{g_i(\xi) - g_i(1)}{\xi^2 - 1} = \left(\frac{n-2}{2}\right)^2 - \frac{2[w_i(\bar{t}_i)]^{2^\star(s)-2}}{2^\star(s)}\, \frac{\xi^{2^\star(s)} - 1}{\xi^2 - 1}. \tag{5.35}$$

Since $(2^\star(s) - 1)\, \xi^{2^\star(s)} - 2^\star(s)\, \xi^{2^\star(s)-2} + 1$ increases for $\xi \geq 1$, we get that $\xi^{2^\star(s)} - 1$ is bounded from above by $2^\star(s)\, \xi^{2^\star(s)-2}(\xi^2 - 1)$ for all $\xi \geq 1$. Hence, for any $1 < \xi \leq \varepsilon_0/w_i(\bar{t}_i)$, we find that

$$\frac{[w(\bar{t}_i)]^{2^\star(s)-2}}{2^\star(s)}\, \frac{\xi^{2^\star(s)} - 1}{\xi^2 - 1} \leq [w_i(\bar{t}_i)\, \xi]^{2^\star(s)-2} \leq \varepsilon_0^{2^\star(s)-2}. \tag{5.36}$$

Since we fix $\varepsilon_0>0$ small, there exists a positive constant k, independent of ε_0 and i, such that

$$\left[\frac{\xi^2-1}{g_i(\xi)-g_i(1)}\right]^{1/2}\leq \frac{2}{n-2}+2k\,[w_i(\bar t_i)\,\xi]^{2^\star(s)-2} \tag{5.37}$$

for every $1<\xi\leq \varepsilon_0/w_i(\bar t_i)$. Since $w_i(t)\leq w_i(t_i)\leq\varepsilon_0$ for each $t\in[t_i,\bar t_i)$, using (5.37) in (5.34), we get

$$\mathscr{D}_i(t)\leq \frac{2}{n-2}\int_1^{h_i(t)}\frac{d\xi}{[\xi^2-1]^{1/2}}+2k\,[w_i(\bar t_i)]^{2^\star(s)-2}\,\mathscr{E}_i(t), \tag{5.38}$$

where for every $t\in[t_i,\bar t_i)$, we define $h_i(t)$ and $\mathscr{E}_i(t)$ by

$$h_i(t):=\frac{w_i(t)}{w_i(\bar t_i)}\quad\text{and}\quad \mathscr{E}_i(t):=\int_1^{h_i(t)}\frac{\xi^{2^\star(s)-2}}{(\xi^2-1)^{1/2}}\,d\xi. \tag{5.39}$$

A simple calculation gives that there exists $C>0$ such that for every $t\in[t_i,\bar t_i)$, we have

$$\mathscr{E}_i(t)\leq C h_i^{2^\star(s)-2}(t). \tag{5.40}$$

Using (5.39) and (5.40) into (5.38), we reach (5.32) with $\tilde k$ large enough. This completes the proof of the inequalities in (5.21). □

Proof of (5.22) *in Case 2 (when $\bar t_i\in(t_i,\log r_0)$).* Recall that w_i is increasing on $[\bar t_i,\log r_0]$ so that using (5.16), we get that $w_i(t)\leq w_i(\log r_0)<\varepsilon_0/2$ for all $t\in[\bar t_i,\log r_0]$. Moreover, the function in (5.24) is non-decreasing on $[\bar t_i,\log r_0]$. Hence, we recover (5.29) for all $t\in(\bar t_i,\log r_0]$. Since this time $w_i'>0$ on $(\bar t_i,\log r_0]$, instead of (5.31), we find that

$$w_i'(t)\,[F_0(w_i(t))-F_0(w_i(\bar t_i))]^{-1/2}\geq 1\quad\text{for every } t\in(\bar t_i,\log r_0]. \tag{5.41}$$

Using $\mathscr{D}_i(t)$ given by (5.31), we see that by integrating (5.41) over $[\bar t_i,t]$, we obtain that

$$t-\bar t_i\leq \mathscr{D}_i(t)\quad\text{for all } t\in(\bar t_i,\log r_0]. \tag{5.42}$$

Similar to the case $t\in[t_i,\bar t_i)$, we can prove (5.32) for all $t\in(\bar t_i,\log r_0]$, which jointly with (5.42), gives the existence of a constant $c_3>0$ independent of ε_0 and i such that

$$t-\bar t_i\leq \frac{2}{n-2}\log\frac{w_i(t)}{w_i(\bar t_i)}+c_3\quad\text{for all } t\in(\bar t_i,\log r_0]. \tag{5.43}$$

By letting $t=\log r_0$ in (5.43), we conclude (5.22). This proves the assertions of Step 5.3. □

Step 5.4. *Proof of the Claim concluded in Case 1 of Step 5.3: $\bar t_i=\log r_0$.*

Proof of Step 5.4. Suppose that $w_i'<0$ on $[t_i,\log r_0)$.

The second inequality in (5.21) of Step 5.3 reads as follows

$$\log r_0-t_i\leq\frac{2}{n-2}\log\frac{\varepsilon_0}{w_i(\log r_0)}+c_2. \tag{5.44}$$

The first inequality in (5.21) and (5.16) give that $r_0^{\frac{n-2}{2}}u_0\geq c_1\varepsilon_0^{\frac{q+2}{2}}e^{\frac{\lambda t_i}{2}}$. By applying log to this inequality and to (5.28) (with $\bar t_i=\log r_0$), respectively, we find that

$$\lambda t_i/2\leq[(n-2)/2]\log r_0+c_4\,(\log u_0+\log(1/\varepsilon_0)) \tag{5.45}$$

for some constant $c_4>0$ independent of ε_0 and i, respectively

$$\log(w_i(\log r_0))\geq \lambda t_i/2+[(q+2)/2]\log\varepsilon_0+\log c_1. \tag{5.46}$$

Using (5.46) into (5.44), we deduce that

$$\log r_0 \le [1-\lambda/(n-2)]\, t_i + c_5 \log(1/\varepsilon_0) \tag{5.47}$$

for a constant $c_5 > 0$ independent of ε_0 and i. We have

$$\Theta := 2(q-2^\star+2)/(2^\star-1-q) > 0 \quad \text{since } q \in (2^\star-2, 2^\star-1). \tag{5.48}$$

Plugging into (5.47) the estimate on t_i from (5.45), we conclude that

$$-\Theta \log r_0 \le c_6 \left[\log u_0 + \log(1/\varepsilon_0)\right], \tag{5.49}$$

where c_6 is a positive constant independent of ε_0 and i. Since $\Theta > 0$, we can choose $r_0 > 0$ small so that the left-hand side of (5.49) is bigger than twice the right-hand side of (5.49), which is a contradiction with (5.49). This completes Step 5.4. □

Step 5.5. *Proof of the Claim in Case 2 of Step 5.3: $\bar{t}_i \in (t_i, \log r_0)$.*

Proof of Step 5.5. We have $w_i' < 0$ on $[t_i, \bar{t}_i)$ and $w_i' > 0$ on $(\bar{t}_i, \log r_0]$. The first inequality of (5.21) yields

$$2\log w_i(\bar{t}_i) \ge (q+2)\log \varepsilon_0 + \lambda t_i + 2\log c_1. \tag{5.50}$$

By adding the second inequality of (5.21) to that of (5.22), we get

$$\log r_0 - t_i \le \frac{2}{n-2}\left[\log \varepsilon_0 + \log w_i(\log r_0) - 2\log w_i(\bar{t}_i)\right] + C_1, \tag{5.51}$$

where $C_1 > 0$ is a constant independent of ε_0 and i. By (5.16), we have

$$\log w_i(\log r_0) \le \log u_0 + [(n-2)/2]\,\log r_0. \tag{5.52}$$

Using (5.50) and (5.52) into (5.51), we obtain that

$$[2\lambda/(n-2) - 1]\, t_i \le C_2 \left[\log(1/\varepsilon_0) + \log u_0\right], \tag{5.53}$$

where $C_2 > 0$ is a a constant independent of ε_0 and i. Since the coefficient of t_i in (5.53) equals $2^\star - 2 - q$, which is negative from the assumption $q > 2^\star - 2$, using that $t_i < \log r_0$, we infer that

$$(2^\star - 2 - q)\log r_0 \le C_2 \left[\log(1/\varepsilon_0) + \log u_0\right]. \tag{5.54}$$

By choosing $r_0 > 0$ small so that the left-hand side of (5.54) is greater than twice the right-hand side of (5.54), we reach a contradiction. This proves Step 5.5. □

From Steps 5.4 and 5.5 above, we conclude the proof of the Claim. □

6. (CGS) SOLUTIONS

This section is devoted to the proof of part (iii) of Theorem 1.3, restated below.

Proposition 6.1. *Let $q \in (1, 2^\star - 1)$. There exists $R_0 > 0$ such that for every $R \in (0, R_0)$ and any positive singular solution U of (1.2), there exists a unique positive radial (CGS) solution u of (1.3) with asymptotic profile U near zero.*

Proof. Let f be given by (5.10). Let U be a positive singular solution of (1.2). Then, by defining $\varphi(t) = e^{-(n-2)t/2} U(e^{-t})$ for $t \in \mathbb{R}$, we see that $\varphi \in C^\infty(\mathbb{R})$ is a positive periodic solution of

$$\varphi''(t) = f(\varphi(t)) \quad \text{for all } t \in \mathbb{R}. \tag{6.1}$$

Let $\mathscr{P}$ denote the set of all positive smooth periodic solutions of (6.1) to be described in Sect. 6.1. We next show that Proposition 6.1 is equivalent to Lemma 6.2, the proof of which will be given in Sect. 6.2.

Lemma 6.2. *Let $q \in (1, 2^\star - 1)$. For every $\varphi \in \mathscr{P}$, there exists $T_0 = T_0(\varphi) > 0$ large for which the non-autonomous first order system*

$$\begin{cases} (V', W') = (W, f(V) + \mu e^{-\lambda t} V^q) & \text{in } [T_0, \infty), \\ V > 0 \text{ on } [T_0, \infty), \end{cases} \tag{6.2}$$

has a unique solution satisfying

$$(V(t), W(t)) - \big(\varphi(t), \varphi'(t)\big) \to (0,0) \quad \text{as } t \to \infty. \tag{6.3}$$

Indeed, assuming that Proposition 6.1 holds, then for every $\varphi \in \mathscr{P}$, we use the transformation

$$\varphi(t) = r^{\frac{n-2}{2}} U(r), \quad V(t) = r^{\frac{n-2}{2}} u(r), \quad W(t) = V'(t) \quad \text{with } t = \log(1/r), \tag{6.4}$$

where u is the unique positive radial (CGS) solution of (1.3) satisfying $\lim_{r\to 0^+} u(r)/U(r) = 1$. Hence, we obtain that (V, W) is a solution of (6.2) for any $T_0 > \log(1/R)$ and, moreover, $V(t) - \varphi(t) \to 0$ as $t \to \infty$. Using (6.1), we find that $W'(t) - \varphi''(t) \to 0$ as $t \to \infty$. Hence, $W - \varphi'$ is uniformly continuous on $[T_0, +\infty)$. Since $\lim_{t\to\infty}(V - \varphi)(t) = 0$, we get that $\lim_{t\to+\infty}(W - \varphi')(t) = 0$. This proves Lemma 6.2. We prove the reverse implication. If Lemma 6.2 holds, then for every positive singular solution U of (1.2), using (6.4) and Proposition 3.1, we get a unique positive radial (CGS) solution u of (1.3) satisfying $\lim_{r\to 0^+} u(r)/U(r) = 1$.

6.1. **Description of $\mathscr{P}$.** We show that the set $\mathscr{P}$ of all positive smooth periodic solutions of (6.1) is given by (6.7). This is basically standard ODE theory. We state only the essential steps and leave the details to the reader. The function F_0 in (3.6) is increasing on $[0, M_0]$ and decreasing on $[M_0, \infty)$ with M_0 given by (6.5). Thus F_0 reaches its maximum $\overline{\sigma}$ at M_0, where

$$M_0 := \left(\frac{n-2}{2}\right)^{\frac{n-2}{2-s}} \quad \text{and} \quad \overline{\sigma} := F_0(M_0) = \frac{2-s}{n-s}\left(\frac{n-2}{2}\right)^{\frac{2(n-2)}{2-s}}. \tag{6.5}$$

Note that M_0 is the only positive zero of $f(\xi) = 0$. Let $\varphi \in \mathscr{P}$. Since $F_0(\xi) = 2\int_0^\xi f(t)\,dt$ for all $\xi \geq 0$, from (6.1), there exists a constant $\sigma > 0$ such that

$$\big(\varphi'(t)\big)^2 = F_0(\varphi(t)) - \sigma \quad \text{for all } t \in \mathbb{R}. \tag{6.6}$$

In fact, by taking $\mu = 0$ in (3.16) for $u = U$ with U given by (6.4), we precisely obtain that $\sigma = 2P(U)/\omega_{n-1} > 0$, where $P(U)$ is the Pohozaev invariant associated to the positive singular solution U of (1.2). From (6.5) and (6.6), we must have

$$0 < \sigma \leq \overline{\sigma} \quad \text{and} \quad \varphi \equiv M_0 \text{ on } \mathbb{R} \text{ if } \sigma = \overline{\sigma}.$$

Let $\sigma \in (0, \overline{\sigma})$ be fixed. Let a_σ and b_σ denote the two positive solutions of $F_0(\xi) = \sigma$ with $0 < a_\sigma < M_0 < b_\sigma$. It follows from standard analysis of the ODE (6.1) that for any $\sigma \in (0, \bar{\sigma})$, there is a unique solution φ_σ to (6.1) such that $\min_{\mathbb{R}} \varphi_\sigma = \varphi_\sigma(0) = a_\sigma < b_\sigma = \max_{\mathbb{R}} \varphi_\sigma$. Moreover, φ_σ is periodic and we let $2t_\sigma > 0$ be its principal period.

For every $\tau \in \mathbb{S}^1$, let $\varphi_{\sigma,\tau}$ denote the function whose graph is obtained from that of φ_σ by a horizontal shift with $(t_\sigma/\pi)\mathrm{Arg}\ \tau$ units, where $\mathrm{Arg}\,\tau$ denotes the principal argument of τ. Note that $\varphi_\sigma = \varphi_{\sigma,\tau_0}$ with $\tau_0 = (1,0) \in \mathbb{S}^1$. It follows that

$$\mathscr{P} = \{\varphi_{\overline{\sigma}}\} \cup \{\varphi_{\sigma,\tau}\}_{(\sigma,\tau)\in(0,\overline{\sigma})\times\mathbb{S}^1}, \tag{6.7}$$

where $\varphi_{\overline{\sigma}} \equiv M_0$ and $\varphi_{\sigma,\tau}(t) = \varphi_\sigma\left(t - (t_\sigma/\pi)\mathrm{Arg}\,\tau\right)$ for all $t \in \mathbb{R}$.

6.2. **Proof of Lemma 6.2.** We first prove Lemma 6.2 for $\varphi\in\cup\{\varphi_{\sigma,\tau}\}_{(\sigma,\tau)\in(0,\sigma_0)\times\mathbb{S}^1}$ with $\sigma_0\in(0,\overline{\sigma})$ and second for $\varphi\in\{\varphi_{\overline{\sigma}}\}\cup\{\varphi_{\sigma,\tau}\}_{(\sigma,\tau)\in[\sigma_0,\overline{\sigma})\times\mathbb{S}^1}$ with $\sigma_0\in(0,\overline{\sigma})$ close enough to $\overline{\sigma}$.

Step 6.1. *For any $\sigma_*\in(0,\sigma_0)$ fixed, there exists $T_0>0$ large such that for every $\varphi=\varphi_{\sigma,\tau}$ with $(\sigma,\tau)\in(\sigma_*,\sigma_0)\times\mathbb{S}^1$, the system (6.2), subject to (6.3), admits a unique solution $(V_{\sigma,\tau},W_{\sigma,\tau})$.*

Proof of Step 6.1. For the existence proof, we make a suitable transformation and use the Fixed Point Theorem for a contraction mapping. Let I_0 be an open interval such that $(\sigma_*,\sigma_0)\Subset I_0\Subset(0,\overline{\sigma})$. The key here is that for every $(\sigma,\tau)\in I_0\times\mathbb{S}^1$, both $\varphi_{\sigma,\tau}$ and $\partial_t\varphi_{\sigma,\tau}=\varphi'_{\sigma,\tau}$ are differentiable with respect to σ. This does not hold for $\sigma=\overline{\sigma}$. By differentiating (6.6) with respect to σ and using (6.1), we get

$$f(\varphi_{\sigma,\tau}(t))\frac{d\,[\varphi_{\sigma,\tau}(t)]}{d\sigma}-\partial_t\varphi_{\sigma,\tau}(t)\frac{d\,[\partial_t\varphi_{\sigma,\tau}(t)]}{d\sigma}=\frac{1}{2}\quad\text{for all } t\in\mathbb{R}.$$

We see that there exists $C_*>0$ such that for every $(\sigma,\tau)\in I_0\times\mathbb{S}^1$, we have

$$|\partial_t\varphi_{\sigma,\tau}(t)|+\left|\frac{d\,[\varphi_{\sigma,\tau}(t)]}{d\sigma}\right|\le C_*\quad\text{for all } t\in\mathbb{R}. \tag{6.8}$$

Moreover, there exists $T_0>0$ such that $C_*e^{-\lambda T_0/2}<a_0:=\inf\{a_\sigma:\,\sigma\in I_0\}$, where a_σ is the smallest positive root of $F_0(\xi)=\sigma$. Let $\mathscr{X}_{T_0}$ denote the set of all continuous functions $(f_1,f_2):[T_0,\infty)\to\mathbb{R}^2$ with $e^{\lambda t/2}(|f_1(t)|+|f_2(t)|)\le 1$ for all $t\ge T_0$. If we define

$$\|(f_1,f_2)\|:=\sup_{t\ge T_0}\left\{e^{\lambda t/2}\left(|f_1(t)|+|f_2(t)|\right)\right\},$$

then $(\mathscr{X}_{T_0},\|\cdot\|)$ is a complete metric space. For $(\widehat{V},\widehat{W})\in\mathscr{X}_{T_0}$ and recalling that $\varphi''_{\sigma,\tau}=f(\varphi_{\sigma,\tau})$ on $\mathbb{R}$, we consider the following transformation:

$$\begin{bmatrix}V(t)-\varphi_{\sigma,\tau}(t)\\ W(t)-\varphi'_{\sigma,\tau}(t)\end{bmatrix}=\begin{bmatrix}\partial_t\varphi_{\sigma,\tau}(t) & \frac{d\left[\varphi_{\sigma,\tau}(t)\right]}{d\sigma}\\ f(\varphi_{\sigma,\tau}(t)) & \frac{d\left[\partial_t\varphi_{\sigma,\tau}(t)\right]}{d\sigma}\end{bmatrix}\begin{bmatrix}\widehat{V}(t)\\ \widehat{W}(t)\end{bmatrix}\quad\text{for } t\in[T_0,\infty). \tag{6.9}$$

Note that the matrix and its inverse are both uniformly bounded with respect to $(\sigma,\tau)\in I_0\times\mathbb{S}^1$. In particular, (6.9) yields that

$$V(t):=\varphi_{\sigma,\tau}(t)+\widehat{V}(t)\partial_t\varphi_{\sigma,\tau}(t)+\widehat{W}(t)\frac{d\,[\varphi_{\sigma,\tau}(t)]}{d\sigma}. \tag{6.10}$$

From (6.8), (6.10) and our choice of T_0, we find that

$$|V(t)-\varphi_{\sigma,\tau}(t)|\le C_*e^{-\lambda t/2}\|(\widehat{V},\widehat{W})\|\le C_*e^{-\lambda T_0/2}<a_0\quad\text{for all } t\ge T_0. \tag{6.11}$$

For every $t\ge T_0$ and $(\sigma,\tau)\in I_0\times\mathbb{S}^1$, we have $\varphi_{\sigma,\tau}(t)\ge a_\sigma\ge a_0$ since a_σ is increasing in σ. Thus, (6.11) proves that V in (6.10) is positive on $[T_0,\infty)$ for all $(\widehat{V},\widehat{W})\in\mathscr{X}_{T_0}$. For simplicity of reference, using V in (6.10) for $(\widehat{V},\widehat{W})\in\mathscr{X}_{T_0}$, we define

$$g_{\sigma,\tau,\widehat{V},\widehat{W}}(t):=f(V(t))-f(\varphi_{\sigma,\tau}(t))-(V(t)-\varphi_{\sigma,\tau}(t))f'(\varphi_{\sigma,\tau}(t))+\mu e^{-\lambda t}V^q(t).$$

By (6.11), there exist positive constants C_0 and C_1 such that for all $(\widehat{V},\widehat{W})\in\mathscr{X}_{T_0}$,

$$|g_{\sigma,\tau,\widehat{V},\widehat{W}}(t)|\le C_0|V(t)-\varphi_{\sigma,\tau}(t)|^2+\mu e^{-\lambda t}V^q(t)\le C_1e^{-\lambda t} \tag{6.12}$$

for every $t\in[T_0,\infty)$ and $(\sigma,\tau)\in I_0\times\mathbb{S}^1$. Remark that (6.2) is equivalent to the system

$$(\widehat{V}'(t),\widehat{W}'(t))=\left(2g_{\sigma,\tau,\widehat{V},\widehat{W}}(t)\frac{d\,[\varphi_{\sigma,\tau}(t)]}{d\sigma},-2g_{\sigma,\tau,\widehat{V},\widehat{W}}(t)\,\partial_t\varphi_{\sigma,\tau}(t)\right)\text{ on } [T_0,\infty). \tag{6.13}$$

For every $(\widehat{V},\widehat{W})\in\mathscr{X}_{T_0}$ and $t\geq T_0$, we define

$$\Phi_{\sigma,\tau}(\widehat{V},\widehat{W})(t)=\left(-2\int_t^\infty g_{\sigma,\tau,\widehat{V},\widehat{W}}(y)\frac{d\left[\varphi_{\sigma,\tau}(y)\right]}{d\sigma}\,dy,2\int_t^\infty g_{\sigma,\tau,\widehat{V},\widehat{W}}(y)\,\partial_t\varphi_{\sigma,\tau}(y)\,dy\right).$$

We next prove the existence of $T_0>0$ large such that $\Phi_{\sigma,\tau}$ maps $\mathscr{X}_{T_0}$ into $\mathscr{X}_{T_0}$ and $\Phi_{\sigma,\tau}$ is a contraction mapping on $\mathscr{X}_{T_0}$ for every $(\sigma,\tau)\in I_0\times\mathbb{S}^1$. From (6.8), (6.12) and the definition of $(\mathscr{X}_{T_0},\|\cdot\|)$, we have

$$\|\Phi_{\sigma,\tau}(\widehat{V},\widehat{W})\|\leq 2C_*\sup_{t\geq T_0}\left\{e^{\lambda t/2}\int_t^\infty|g_{\sigma,\tau,\widehat{V},\widehat{W}}(y)|\,dy\right\}\leq\frac{2C_*C_1}{\lambda}e^{-\lambda T_0/2}. \tag{6.14}$$

Thus, for large $T_0>0$, we find that $\Phi_{\sigma,\tau}(\widehat{V},\widehat{W})\in\mathscr{X}_{T_0}$ for every $(\widehat{V},\widehat{W})\in\mathscr{X}_{T_0}$ and all $(\sigma,\tau)\in I_0\times\mathbb{S}^1$. For every $(\widehat{V}_1,\widehat{W}_1)$ and $(\widehat{V}_2,\widehat{W}_2)$ in $\mathscr{X}_{T_0}$, we have

$$\|(\widehat{V}_1,\widehat{W}_1)-(\widehat{V}_2,\widehat{W}_2)\|=\sup_{t\geq T_0}\left\{e^{\frac{\lambda t}{2}}\widehat{S}(t)\right\}\text{ with }\widehat{S}(t):=|(\widehat{V}_1-\widehat{V}_2)(t)|+|(\widehat{W}_1-\widehat{W}_2)(t)|.$$

Hence, there exist positive constants C_2 and C_3 such that for every $(\sigma,\tau)\in I_0\times\mathbb{S}^1$

$$\begin{aligned}\|\Phi_{\sigma,\tau}(\widehat{V}_1,\widehat{W}_1)-\Phi_{\sigma,\tau}(\widehat{V}_2,\widehat{W}_2)\|&\leq 2C_*\sup_{t\geq T_0}\left\{e^{\frac{\lambda t}{2}}\int_t^\infty\left|g_{\sigma,\tau,\widehat{V}_1,\widehat{W}_1}(y)-g_{\sigma,\tau,\widehat{V}_2,\widehat{W}_2}(y)\right|dy\right\}\\&\leq C_2\sup_{t\geq T_0}\left\{e^{\frac{\lambda t}{2}}\int_t^\infty\left[(\widehat{S}(y))^2+e^{-\lambda y}\widehat{S}(y)\right]dy\right\}\\&\leq C_3e^{-\frac{\lambda T_0}{2}}\|(\widehat{V}_1,\widehat{W}_1)-(\widehat{V}_2,\widehat{W}_2)\|\end{aligned}$$

for all $(\widehat{V}_1,\widehat{W}_1),(\widehat{V}_2,\widehat{W}_2)\in\mathscr{X}_{T_0}$. Thus, for $T_0>0$ large, $\Phi_{\sigma,\tau}$ is a contraction on $\mathscr{X}_{T_0}$ for all $(\sigma,\tau)\in I_0\times\mathbb{S}^1$. Hence, $\Phi_{\sigma,\tau}$ has a unique fixed point in $\mathscr{X}_{T_0}$, say $(\widehat{V}_{\sigma,\tau},\widehat{W}_{\sigma,\tau})$, which gives a unique solution in $\mathscr{X}_{T_0}$ of (6.13) such that $\lim_{t\to\infty}(\widehat{V}_{\sigma,\tau},\widehat{W}_{\sigma,\tau})(t)=(0,0)$. By (6.8) and $(\widehat{V},\widehat{W})=(\widehat{V}_{\sigma,\tau},\widehat{W}_{\sigma,\tau})$ in (6.9), we obtain a solution $(V_{\sigma,\tau},W_{\sigma,\tau})$ of (6.2) satisfying (6.3) with $\varphi=\varphi_{\sigma,\tau}$. Moreover, $(V_{\sigma,\tau},W_{\sigma,\tau})$ is continuous in (σ,τ) since $\Phi_{\sigma,\tau}$ is continuous in (σ,τ).

To prove uniqueness, on $\Omega_0:=I_0\times\mathbb{S}^1\times(0,e^{-T_0})$, we define the functions $H,G:\Omega_0\to\mathbb{R}^3$ by

$$H(\sigma,\tau,r):=(V_{\sigma,\tau}(t(r)),W_{\sigma,\tau}(t(r)),r)\text{ and }G(\sigma,\tau,r):=\left(\varphi_{\sigma,\tau}(t(r)),\varphi'_{\sigma,\tau}(t(r)),r\right)$$

for every $(\sigma,\tau,r)\in\Omega_0$, where $t(r):=\log(1/r)$. From our construction, H is continuous. Since $V_{\sigma,\tau}$ is a solution to a second-order ODE and $W_{\sigma,\tau}=V'_{\sigma,\tau}$, the uniqueness theorem for ODEs yields that H is one-to-one in Ω_0. Clearly, G is also continuous and one-to-one in Ω_0. Thus, by applying the Domain Invariance Theorem, we obtain that H and G are open. Moreover, since the functions $\{\varphi_{\sigma,\tau}\}_{(\sigma,\tau)\in I_0\times\mathbb{S}^1}$ are periodic, we see that $G(\Omega_0)=\Sigma_0\times(0,e^{-T_0})$ for some domain Σ_0 in $\mathbb{R}^2$. Let $H_0:\Sigma_0\times(-e^{-T_0},e^{-T_0})\to\mathbb{R}^3$ be the function defined as

$$H_0(\xi_1,\xi_2,r):=\begin{cases}H\left(G^{-1}(\xi_1,\xi_2,r)\right) & \text{if } r>0\\ (\xi_1,\xi_2,0) & \text{if } r=0\\ J\left(H\left(G^{-1}(\xi_1,\xi_2,-r)\right)\right) & \text{if } r<0,\end{cases}$$

where $J(\xi_1,\xi_2,r):=(\xi_1,\xi_2,-r)$. Since H and G are one-to-one in Ω_0, we obtain that H_0 is one-to-one in $\Sigma_0\times(-e^{-T_0},e^{-T_0})$. Moreover, since H and G^{-1} are continuous in Ω_0, we obtain that H_0 is continuous in $\Sigma_0\times[(-e^{-T_0},e^{-T_0})\setminus\{0\}]$. As regards the continuity of H_0 on $\Sigma_0\times\{0\}$, for every $(\zeta_1,\zeta_2)\in\Sigma_0$ and

$(\xi_1,\xi_2,r)\in\Sigma_0\times[(-e^{-T_0},e^{-T_0})\setminus\{0\}]$, we write

$$|H_0(\xi_1,\xi_2,r)-H_0(\zeta_1,\zeta_2,0)|\le|H_0(\xi_1,\xi_2,r)-(\xi_1,\xi_2,r)|+|(\xi_1,\xi_2,r)-(\zeta_1,\zeta_2,0)| \\ \le\left|\left(V_{\sigma,\tau}(t(|r|))-\varphi_{\sigma,\tau}(t(|r|)),W_{\sigma,\tau}(t(|r|))-\varphi'_{\sigma,\tau}(t(|r|))\right)\right|+|(\xi_1,\xi_2,r)-(\zeta_1,\zeta_2,0)|, \tag{6.15}$$

where $(\sigma,\tau,|r|)=G^{-1}(\xi_1,\xi_2,|r|)$. Since $(\widehat{V}_{\sigma,\tau},\widehat{W}_{\sigma,\tau})\in\mathscr{X}_{T_0}$, it follows from (6.8), (6.9) and (6.15) that for every $(\zeta_1,\zeta_2,0)\in\Sigma_0\times\{0\}$, $H_0(\xi_1,\xi_2,r)\to H_0(\zeta_1,\zeta_2,0)$ as $(\xi_1,\xi_2,r)\to(\zeta_1,\zeta_2,0)$ i.e., H_0 is continuous at $(\zeta_1,\zeta_2,0)$. By another application of the Domain Invariance Theorem, we obtain that H_0 is open. We let Σ_* be the domain such that $\Sigma_*\times\{r\}=G((\sigma_*,\sigma_0)\times\mathbb{S}^1,r)$ for every $r>0$. In particular, since Σ_* is open, we obtain that for every $(\sigma,\tau)\in(\sigma_*,\sigma_0)\times\mathbb{S}^1$, every solution $(V(t),W(t))$ of (6.2) satisfying

$$(V(t),W(t))-\left(\varphi_{\sigma,\tau}(t),\varphi'_{\sigma,\tau}(t)\right)\to(0,0)\quad\text{as } t\to\infty \tag{6.16}$$

must satisfy $(X(t(r)),Y(t(r)))\in\Sigma_*$ for small $r>0$. Since $\Sigma_*\times\{0\}\Subset H_0(\Sigma_0\times(-e^{-T_0},e^{-T_0}))$ and H_0 is open, we obtain that there exists $R_0\in(0,e^{-T_0})$ such that $\Sigma_*\times[-R_0,R_0]\subset H_0(\Sigma_0\times(-e^{-T_0},e^{-T_0}))$. It then follows from the definitions of H_0 and Σ_* that $\Sigma_*\times(0,R_0]\subset H(I_0\times\mathbb{S}^1\times(0,R_0])$. Hence, for every solution $(X(t),Y(t))$ of (6.2) satisfying (6.16) for some $(\sigma,\tau)\in(\sigma_*,\sigma_0)\times\mathbb{S}^1$, we obtain $(X(t(r)),Y(t(r)),r)\in H(I_0\times\mathbb{S}^1\times(0,R_0])$ and so $(X(t(r)),Y(t(r)))=(V_{\sigma,\tau}(t(r)),W_{\sigma,\tau}(t(r)))$ for small $r>0$. Hence, for every $\varphi=\varphi_{\sigma,\tau}$ with $(\sigma,\tau)\in(\sigma_*,\sigma_0)\times\mathbb{S}^1$, we conclude that $(V_{\sigma,\tau},W_{\sigma,\tau})$ is the unique solution of (6.2) satisfying (6.3). This ends Step 6.1. ☐

Step 6.2. *Proof of Lemma 6.2 if $\varphi\in\cup\{\varphi_{\sigma,\tau}\}_{(\sigma,\tau)\in[\sigma_0,\overline{\sigma}]\times\mathbb{S}^1}$ for $\sigma_0\in(0,\overline{\sigma})$ close enough to $\overline{\sigma}$.*

Proof of Step 6.2. For $(\sigma,\tau)\in(0,\overline{\sigma}]\times\mathbb{S}^1$, we search for $T_0\in\mathbb{R}$ and $V,W\in C^1([T_0,+\infty))$ such that (6.2) holds and $(V(t),W(t))-(\varphi_{\sigma,\tau}(t),\varphi'_{\sigma,\tau}(t))\to(0,0)$ as $t\to\infty$. Writing $\tilde V=V-\varphi_{\sigma,\tau}$ and $\tilde W=W-\varphi'_{\sigma,\tau}$, this is equivalent to finding $T_0\in\mathbb{R}$ and $\tilde V,\tilde W\in C^1([T_0,+\infty))$ such that

$$\begin{cases}(\tilde V',\tilde W')=(\tilde W,f(\varphi_{\sigma,\tau}+\tilde V)-f(\varphi_{\sigma,\tau})+\mu e^{-\lambda t}(\varphi_{\sigma,\tau}+\tilde V)^q) & \text{in } [T_0,\infty),\\ (\tilde V(t),\tilde W(t))\to(0,0)\ \text{ as } t\to+\infty.\end{cases} \tag{6.17}$$

We define $L(\varphi_{\sigma,\tau},\varphi_{\overline{\sigma}})(t):=f'(\varphi_{\sigma,\tau}(t))-f'(\varphi_{\overline{\sigma}}(t))$ for $t\in\mathbb{R}$ and

$$A:=\begin{pmatrix}0 & 1\\ f'(M_0) & 0\end{pmatrix}.$$

Since $\varphi_{\overline{\sigma}}\equiv M_0$ and $\varphi_{\sigma,\tau}\to\varphi_{\overline{\sigma}}$ as $\sigma\to\overline{\sigma}$ uniformly with respect to $\tau\in\mathbb{S}^1$, we get that

$$\lim_{\sigma\to\overline{\sigma}}\sup_{\tau\in\mathbb{S}^1,t\in\mathbb{R}}|L(\varphi_{\sigma,\tau},\varphi_{\overline{\sigma}})(t)|=0. \tag{6.18}$$

With a Taylor expansion, we write

$$f(\varphi_{\sigma,\tau}+\tilde V)-f(\varphi_{\sigma,\tau}) \;=\; f'(\varphi_{\overline{\sigma}})\tilde V+L(\varphi_{\sigma,\tau},\varphi_{\overline{\sigma}})\tilde V+Q(\varphi_{\sigma,\tau},\tilde V)$$

with $|Q(\varphi_{\sigma,\tau},\tilde V)|\le C|\tilde V|^2$. Therefore, the system in (6.17) rewrites as follows

$$\begin{pmatrix}\tilde V\\ \tilde W\end{pmatrix}'=A\begin{pmatrix}\tilde V\\ \tilde W\end{pmatrix}+\begin{pmatrix}0\\ L(\varphi_{\sigma,\tau},\varphi_{\overline{\sigma}})\tilde V+Q(\varphi_{\sigma,\tau},\tilde V)+\mu e^{-\lambda t}(\varphi_{\sigma,\tau}+\tilde V)^q\end{pmatrix}. \tag{6.19}$$

Since $f'(M_0)<0$, we get that A has two conjugate pure imaginary eigenvalues. Therefore, there exists $C>0$ such that $\|e^{tA}\|+\|e^{-tA}\|\le C$ for all $t\in\mathbb{R}$, where $\|\cdot\|$ is any operator norm on $\mathbb{R}^2$. For all $t\ge T_0$, we define

$\tilde{X}(t) := e^{-tA}\begin{pmatrix} \tilde{V}(t) \\ \tilde{W}(t) \end{pmatrix}$ and

$$\Phi_{\varphi_{\sigma,\tau}}(t,\tilde{X}) := e^{-tA}\begin{pmatrix} 0 \\ L(\varphi_{\sigma,\tau},\varphi_{\overline{\sigma}})(e^{tA}\tilde{X})_1 + Q(\varphi_{\sigma,\tau},(e^{tA}\tilde{X})_1) + \mu e^{-\lambda t}(\varphi_{\sigma,\tau}+(e^{tA}\tilde{X})_1)^q \end{pmatrix},$$

where $(e^{tA}\tilde{X})_1$ denotes the first coordinate of $e^{tA}\tilde{X} \in \mathbb{R}^2$. Then getting a solution to (6.17) amounts to finding a solution $\tilde{X} \in C^1([T_0,+\infty),\mathbb{R}^2)$ to

$$\tilde{X}'(t) = \Phi_{\varphi_{\sigma,\tau}}(t,\tilde{X}(t)) \text{ for } t \geq T_0 \text{ and } \lim_{t\to+\infty}\tilde{X}(t) = 0. \tag{6.20}$$

As in Step 6.1, we find a solution to (6.20) via the Fixed Point Theorem for contracting maps on a complete metric space. Since $Q(\varphi_{\sigma,\tau},\tilde{V})$ is quadratic in $\tilde{V}$, the last two terms of the second coordinate of $\Phi_{\varphi_{\sigma,\tau}}(t,\tilde{X})$ are tackled as in Step 6.1. The first term is linear in $\tilde{V}$ and controled by $L(\varphi_{\sigma,\tau},\varphi_{\overline{\sigma}})$: with (6.18), this term is contracting for σ close enough to $\overline{\sigma}$. Mimicking the existence proof of Step 6.1, we get the following:

There exist $\varepsilon > 0$ and $T_0 > 0$ such that for every $(\sigma,\tau) \in [\overline{\sigma}-3\varepsilon,\overline{\sigma}]\times\mathbb{S}^1$, there exists a solution $(V_{\sigma,\tau},W_{\sigma,\tau}) \in C^1([T_0,+\infty),\mathbb{R}^2)$ to (6.2) such that (6.3) holds for $\varphi = \varphi_{\sigma,\tau}$. Moreover, since $(\sigma,\tau)\longmapsto(\varphi_{\sigma,\tau},\varphi'_{\sigma,\tau})$ is continuous on $(0,\overline{\sigma}]\times\mathbb{S}^1$ (despite the issue for $\overline{\sigma}$), the continuity of the fixed points depending on a parameter yields that $(\sigma,\tau)\longmapsto(V_{\sigma,\tau},W_{\sigma,\tau})$ is continuous on $[\overline{\sigma}-3\varepsilon,\overline{\sigma}]\times\mathbb{S}^1$. Here we have taken the supremum norm on $C^0([T_0,+\infty),\mathbb{R}^2)$: via the fixed point construction, we also get that this holds with a weighted norm.

We only sketch the uniqueness proof. For $\tau_0 = (1,0) \in \mathbb{S}^1$ and every $\xi \in B(0,2\varepsilon) \subset \mathbb{R}^2$, we define

$$\sigma(\xi) := \overline{\sigma} - |\xi| \text{ and } \{\tau(\xi) := \xi/|\xi| \text{ if } \xi \neq 0 \text{ and } \tau(0) = \tau_0\}.$$

Due to the uniqueness of solution for $\sigma = \overline{\sigma}$, as one checks, we have the continuity of the mappings $\xi \longmapsto (\varphi_{\sigma(\xi),\tau(\xi)},\varphi'_{\sigma(\xi),\tau(\xi)})$ and $\xi \longmapsto (V_{\sigma(\xi),\tau(\xi)},W'_{\sigma(\xi),\tau(\xi)})$ on $B(0,2\varepsilon)$. We introduce the domain $\Omega_0 := B(0,2\varepsilon)\times(0,e^{-T_0})$ and the functions $H,G:\Omega_0\to\mathbb{R}^3$ defined as

$$H(\xi,r) := (V_{\sigma,\tau}(t),W_{\sigma,\tau}(t),r) \text{ and } G(\xi,r) := \left(\varphi_{\sigma,\tau}(t),\varphi'_{\sigma,\tau}(t),r\right)$$

for every $(\xi,r)\in\Omega_0$, where $t(r) := \log(1/r)$, $\sigma = \sigma(\xi)$ and $\tau = \tau(\xi)$. Arguing as in Step 6.1 and with some extra care for the case $\xi = 0$, we get the uniqueness of the solution of (6.2) satisfying (6.3) for $\varphi = \varphi_{\sigma,\tau}$. This ends the proof of Step 6.2. □

This completes the proof of Lemma 6.2 and thus of Proposition 6.1. □

7. Appendix

Here, we establish Theorem 7.1, a critical result that was used in the proof of Lemma 2.2. The proof of Theorem 7.1 is strongly inspired by Kelley's paper [15]. We denote by $B_\delta(0) \subset \mathbb{R}^3$ the ball centered at 0 with radius $\delta > 0$. For any $r_0 > 0$, we set $D_{r_0} := [0,r_0]\times[-r_0,r_0]$.

Theorem 7.1. *Let $h_j \in C^1(B_\delta(0))$ for some $\delta > 0$ with $j = 1,2,3$. Suppose there exist constants $C_1 > 0$ and $p > 1$ such that for all $\vec{\xi} = (\xi_1,\xi_2,\xi_3) \in B_\delta(0)$, we have*

$$\begin{cases} \sum_{j=1}^{3}|h_j(\vec{\xi})| \leq C_1\sum_{j=1}^{3}\xi_j^2 \text{ and } \sum_{j=1}^{3}|\nabla h_j(\vec{\xi})| \leq C_1\sum_{j=1}^{3}|\xi_j|, \\ h_2(\vec{\xi}) \leq -C_1|\xi_2|^p \text{ and } h_2(\xi_1,0,\xi_3) = 0. \end{cases} \tag{7.1}$$

For fixed $a>0$ and $c<0$, we consider the system

$$\begin{cases} \vec{\mathscr{Z}}'(t) = (a\mathscr{Z}_1(t)+h_1(\vec{\mathscr{Z}}(t)), h_2(\vec{\mathscr{Z}}(t)), c\mathscr{Z}_3(t)+h_3(\vec{\mathscr{Z}}(t))) & \textit{for } t\geq 0, \\ \vec{\mathscr{Z}}(0) = (x_0,y_0,z_0). \end{cases} \tag{7.2}$$

Then there exist $r_0\in(0,\delta/2)$ and a Lipschitz function $w: D_{r_0}\to[-r_0,r_0]$ such that for all $(y_0,z_0)\in D_{r_0}$ and $x_0=w(y_0,z_0)$, the initial value system (7.2) has a solution $\vec{\mathscr{Z}}$ on $[0,\infty)$ and

$$\lim_{t\to+\infty} \vec{\mathscr{Z}}(t) = (0,0,0). \tag{7.3}$$

Moreover, we have that the parametrized surface $(\mathscr{Z}_2,\mathscr{Z}_3)\longmapsto(w(\mathscr{Z}_2,\mathscr{Z}_3),\mathscr{Z}_2,\mathscr{Z}_3)$ is stable in the sense that $\mathscr{Z}_1(t)=w(\mathscr{Z}_2(t),\mathscr{Z}_3(t))$ for all $t\geq 0$.

Proof. Since $h_j\in C^1(B_\delta(0))$ for $1\leq j\leq 3$, the Cauchy–Lipschitz theory applies to the system. For $r_0\in(0,\delta/2)$ and $C_2>0$, we define $\mathscr{X}$ as the set of all continuous functions $w: D_{r_0}\to[-r_0,r_0]$ such that $w(0,0)=0$ and w is C_2-Lipschitz. Note that $(\mathscr{X},\|\cdot\|_\infty)$ is a complete metric space. For any $w\in\mathscr{X}$, we consider the system

$$\begin{cases} (y',z') = (h_2(w(y,z),y,z), cz+h_3(w(y,z),y,z)) & \text{on } [0,\infty), \\ (y(0),z(0)) = (y_0,z_0). \end{cases} \tag{S_w}$$

We now divide the proof of Theorem 7.1 in five Steps.

Step 7.1. *Let $r_0\in(0,\delta/2)$ be such that $4C_1(1+C_2^2)r_0\leq|c|$. If $(y_0,z_0)\in D_{r_0}$, then the flow $\Phi_t^w(y_0,z_0)$ associated to (S_w) is defined for all $t\in[0,+\infty)$. If we set*

$$(y(t),z(t)) := \Phi_t^w(y_0,z_0) \quad \textit{for all } t\in[0,\infty), \tag{7.4}$$

then $0\leq y(t)\leq y_0$ and $|z(t)|\leq\max\{y_0,|z_0|\}$ on $[0,\infty)$. Moreover, we have

$$\lim_{t\to\infty}(y(t),z(t)) = (0,0). \tag{7.5}$$

Proof of Step 7.1. Let $(y_0,z_0)\in D_{r_0}$ be arbitrary. Since the Cauchy–Lipschitz theory applies, the initial value problem (S_w) has a unique solution (y,z) on an interval $[0,b)$ with $b>0$. We prove the following:

(i) $y\equiv 0$ if $y_0=0$ and $0<y(t)\leq y_0$ for all $t\in[0,b)$ if $y_0\in(0,r_0]$;

(ii) $|z(t)|\leq\max\{y_0,|z_0|\}$ for every $t\in[0,b)$.

Proof of (i). We write $h_2(w(y(t),z(t)),y(t),z(t))=\widehat{h}_2(t,y(t))$ for $t\in[0,b)$, where $\widehat{h}_2(t,y)$ is continuous in $t\in[0,b)$ and Lipschitz with respect to $y\in[0,r_0]$. The assumption (7.1) yields $\widehat{h}_2(\cdot,0)=0$ on $[0,b)$ and $\widehat{h}_2(t,y(t))\leq 0$ for all $t\in[0,b)$. The claim of (i) holds since $y'(t)\leq 0$ on $[0,b)$ and y is the unique solution of $y'(t)=\widehat{h}_2(t,y(t))$ for $t\in[0,b)$, subject to $y(0)=y_0$. □

Proof of (ii). Since $c<0$, using the system (S_w), we find that

$$(z^2)' = 2\left(-|c|z^2+zh_3(w(y,z),y,z)\right) \quad \text{on } [0,b). \tag{7.6}$$

Since w is a C_2-Lipschitz function, using the hypothesis on h_3 in (7.1), we have

$$|zh_3(w(y,z),y,z)|\leq C_1|z|\left[w^2(y,z)+y^2+z^2\right]\leq C_1\left(1+C_2^2\right)|z|\left(y^2+z^2\right). \tag{7.7}$$

Using $|z|\leq r_0$ and the choice of $r_0>0$, from (7.7) we obtain that

$$|zh_3(w(y,z),y,z)|\leq|c|\max\{y^2,z^2\}/2 \quad \text{on } [0,b). \tag{7.8}$$

If $y_0=0$, then $y\equiv 0$ on $[0,b)$ by (i). From (7.8) and (7.6), we have $(z^2)'\leq 0$ on $[0,b)$, which yields $|z(t)|\leq|z_0|$ for all $t\in[0,b)$, proving (ii) if $y_0=0$.

We now prove (ii) when $y_0 > 0$. If there exists $t_0 \in [0,b)$ such that $|z(t_0)| = y_0$, then using (i) and (7.8), we find that $|z h_3(w(y,z),y,z)|(t_0) < |c| z^2(t_0)$. Thus, (7.6) yields that $(z^2)'(t_0) < 0$. This means that $|z(t)| = y_0$ has at most a solution in $[0,b)$. Hence, one of the following holds:

(a) $|z(t)| \le y_0$ for all $t \in [0,b)$, which immediately yields (ii);
(b) $|z(t)| \ge y_0$ for all $t \in [0,b)$;
(c) For some $t_0 \in (0,b)$, we have $|z| > y_0$ on $t \in [0,t_0)$ and $|z| < y_0$ on (t_0,b).

Using (7.8) into (7.6), we get $(z^2)' < 0$ on $[0,b)$ in case (b) and on $[0,t_0)$ in case (c) since $\max\{y^2,z^2\} = z^2$. Thus in case (b) and (c) respectively, we find that $|z| \le |z_0|$ on $[0,b)$ and $[0,t_0)$, respectively. This proves (ii) when $y_0 > 0$. □

By (i), (ii) and the finite-time blow-up of solutions of ODEs, the flow $\Phi_t^w(y_0,z_0)$ associated to (S_w) is defined for all $t \in [0,+\infty)$. Let $(y(t),z(t))$ be as in (7.4).

Proof of (7.5). If $y_0 = 0$, then $y \equiv 0$ on $[0,\infty)$. Assuming $y_0 > 0$, then $y > 0$ on $[0,\infty)$. The hypothesis on h_2 in (7.1) implies that $(y^{1-p})'(t) \ge (p-1)C_1$ for all $t \ge 0$. By integration, we get that $\lim_{t\to+\infty} y(t) = 0$. Hence, for every $\varepsilon > 0$, there exists $t_\varepsilon > 0$ large such that $0 \le y \le \varepsilon$ on $[t_\varepsilon,\infty)$. To prove that $\lim_{t\to+\infty} z(t) = 0$, we show that there exists $\widetilde{t_\varepsilon} \ge t_\varepsilon$ such that $|z(t)| \le \varepsilon$ for all $t \ge \widetilde{t_\varepsilon}$. Indeed, with a similar argument to the proof of (ii), it can be shown that $|z(t)| = \varepsilon$ has at most one zero on $[t_\varepsilon,\infty)$. The option $|z| \ge \varepsilon$ on $[t_\varepsilon,\infty)$ is not viable here. Indeed, if $|z| \ge \varepsilon$ on $[t_\varepsilon,\infty)$, then again from (7.6) and (7.8), we would have $(z^2)' \le -|c|z^2 \le -|c|\varepsilon^2$ on $[t_\varepsilon,\infty)$, leading to a contradiction. Hence, either $|z| \le \varepsilon$ on $[t_\varepsilon,\infty)$ or there exists $\widetilde{t_\varepsilon} \in (t_\varepsilon,\infty)$ such that $|z| > \varepsilon$ on $[t_\varepsilon,\widetilde{t_\varepsilon})$ and $|z| < \varepsilon$ on $(\widetilde{t_\varepsilon},\infty)$. In either of these cases, the conclusion $\lim_{t\to+\infty} z(t) = 0$ follows. This proves (7.5). □

The proof of Step 7.1 is now complete. □

Step 7.2. *For any $\rho > 0$, let $r_0 \in (0,\delta/2)$ be as in Step 7.1 and $3C_1(3+2C_2)r_0 < \rho$. Then for any $w_j \in \mathscr{X}$ and $\left(y_0^{(j)}, z_0^{(j)}\right) \in D_{r_0}$ with $j = 1,2$, we have*

$$|(y_1,z_1) - (y_2,z_2)|(t) \le e^{\rho t}\left(\|w_1 - w_2\|_\infty + \left|(y_0^{(1)},z_0^{(1)}) - (y_0^{(2)},z_0^{(2)})\right|\right) \tag{7.9}$$

for all $t \in [0,\infty)$, where we denote $(y_j(t),z_j(t)) := \Phi_t^{w_j}(y_0^{(j)},z_0^{(j)})$ for $j = 1,2$.

Proof of Step 7.2. We denote $Y := y_1 - y_2$ and $Z := z_1 - z_2$. It suffices to prove that

$$e^{-2\rho t}(Y^2+Z^2)(t) \le \|w_1 - w_2\|_\infty^2 + (Y^2+Z^2)(0) \quad \text{for all } t \ge 0. \tag{7.10}$$

When clear, we drop the dependence on t in notation. For $j = 1,2$, we set

$$P_j := (w_j(y_j,z_j), y_j, z_j) \text{ and } L := Y\left[h_2(P_1) - h_2(P_2)\right] + Z\left[h_3(P_1) - h_3(P_2)\right]. \tag{7.11}$$

By a simple calculation, we see that

$$\left(e^{-2\rho t}(Y^2+Z^2)\right)' = 2e^{-2\rho t}\left[-\rho\left(Y^2+Z^2\right) - |c|Z^2 + L\right]. \tag{7.12}$$

We show that L in (7.11) satisfies

$$|L| \le 3C_1 r_0\left[(3+2C_2)(Y^2+Z^2) + \|w_1 - w_2\|_\infty^2\right]. \tag{7.13}$$

Proof of (7.13). Since $\max\{|y_j|,|z_j|,|w_j(y_j,z_j)|\} \le r_0$ for $j = 1,2$, by the assumption on $|\nabla h_2|$ and $|\nabla h_3|$ in (7.1), we infer that

$$\sup_{\xi\in[0,1]} |(\nabla\phi)(\xi P_1 + (1-\xi)P_2)| \le 3C_1 r_0 \tag{7.14}$$

with $\phi = h_2$ and $\phi = h_3$. Therefore, we get that

$$|L| \leq 3C_1 r_0 |P_1 - P_2| \left(|Y| + |Z|\right) \leq 3\sqrt{2} C_1 r_0 |P_1 - P_2| \sqrt{Y^2 + Z^2}. \tag{7.15}$$

Set $a_1 = \|w_1 - w_2\|_\infty$ and $a_2 = \sqrt{Y^2 + Z^2}$. Using (7.11) and that w_1 is C_2-Lipschitz, we get

$$\begin{aligned} |P_1 - P_2| &\leq |w_1(y_1, z_1) - w_2(y_2, z_2)| + a_2 \\ &\leq |w_1(y_1, z_1) - w_1(y_2, z_2)| + a_1 + a_2 \leq (1 + C_2) a_2 + a_1. \end{aligned} \tag{7.16}$$

Plugging (7.16) into (7.15), then using the inequality $2a_1 a_2 \leq a_1^2 + a_2^2$, we conclude (7.13). □

Using (7.13) into (7.12), we get that

$$\left(e^{-2\rho t} \left(Y^2 + Z^2\right)(t)\right)' \leq 2e^{-2\rho t} \left[\alpha_0 (Y^2 + Z^2)(t) + 3C_1 r_0 \|w_1 - w_2\|_\infty^2\right], \tag{7.17}$$

where $\alpha_0 := 3C_1(3 + 2C_2) r_0 - \rho$ is negative from our choice of r_0. Hence, from (7.17), for every $t \in (0, \infty)$, we deduce that

$$\left(e^{-2\rho t} \left(Y^2 + Z^2\right)(t)\right)' \leq 2\rho\, e^{-2\rho t} \|w_1 - w_2\|_\infty^2. \tag{7.18}$$

By integrating (7.18), we obtain (7.10), which completes Step 7.2. □

Step 7.3. *Let $r_0 \in (0, \delta/2)$ be as in Step 7.2 with $\rho = a/2$ and $6C_1(1 + C_2) r_0 < aC_2$. Then, the map $T : \mathscr{X} \to \mathscr{X}$ is well-defined, where for every $w \in \mathscr{X}$, we put*

$$Tw(y_0, z_0) := -\int_0^\infty e^{-at} h_1\left(w\left(\Phi_t^w(y_0, z_0)\right), \Phi_t^w(y_0, z_0)\right) dt \ \ \textit{for all } (y_0, z_0) \in D_{r_0}.$$

Proof of Step 7.3. For all $(y_0, z_0) \in D_{r_0}$, we define $(y(t), z(t)) := \Phi_t^w(y_0, z_0)$. We now observe that for all $t \geq 0$, $h_1(w(y(t), z(t)), y(t), z(t))$ stays bounded since $\max\{|w(y(t), z(t))|, |y(t)|, |z(t)|\} \leq r_0$. Then, $Tw(y_0, z_0)$ is well-defined since $a > 0$. From $w(0,0) = 0$, we have $\Phi_t^w(0,0) = (0,0)$ for all $t \geq 0$, which yields $Tw(0,0) = 0$. To prove that $Tw \in \mathscr{X}$, it remains to show that Tw ranges in $[-r_0, r_0]$ and Tw is C_2-Lipschitz. Indeed, using (7.1), for every $(y_0, z_0) \in D_{r_0}$, we find that

$$|Tw(y_0, z_0)| \leq C_1 \int_0^\infty e^{-at} (w^2(y, z) + y^2 + z^2)\, dt \leq \frac{3C_1 r_0^2}{a}. \tag{7.19}$$

Since $3C_1 r_0 < a$, we have $|Tw(y_0, z_0)| \leq r_0$ so that Tw ranges in $[-r_0, r_0]$.

We prove that Tw is C_2-Lipschitz. We fix $(y_0^{(j)}, z_0^{(j)}) \in D_{r_0}$ for $j = 1, 2$, then define $(y_j(t), z_j(t)) := \Phi_t^w(y_0^{(j)}, z_0^{(j)})$ and $P_j(t) := (w(y_j, z_j), y_j, z_j)(t)$ for all $t \geq 0$. By the definition of Tw, we see that

$$\left|Tw(y_0^{(1)}, z_0^{(1)}) - Tw(y_0^{(2)}, z_0^{(2)})\right| \leq \int_0^\infty e^{-at} |h_1(P_1) - h_1(P_2)|\, dt. \tag{7.20}$$

Since (7.14) holds for $\phi = h_1$, using $w_1 = w_2 = w$ in (7.16), we get that

$$|h_1(P_1) - h_2(P_2)| \leq 3C_1(1 + C_2) r_0 \left|(y_1, z_1) - (y_2, z_2)\right|.$$

Using that $6C_1(1 + C_2) r_0 < aC_2$ and taking $\rho = a/2$ in Step 7.2, we arrive at

$$|h_1(P_1) - h_1(P_2)| \leq \frac{aC_2}{2} e^{\frac{at}{2}} \left|(y_0^{(1)}, z_0^{(1)}) - (y_0^{(2)}, z_0^{(2)})\right|. \tag{7.21}$$

From (7.20) and (7.21), we see that Tw is C_2-Lipschitz, completing Step 7.3. □

Step 7.4. *If also $12C_1 r_0 (2 + C_2) < a$ in Step 7.3, then T is a contraction on $\mathscr{X}$.*

Proof of Step 7.4. For $w_1, w_2 \in \mathscr{X}$ and $(y_0, z_0) \in D_{r_0}$, we define

$$(y_j(t), z_j(t)) := \Phi_t^{w_j}(y_0, z_0) \quad \text{and} \quad P_j(t) := (w_j(y_j(t), z_j(t)), y_j(t), z_j(t))$$

for all $t \geq 0$ and $j = 1, 2$. As in Step 7.2 with $\rho = a/2$, we obtain that

$$\begin{aligned} |h_1(P_1) - h_1(P_2)| &\leq 3C_1 r_0 |P_1 - P_2| \leq 3C_1 r_0 \left[(1 + C_2)|(y_1, z_1) - (y_2, z_2)| + \|w_1 - w_2\|_\infty\right] \\ &\leq 3C_1 r_0 (2 + C_2) e^{\frac{at}{2}} \|w_1 - w_2\|_\infty. \end{aligned} \tag{7.22}$$

Then, using (7.22) and our choice of r_0, we see that

$$|(Tw_1 - Tw_2)(y_0, z_0)| \leq \int_0^\infty e^{-at} |h_1(P_1) - h_1(P_2)| \, dt < \frac{1}{2} \|w_1 - w_2\|_\infty.$$

Therefore, T is $1/2$-Lipschitz, so it is a contraction mapping. This ends Step 7.4. □

Step 7.5. *Let $r_0 \in (0, \delta/2)$ be as in Step 7.4. Then, there exists $w \in \mathscr{X}$ such that for every $(y_0, z_0) \in D_{r_0}$ and $x_0 = w(y_0, z_0)$, the initial value system (7.2) has a solution $\vec{\mathscr{Z}} = (x, y, z)$ defined on $[0, \infty)$ and satisfying (7.3).*

Proof of Step 7.5. The choice of r_0 in Step 7.4 depends only on $a, |c|, C_1, C_2$. Picard's fixed point theorem yields the existence of $w \in \mathscr{X}$ such that $Tw = w$, where T is given by Step 7.3, that is,

$$w(y, z) = -\int_0^\infty e^{-a\xi} h_1\left(w\left(\Phi_\xi^w(y, z)\right), \Phi_\xi^w(y, z)\right) d\xi \tag{7.23}$$

for all $(y, z) \in D_{r_0}$. We fix $(y_0, z_0) \in D_{r_0}$ arbitrarily. We show that $\vec{\mathscr{Z}}(t) = (x(t), y(t), z(t))$ is a solution to the initial system (7.2), subject to (7.3), where we define

$$(y(t), z(t)) := \Phi_t^w(y_0, z_0) \quad \text{and} \quad x(t) := w(y(t), z(t)) \text{ for all } t \geq 0. \tag{7.24}$$

Indeed, Step 7.1 yields that $(y', z') = (h_2(x, y, z), cz + h_3(x, y, z))$ on $[0, \infty)$. In view of $w(0, 0) = 0$ and $\lim_{t \to +\infty} (y(t), z(t)) = (0, 0)$, we get $\lim_{t \to +\infty} x(t) = 0$, proving (7.3). Since

$$\Phi_\xi^w(y(t), z(t)) = \Phi_\xi^w \circ \Phi_t^w(y_0, z_0) = \Phi_{t+\xi}^w(y_0, z_0) = (y(t + \xi), z(t + \xi))$$

for all $\xi, t \geq 0$, from (7.23) and (7.24), we obtain that

$$\begin{aligned} x(t) &= -\int_0^\infty e^{-a\xi} h_1\left(w\left(\Phi_\xi^w(y(t), z(t))\right), \Phi_\xi^w(y(t), z(t))\right) d\xi \\ &= -\int_0^\infty e^{-a\xi} h_1\left(w(y(t + \xi), z(t + \xi)), y(t + \xi), z(t + \xi)\right) d\xi \\ &= -e^{at} \int_t^\infty e^{-a\theta} h_1\left(w(y(\theta), z(\theta)), y(\theta), z(\theta)\right) d\theta \end{aligned} \tag{7.25}$$

for all $t \geq 0$. This ends Step 7.5 since $x \in C^1[0, +\infty)$ and $x' = ax + h_1(x, y, z)$ on $[0, \infty)$. □

Using the definition of $\mathscr{X}$ and Step 7.5, we finish the proof of Theorem 7.1. □

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