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LARGE DEVIATIONS FOR STOCHASTIC GEOMETRIC WAVE EQUATION

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Abstract. We consider stochastic wave map equation on real line with solutions taking values in a d-dimensional compact Riemannian manifold. We show first that this equation has unique, global, strong in PDE sense, solution in local Sobolev spaces. The main result of the paper is a proof of the Large Deviations Principle for solutions in the case of vanishing noise. Our proof relies on a new version of the weak convergence approach by Budhiraja and Dupuis (Probab. Math. Statist., 2000) suitable for the analysis of stochastic wave maps in local Sobolev spaces.

Contents

1. Introduction

Stochastic PDEs for manifold-valued processes has attracted a great deal of attention due to its wide range of applications in the kinetic theory of phase transitions and the theory of stochastic quantization, see e.g. [6], [7]-[9], [13]-[17], [21, 33, 51] and references therein. In this paper we are dealing with a particular example of such an equation, known as stochastic geometric wave equation (SGWE), that was

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introduced and studied by the first named author et al. in a series of papers [13], [15, 17], see also [16].

The aim of this paper is to prove a large deviations principle (LDP) for the one-dimensional stochastic wave equation with solutions taking values in a d-dimensional compact Riemannian manifold M. More precisely we will consider the equation

$$\mathbf{D}_t\partial_t u^\varepsilon = \mathbf{D}_x\partial_x u^\varepsilon + \sqrt{\varepsilon}Y_{u^\varepsilon}(\partial_t u^\varepsilon, \partial_x u^\varepsilon)\,\dot{W}, \tag{1.1}$$

where $\varepsilon \in (0,1]$ approaches zero. Here $\mathbf{D}$ is the connection on the pull-back bundle $u^{-1}TM$ of the tangent bundle over M induced by the Riemannian connection on M, see e.g. [14, 53], and W is a spatially homogeneous Wiener process on $\mathbb{R}$. A precise formulation is provided in Section 3. Here we only note that we will work with the extrinsic formulation of (1.1), that is, we assume M to be isometrically embedded into a certain Euclidean space $\mathbb{R}^n$, which holds true due to the celebrated Nash isometric embedding theorem [42]. Then, in view of Remark 2.5 in [13], equation (1.1) can be written in the form

$$\partial_{tt}u^\varepsilon = \partial_{xx}u^\varepsilon + A_{u^\varepsilon}(\partial_t u^\varepsilon, \partial_t u^\varepsilon) - A_{u^\varepsilon}(\partial_x u^\varepsilon, \partial_x u^\varepsilon) + \sqrt{\varepsilon}Y_{u^\varepsilon}(\partial_t u^\varepsilon, \partial_x u^\varepsilon)\,\dot{W}, \tag{1.2}$$

where A is the second fundamental form of the submanifold $M \subseteq \mathbb{R}^n$. More details about the equivalence of extrinsic and intrinsic formulations of stochastic PDEs can be found in Sections 2 and 12 of [13].

Due to its importance for applications, LDP for stochastic PDEs has been widely studied by many authors. However, analysis of large deviations for stochastic PDEs for manifold-valued processes is very little understood. To the best of our knowledge, LDP has only been established for the stochastic Landau-Lifshitz-Gilbert equation with solutions taking values in the two dimensional sphere [9]. Our paper is the first to study LDP for SGWE.

If $\varepsilon = 0$ then equation (1.2) reduces to a deterministic equation for wave maps. It has been intensely studied in recent years due to its importance in field theory and general relativity, see for example [34] and references therein. It turns out that solutions to the deterministic geometric wave equation can exhibit a very complex behaviour including (multiple) blowups and shrinking and expanding bubbles, see [3, 4]. In some cases the Soliton Resolution Conjecture has been proved, see [37]. Various concepts of stability of these phenomena, including the stability of soliton solutions has also been intensely studied [27]. It seems natural to investigate stability for wave maps by investigating the impact of small random perturbations and this idea leads to equation (1.2). Let us recall that the stability of solitons under the influence of noise has already been studied by means of LDP for the Schrödinger equations, see [26]. LDP, once established, will provide a tool for more precise analysis of the stability of wave maps.

Another motivation for studying equation (1.2) with $\epsilon > 0$ comes from the Hamiltonian structure of deterministic wave equation. Deterministic Hamiltonian systems may have infinite number of invariant measures and are not ergodic, see the discussion of this problem in [30]. Characterisation of such systems is a long standing problem. The main idea, which goes back to Kolmogorov-Eckmann-Ruelle, is to choose a suitable small random perturbation such that the solution to stochastic system is a Markov process with the unique invariant measure and then one can select a "physical" invariant measure of the deterministic system by taking the limit of vanishing noise, see for example [25], where this idea is applied to wave maps. A finite dimensional toy example was studied in [2].

In the present work, our proof of verifying the large deviations principle relies on the weak convergence method introduced in [19]. It is based on a variational representation formula for certain functionals of the driving infinite dimensional Brownian motion, and was applied to stochastic PDEs in [9], [23], [28] and [57]. In order to apply the result of [19] we have, differently to the aforementioned papers, to work in Fréchet spaces associated to the local Sobolev spaces. Recently in [54] the authors have established a LDP for a general class of Banach space valued stochastic differential equations by a different, but still based on Laplace principle, approach. However, their result does not apply to SGWE studied in this paper because the wave operator does not generate a compact C_0-semigroup.

Finally, we note that the approach we follow in the Section 5 can be applied to the general beam equation studied in [12], and the nonlinear wave equation with polynomial nonlinearity, with spatially homogeneous noise in local Sobolev spaces. In particular, this method would generalize the result of [57] and [45]. Our approach would also lead to an extension of the work of Martirosyan [41] who considers a nonlinear wave equations on a bounded domain. We believe that the methods of the present work would allow us to obtain large deviations principle for the family of stationary measures generated by the flow of stochastic wave equation, with multiplicative white noise, in non-local Sobolev spaces over the full space $\mathbb{R}^d$.

The organisation of the paper is as follows. In Section 2, we introduce our notation and state the definitions used in the paper. Section 3 contains some properties of the nonlinear drift terms and the diffusion coefficient that we need later. In Section 4 we prove the existence of a unique global and strong in PDE sense solution to the skeleton equation associated to (1.2). The proof of Large Deviations Principle, based on weak convergence approach, is provided in Section 5. We conclude the paper with Appendices A and B, where we state modified version of the existing results on global well-posedness of (1.2) and energy inequality from [13] that we use frequently in the paper.

2. Notation

We write $a \lesssim b$ if there exists a universal constant $c > 0$, independent of a, b, such that $a \leq cb$, and we write $a \simeq b$ when $a \lesssim b$ and $b \lesssim a$. In case we want to emphasize the dependence of c on some parameters $a_1, \dots, a_k$, then we write, respectively, $\lesssim_{a_1,\dots,a_k}$ and $\simeq_{a_1,\dots,a_k}$. We will denote by $B_R(a)$, for $a \in \mathbb{R}$ and $R > 0$, the open ball in $\mathbb{R}$ with center at a and we put $B_R = B_R(0)$. Now we list the notations used throughout the whole paper.

- $\mathbb{N} = \{0, 1, \cdots\}$ denotes the set of natural numbers, $\mathbb{R}_+ = [0, \infty)$, Leb denotes the Lebesgue measure.

- Let $I \subseteq \mathbb{R}$ be an open interval. By $L^p(I;\mathbb{R}^n), p \in [1,\infty)$, we denote the classical real Banach space of all (equivalence classes of) $\mathbb{R}^n$-valued p-integrable maps on I. The norm on $L^p(I;\mathbb{R}^n)$ is given by

$$\|u\|_{L^p(I;\mathbb{R}^n)} := \left(\int_I |u(x)|^p \, dx\right)^{\frac{1}{p}}, \qquad u \in L^p(I;\mathbb{R}^n),$$

where $|\cdot|$ is Euclidean norm on $\mathbb{R}^n$. For $p = \infty$, we consider the usual modification to essential supremum.
- For any $p \in [1,\infty]$, $L^p_{\rm loc}(\mathbb{R};\mathbb{R}^n)$ stands for a metrizable topological vector space equipped with a natural countable family of seminorms $\{p_j\}_{j\in\mathbb{N}}$ defined by

$$p_j(u) := \|u\|_{L^p(B_j;\mathbb{R}^n)}, \qquad u \in L^2_{\rm loc}(\mathbb{R};\mathbb{R}^n),\ j \in \mathbb{N}.$$

- By $H^{k,p}(I;\mathbb{R}^n)$, for $p \in [1,\infty]$ and $k \in \mathbb{N}$, we denote the Banach space of all $u \in L^p(I;\mathbb{R}^n)$ for which $D^j u \in L^p(I;\mathbb{R}^n), j = 0,1,\dots,k$, where D^j is the weak derivative of order j. The norm here is given by

$$\|u\|_{H^{k,p}(I;\mathbb{R}^n)} := \left(\sum_{j=0}^{k} \|D^j u\|^p_{L^p(I;\mathbb{R}^n)}\right)^{\frac{1}{p}}, \qquad u \in H^{k,p}(I;\mathbb{R}^n).$$

- We write $H^{k,p}_{\rm loc}(\mathbb{R};\mathbb{R}^n)$, for $p \in [1,\infty]$ and $k \in \mathbb{N}$, to denote the space of all elements $u \in L^p_{\rm loc}(\mathbb{R};\mathbb{R}^n)$ whose weak derivatives up to order k belong to $L^p_{\rm loc}(\mathbb{R};\mathbb{R}^n)$. It is relevant to note that $H^{k,p}_{\rm loc}(\mathbb{R};\mathbb{R}^n)$ is a metrizable topological vector space equipped with the following natural countable family of seminorms $\{q_j\}_{j\in\mathbb{N}}$,

$$q_j(u) := \|u\|_{H^{k,p}(B_j;\mathbb{R}^n)}, \qquad u \in H^{k,p}_{\rm loc}(\mathbb{R};\mathbb{R}^n),\ j \in \mathbb{N}.$$

The spaces $H^{k,2}(I;\mathbb{R}^n)$ and $H^{k,2}_{loc}(\mathbb{R};\mathbb{R}^n)$ are usually denoted by $H^k(I;\mathbb{R}^n)$ and $H^k_{loc}(\mathbb{R};\mathbb{R}^n)$ respectively.
- We set $\mathcal{H} := H^2(\mathbb{R};\mathbb{R}^n) \times H^1(\mathbb{R};\mathbb{R}^n)$, $\mathcal{H}_{\rm loc} := H^2_{\rm loc}(\mathbb{R};\mathbb{R}^n) \times H^1_{\rm loc}(\mathbb{R};\mathbb{R}^n)$.
- To shorten the notation in the calculation we set rules as:
  - if the space where function is taking value, for example $\mathbb{R}^n$, is clear then to save the space we will omit $\mathbb{R}^n$, for example $H^k(I)$ instead $H^k(I;\mathbb{R}^n)$;
  - if $I = (0,T)$ or $(-R,R)$ or $B(x,R)$, for some $T,R > 0$ and $x \in \mathbb{R}$, then instead of $L^p(I;\mathbb{R}^n)$ we write, respectively, $L^p(0,T;\mathbb{R}^n)$, $L^p(B_R;\mathbb{R}^n)$, $L^p(B(x,R);\mathbb{R}^n)$. Similarly for H^k and $H^k_{\rm loc}$ spaces.
  - write $\mathcal{H}(B_R)$ or $\mathcal{H}_R$ for $H^2((-R,R);\mathbb{R}^n) \times H^1((-R,R);\mathbb{R}^n)$.
- For any nonnegative integer j, let $\mathcal{C}^j(\mathbb{R})$ be the space of real valued continuous functions whose derivatives up to order j are continuous on $\mathbb{R}$. We also need the family of spaces $\mathcal{C}^j_b(\mathbb{R})$ defined by

$$\mathcal{C}^j_b(\mathbb{R}) := \left\{u \in \mathcal{C}^j(\mathbb{R}); \forall \alpha \in \mathbb{N}, \alpha \le j, \exists K_\alpha, \|D^j u\|_{L^\infty(\mathbb{R})} < K_\alpha\right\}.$$

- Given $T > 0$ and Banach space E, we denote by $\mathcal{C}([0,T];E)$ the real Banach space of all E-valued continuous functions $u : [0,T] \to E$ endowed with the norm

$$\|u\|_{\mathcal{C}([0,T];E)} := \sup_{t\in[0,T]} \|u(t)\|_E, \qquad u \in \mathcal{C}([0,T];E).$$

By ${}_0\mathcal{C}([0,T],E)$ we mean the set of elements of $\mathcal{C}([0,T];E)$ vanishes at origin, that is,

$$ {}_0\mathcal{C}([0,T],E) := \{u \in \mathcal{C}([0,T],E) : u(0) = 0\}. $$

- For given metric space (X,ρ), by $\mathcal{C}(\mathbb{R};X)$ we mean the space of continuous functions from $\mathbb{R}$ to X which is equipped with the metric

$$ (f,g) \mapsto \sum_{j=1}^{\infty} \frac{1}{2^j} \min\{1, \sup_{t\in[-j,j]} \rho(f(t),g(t))\}. $$

- We denote the tangent and the normal bundle of a smooth manifold M by TM and NM, respectively. Let $\mathfrak{F}(M)$ be the set of all smooth $\mathbb{R}$-valued function on M.
- A map $u : \mathbb{R} \to M$ belongs to $H^k_{loc}(\mathbb{R};M)$ provided that $\theta \circ u \in H^k_{loc}(\mathbb{R};\mathbb{R})$ for every $\theta \in \mathfrak{F}(M)$. We equip $H^k_{loc}(\mathbb{R};M)$ with the topology induced by the mappings

$$ H^k_{loc}(\mathbb{R};M) \ni u \mapsto \theta \circ u \in H^k_{loc}(\mathbb{R};\mathbb{R}), \quad \theta \in \mathfrak{F}(M). $$

Since the tangent bundle TM of a manifold M is also a manifold, this definition covers Sobolev spaces of TM-valued functions too.
- By $\mathscr{L}_2(H_1,H_2)$ we denote the class of Hilbert–Schmidt operators from a separable Hilbert space H_1 to another H_2. By $\mathcal{L}(X,Y)$ we denote the space of all linear continuous operators from a topological vector space X to Y.
- We denote by $\mathcal{S}(\mathbb{R})$ the space of Schwartz functions on $\mathbb{R}$ and write $\mathcal{S}'(\mathbb{R})$ for its dual, which is the space of tempered distributions on $\mathbb{R}$. By L^2_w we denote the weighted space $L^2(\mathbb{R},w,dx)$, where $w(x) := e^{-x^2}, x \in \mathbb{R}$, is an element of $\mathcal{S}(\mathbb{R})$. Let $H^s_w(\mathbb{R}), s \geq 0$, be the completion of $\mathcal{S}(\mathbb{R})$ with respect to the norm

$$ \|u\|_{H^s_w(\mathbb{R})} := \left(\int_{\mathbb{R}} (1+|x|^2)^s |\mathcal{F}(w^{1/2}u)(x)|^2 \, dx \right)^{\frac{1}{2}}, $$

where $\mathcal{F}$ denoted the Fourier transform.

3. Preliminaries

In this section we discuss all the required preliminaries about the nonlinearity and the diffusion coefficient that we need in Section 4. We are following Sections 3 to 5 of [13] very closely here.

3.1. **Wiener process.** Let μ be a symmetric Borel measure on $\mathbb{R}$, such that

$$ \int_{\mathbb{R}} (1+|x|^2)^2 \, \mu(dx) < \infty. \tag{3.1} $$

An $\mathcal{S}'(\mathbb{R})$-valued process $W = \{W(t), t \geq 0\}$, on a given stochastic basis $(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t\geq 0}, \mathbb{P})$, is called a spatially homogeneous Wiener process with spectral measure μ provided that

(1) for every $\varphi \in \mathcal{S}(\mathbb{R})$, $\{W(t)(\varphi), t \geq 0\}$ is a real-valued $(\mathfrak{F}_t)$-adapted Wiener process,
(2) $W(t)(a\varphi + \psi) = aW(t)(\varphi) + W(t)(\psi)$ holds almost surely for every $t \geq 0$, $a \in \mathbb{R}$ and $\varphi, \psi \in \mathcal{S}(\mathbb{R})$,
(3) $\mathbb{E}\{W(t)(\varphi)W(t)(\psi)\} = t\langle \widehat{\varphi}, \widehat{\psi} \rangle_{L^2(\mu)}$ holds for every $t \geq 0$ and $\varphi, \psi \in \mathcal{S}(\mathbb{R})$.

It is shown in [49] that the Reproducing Kernel Hilbert Space (RKHS) H_μ of the Gaussian distribution of $W(1)$ is given by

$$H_\mu := \left\{ \widehat{\psi\mu} : \psi \in L^2(\mathbb{R}^d, \mu, \mathbb{C}), \psi(x) = \overline{\psi(-x)}, x \in \mathbb{R} \right\},$$

where $L^2(\mathbb{R}^d, \mu, \mathbb{C})$ is the Banach space of complex-valued functions that are square integrable with respect to the measure μ. Note that the Hilbert space H_μ is endowed with the inner-product

$$\left\langle \widehat{\psi_1\mu}, \widehat{\psi_2\mu} \right\rangle_{H_\mu} := \int_{\mathbb{R}} \psi_1(x)\overline{\psi_2(x)}\, \mu(dx)\,.$$

Recall from [49, 50] that W can be regarded as a cylindrical Wiener process on H_μ and it takes values in any Hilbert space E, such that the embedding $H_\mu \hookrightarrow E$ is Hilbert-Schmidt. Since we explicitly know the structure of H_μ, in the next result, whose proof is based on [47, Lemma 2.2] and discussion with Szymon Peszat [48], we provide an example of E such that the paths of W can be considered in $\mathcal{C}([0,T];E)$. Below we also use the notation $\mathcal{F}(\cdot)$, along with $\widehat{\cdot}$, to denote the Fourier transform.

Lemma 3.1. *Let us assume that the measure μ satisfies* (3.1). *Then the imbedding $i : H_\mu \to H^2_w(\mathbb{R})$ is a Hilbert-Schmidt operator.*

Proof of Lemma 3.1. To simplify the notation we set $L^2_{(s)}(\mathbb{R},\mu)$ to be the space of all $f \in L^2(\mathbb{R},\mu;\mathbb{C})$ such that $f(x) = \overline{f(-x)}$, $x \in \mathbb{R}$. Let $\{e_k\}_{k\in\mathbb{N}} \subset \mathcal{S}(\mathbb{R})$ be an orthonormal basis of $L^2_{(s)}(\mathbb{R},\mu)$. Then, by the definition of H_μ, $\{\mathcal{F}(e_k\mu)\}_{k\in\mathbb{N}}$ is an orthonormal basis of H_μ. Invoking the convolution property of the Fourier transform and the Bessel inequality, we obtain,

$$\begin{aligned}
\sum_{k=1}^{\infty} \|\widehat{e_k\mu}\|^2_{H^2_w} &= \sum_{k=1}^{\infty} \int_{\mathbb{R}} (1+|x|^2)|\mathcal{F}\left(w^{1/2}\mathcal{F}(e_k\mu)\right)(x)|^2\,dx \\
&= \int_{\mathbb{R}} (1+|x|^2)^2 \left(\sum_{k=1}^{\infty} |\mathcal{F}\left(w^{1/2}\mathcal{F}(e_k\mu)\right)(x)|^2 \right) dx \\
&= \int_{\mathbb{R}} (1+|x|^2)^2 \left(\sum_{k=1}^{\infty} \left| \int_{\mathbb{R}} \mathcal{F}\left(w^{1/2}\right)(x-z) e_k(z)\,\mu(dz) \right|^2 \right) dx \\
&\leq \int_{\mathbb{R}^2} (1+|x|^2)^2 |\mathcal{F}\left(w^{1/2}\right)(x-z)|^2\,\mu(dz)\,dx \\
&= \int_{\mathbb{R}^2} (1+|x+z|^2)^2 |\mathcal{F}\left(w^{1/2}\right)(x)|^2\,\mu(dz)\,dx \\
&\lesssim \|w^{1/2}\|^2_{H^1_w(\mathbb{R})} \int_{\mathbb{R}} (1+|z|^2)^2\,\mu(dz).
\end{aligned}$$

Hence Lemma 3.1. □

It is relevant to note here that $H^2_w(\mathbb{R})$ is a subset of $H^2_{\mathrm{loc}}(\mathbb{R})$. The next result, whose detailed proof can be found in [44, Lemma 1], plays very important role in deriving the required estimates for the terms involving diffusion coefficient.

Lemma 3.2. *If the measure μ satisfies* (3.1)*, then H_μ is continuously embedded in $\mathcal{C}_b^2(\mathbb{R})$. Moreover, for any $g \in H^j(B(x,R);\mathbb{R}^n)$, where $x \in \mathbb{R}, R > 0$ and $j \in \{0,1,2\}$, the multiplication operator*

$$H_\mu \ni \xi \mapsto g \cdot \xi \in H^j(B(x,R);\mathbb{R}^n),$$

is Hilbert-Schmidt and $\exists\ c > 0$, independent of R, x, g, ξ and j, such that

$$\|\xi \mapsto g \cdot \xi\|_{\mathscr{L}_2(H_\mu, H^j(B(x,R);\mathbb{R}^n))} \leq c\|g\|_{H^j(B(x,R);\mathbb{R}^n)}.$$

Remark 3.3. *It is a crucial observation that in the case of spatially homogeneous noise the constant c in Lemma 3.2 des not depend on the size and position of the ball. However, if we consider a cylindrical Wiener process, then this constant will also depend on the centre x but will be bounded on bounded sets with respect to x.*

3.2. **Extensions of non-linear term.** By definition $A_p : T_pM \times T_pM \to N_pM$, $p \in M$, where $T_pM \subseteq \mathbb{R}^n$ and $N_pM \subseteq \mathbb{R}^n$ are the tangent and the normal vector spaces at $p \in M$, respectively. It is well known, see e.g. [36], that A_p, $p \in M$, is a symmetric bilinear form.

Since we are following the approach of [7], [13], and [35], one of the main steps in the proof of the existence theorem is to consider the problem (1.2) in the ambient space $\mathbb{R}^n$ with an appropriate extension of A from their domain to $\mathbb{R}^n$. In this section we discuss two extensions of A which work fine in the context of stochastic wave map, as demonstrated in [13].

Let us denote by $\mathcal{E}$ the exponential function

$$T\mathbb{R}^n \ni (p,\xi) \mapsto p + \xi \in \mathbb{R}^n,$$

relative to the Riemannian manifold $\mathbb{R}^n$ equipped with the standard Euclidean metric. The proof of the following proposition about the existence of an open set O containing M, with some essential required features, which is called a tubular neighbourhood of M, can be found in [46, Proposition 7.26, p. 200].

Proposition 3.4. *There exists an $\mathbb{R}^n$-open neighbourhood O around M and an NM-open neighbourhood V around the set $\{(p,0) \in NM : p \in NM\}$ such that the restriction of the exponential map $\mathcal{E}|_V : V \to O$ is a diffeomorphism. Moreover, the neighbourhood V can be chosen in such a way that $(p, t\xi) \in V$ whenever $t \in [-1,1]$ and $(p,\xi) \in V$.*

If there is no ambiguity, we will denote the diffeomorphism $\mathcal{E}|_V : V \to O$ by $\mathcal{E}$. Using Proposition 3.4, the diffeomorphism $i : NM \ni (p,\xi) \mapsto (p,-\xi) \in NM$ and the standard partition of unity, one can obtain a function $\Upsilon : \mathbb{R}^n \to \mathbb{R}^n$ which identifies the manifold M as its fixed point set. We have the following result.

Lemma 3.5. [13, Corollary 3.4 and Remark 3.5] *There exists a smooth compactly supported function $\Upsilon : \mathbb{R}^n \to \mathbb{R}^n$ which has the following properties:*

(1) *restriction of Υ on O is a diffeomorpshim,*
(2) *$\Upsilon\big|_O = \mathcal{E} \circ i \circ \mathcal{E}^{-1} : O \to O$ is an involution on the tubular neighborhood O of M,*
(3) *$\Upsilon(\Upsilon(q)) = q$ for every $q \in O$,*
(4) *if $q \in O$, then $\Upsilon(q) = q$ if and only if $q \in M$,*

(5) *if $p \in M$, then*

$$\Upsilon'(p)\xi = \begin{cases} \xi, & \text{provided } \xi \in T_pM, \\ -\xi & \text{provided } \xi \in N_pM. \end{cases}$$

The following result is the first extension of the second fundamental form that we use in this paper.

Proposition 3.6. [13, Proposition 3.6] *If we define*

$$B_q(a,b) = \sum_{i,j=1}^{n} \frac{\partial^2 \Upsilon}{\partial q_i \partial q_j}(q) a_i b_j = \Upsilon''_q(a,b), \qquad q \in \mathbb{R}^n, \quad a,b \in \mathbb{R}^n, \tag{3.2}$$

and

$$\mathcal{A}_q(a,b) = \frac{1}{2} B_{\Upsilon(q)}(\Upsilon'(q)a, \Upsilon'(q)b), \qquad q \in \mathbb{R}^n, \quad a,b \in \mathbb{R}^n, \tag{3.3}$$

then, for every $p \in M$,

$$\mathcal{A}_p(\xi,\eta) = A_p(\xi,\eta), \ \xi,\eta \in T_pM,$$

and

$$\mathcal{A}_{\Upsilon(q)}(\Upsilon'(q)a, \Upsilon'(q)b) = \Upsilon'(q)\mathcal{A}_q(a,b) + B_q(a,b), \ q \in O, \ a,b \in \mathbb{R}^n. \tag{3.4}$$

Along with the extension $\mathcal{A}$, defined by formula (3.3), we also need the extension $\mathscr{A}$, defined by formula (3.5), of the second fundamental form tensor A which will be perpendicular to the tangent space.

Proposition 3.7. [13, Proposition 3.7] *Consider the function*

$$\mathscr{A} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \ni (q,a,b) \mapsto \mathscr{A}_q(a,b) \in \mathbb{R}^n,$$

defined by formula

$$\mathscr{A}_q(a,b) = \sum_{i,j=1}^{n} a_i v_{ij}(q) b_j = A_q(\pi_q(a), \pi_q(b)), \qquad q \in \mathbb{R}^n, \quad a \in \mathbb{R}^n, \quad b \in \mathbb{R}^n, \tag{3.5}$$

where π_p, $p \in M$ is the orthogonal projection of $\mathbb{R}^n$ to T_pM, and v_{ij}, for $i,j \in \{1,\dots,n\}$, are smooth and symmetric (i.e. $v_{ij} = v_{ji}$) extensions of $v_{ij}(p) := A_p(\pi_p e_i, \pi_p e_j)$ to ambient space $\mathbb{R}^n$. Then $\mathscr{A}$ satisfies the following:

(1) *$\mathscr{A}$ is smooth in (q,a,b) and symmetric in (a,b) for every q,*
(2) *$\mathscr{A}_p(\xi,\eta) = A_p(\xi,\eta)$ for every $p \in M$, $\xi,\eta \in T_pM$,*
(3) *$\mathscr{A}_p(a,b)$ is perpendicular to T_pM for every $p \in M$, $a,b \in \mathbb{R}^n$.*

3.3. **The C_0-group, extension operators.** Here we recall some facts on infinitesimal generators of the linear wave equation and on the extension operators in various Sobolev spaces. We refer to [13, Section 5] for more details.

Proposition 3.8. *Assume that $k, n \in \mathbb{N}$. The one parameter family of operators defined by*

$$S_t \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cos[t(-\Delta)^{1/2}]u^1 & + & (-\Delta)^{-1/2}\sin[t(-\Delta)^{1/2}]v^1 \\ & \vdots & \\ \cos[t(-\Delta)^{1/2}]u^n & + & (-\Delta)^{-1/2}\sin[t(-\Delta)^{1/2}]v^n \\ -(-\Delta)^{1/2}\sin[t(-\Delta)^{1/2}]u^1 & + & \cos[t(-\Delta)^{1/2}]v^1 \\ & \vdots & \\ -(-\Delta)^{1/2}\sin[t(-\Delta)^{1/2}]u^n & + & \cos[t(-\Delta)^{1/2}]v^n \end{pmatrix}$$

is a C_0-group on

$$\mathcal{H}^k := H^{k+1}(\mathbb{R};\mathbb{R}^n) \times H^k(\mathbb{R};\mathbb{R}^n),$$

and its infinitesimal generator is an operator $\mathcal{G}^k = \mathcal{G}$ defined by

$$\begin{aligned} D(\mathcal{G}^k) &= H^{k+2}(\mathbb{R};\mathbb{R}^n) \times H^{k+1}(\mathbb{R};\mathbb{R}^n), \\ \mathcal{G}\begin{pmatrix} u \\ v \end{pmatrix} &= \begin{pmatrix} v \\ \Delta u \end{pmatrix}. \end{aligned}$$

The following theorem is well known, see e.g. [40] and [31, Section II.5.4].

Proposition 3.9. *Let $k \in \mathbb{N}$. There exists a linear bounded operator*

$$E^k : H^k((-1,1);\mathbb{R}^n) \to H^k(\mathbb{R};\mathbb{R}^n),$$

such that

(i) *$E^k f = f$ almost everywhere on $(-1,1)$ whenever $f \in H^k((-1,1);\mathbb{R}^n)$,*
(ii) *$E^k f$ vanishes outside of $(-2,2)$ whenever $f \in H^k((-1,1);\mathbb{R}^n)$,*
(iii) *$E^k f \in \mathcal{C}^k(\mathbb{R};\mathbb{R}^n))$, if $f \in \mathcal{C}^k([-1,1];\mathbb{R}^n))$,*
(iv) *if $j \in \mathbb{N}$ and $j < k$, then there exists a unique extension of E^k to a bounded linear operator from $H^j((-1,1);\mathbb{R}^n)$ to $H^j(\mathbb{R};\mathbb{R}^n)$.*

Definition 3.10. *For $k \in \mathbb{N}$, $r > 0$ we define the operators*

$$E^k_r : H^j((-r,r);\mathbb{R}^n) \to H^j(\mathbb{R};\mathbb{R}^n), \qquad j \in \mathbb{N}, j \leq k,$$

called as r-scaled E^k operators, by the following formula

$$(E^k_r f)(x) = \{E^k[y \mapsto f(yr)]\}\left(\frac{x}{r}\right), \qquad x \in \mathbb{R}, \tag{3.6}$$

for $r > 0$ and $f \in H^k((-r,r);\mathbb{R}^n)$.

The following remark will be useful in Lemma 4.4.

Remark 3.11. *We can rewrite (3.6) as $(E^k_r f)(x) = (E^k f_r)(\frac{x}{r})$, $f \in H^k((-r,r);\mathbb{R}^n)$ where $f_r : (-1,1) \ni y \mapsto f(yr) \in \mathbb{R}^n$. Also, observe that for $f \in H^1((-r,r);\mathbb{R}^n)$*

$$\|f_r\|^2_{H^1((-1,1);\mathbb{R}^n)} \leq (r^{-1} + r)\|f\|^2_{H^1((-r,r);\mathbb{R}^n)}.$$

3.4. **Diffusion coefficient.** In this subsection we discuss the assumptions on diffusion coefficient Y which we use only in Section 4. We note that due to a technical issue, that is explained in Section 5, we need to impose stricter conditions on Y in establishing the large deviation principle for (1.2). Here $Y_p : T_pM \times T_pM \to T_pM$, for $p \in M$, is a mapping satisfying,

$$|Y_p(\xi, \eta)|_{T_pM} \leq C_Y(1 + |\xi|_{T_pM} + |\eta|_{T_pM}), \qquad p \in M, \quad \xi, \eta \in T_pM,$$

for some constant $C_Y > 0$ which is independent of p. By invoking Lemma 3.5 and [13, Proposition 3.10], we can extend the noise coefficient to map $Y : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \ni (p, a, b) \mapsto Y_p(a, b) \in \mathbb{R}^n$ which satisfies the following:

Y.1 for $q \in O$ and $a, b \in \mathbb{R}^n$,

$$Y_{\Upsilon(q)}\left(\Upsilon'(q)a, \Upsilon'(q)b\right) = \Upsilon'(q)Y_q(a, b), \tag{3.7}$$

Y.2 there exists an compact set $K_Y \subset \mathbb{R}^n$ containing M such that $Y_p(a, b) = 0$, for all $a, b \in \mathbb{R}^n$, whenever $p \notin K_Y$,

Y.3 Y is of $\mathcal{C}^2$-class and there exist positive constants $C_{Y_i}, i \in \{1, 2, 3\}$ such that, with notation $Y(p, a, b) := Y_p(a, b)$, for every $p, a, b \in \mathbb{R}^n$,

$$|Y_p(a, b)| \leq C_{Y_0}(1 + |a| + |b|), \tag{3.8}$$

$$\left|\frac{\partial Y}{\partial p_i}(p, a, b)\right| \leq C_{Y_1}(1 + |a| + |b|), \quad i = 1, \ldots, n \tag{3.9}$$

$$\left|\frac{\partial Y}{\partial a_i}(p, a, b)\right| + \left|\frac{\partial Y}{\partial b_i}(p, a, b)\right| \leq C_{Y_2}, \quad i = 1, \ldots, n \tag{3.10}$$

$$\left|\frac{\partial^2 Y}{\partial x_j \partial y_i}(p, a, b)\right| \leq C_{Y_3}, \quad x, y \in \{p, a, b\} \text{ and } i, j \in \{1, \ldots, n\}. \tag{3.11}$$

4. Skeleton equation

The purpose of this section is to introduce and study the skeleton (deterministic) equation associated to (1.2). Define

$${}_0H^{1,2}(0, T; H_\mu) := \left\{h \in {}_0\mathcal{C}([0, T], E) : \dot{h} \in L^2(0, T; H_\mu)\right\}.$$

Note that ${}_0H^{1,2}(0, T; H_\mu)$ endowed with the norm

$$\|h\|_{{}_0H^{1,2}(0,T;H_\mu)} = \left(\int_0^T \|\dot{h}(t)\|^2_{H_\mu}\, dt\right)^{1/2}$$

, is a Hilbert space and the map

$$L^2(0, T; H_\mu) \ni \dot{h} \mapsto h = \left\{t \mapsto \int_0^t \dot{h}(s)\, ds\right\} \in {}_0H^{1,2}(0, T; H_\mu),$$

is an isometric isomorphism. For $h \in {}_0H^{1,2}(0, T; H_\mu)$, we will consider the so called skeleton equation associated to problem (1.2):

$$\begin{cases} \partial_{tt}u = \partial_{xx}u + A_u(\partial_t u, \partial_t u) - A_u(\partial_x u, \partial_x u) + Y_u(\partial_t u, \partial_x u)\, \dot{h}\,, \\ u(0, \cdot) = u_0, \;\; \partial_t u(0, \cdot) = v_0. \end{cases} \tag{4.1}$$

The main result of this section is the following deterministic version of [13, Theorem 11.1].

Theorem 4.1. *Let $T>0$, $h \in {}_0H^{1,2}(0,T;H_\mu)$ and $(u_0,v_0) \in H^2_{loc} \times H^1_{loc}(\mathbb{R};TM)$ be given. Then for every $R>T$, there exists a function $u:[0,T)\times\mathbb{R}\to M$ such that:*

1. *$[0,T) \ni t \mapsto u(t,\cdot) \in H^2((-R,R);\mathbb{R}^n)$ is continuous,*
2. *$[0,T) \ni t \mapsto u(t,\cdot) \in H^1((-R,R);\mathbb{R}^n)$ is continuously differentiable,*
3. *$u(t,x) \in M$ for every $t \in [0,T)$ and $x \in \mathbb{R}$,*
4. *$u(0,x) = u_0(x)$ and $\partial_t u(0,x) = v_0(x)$ for every $x \in \mathbb{R}$,*
5. *for every $t \in [0,T)$ the following holds in $L^2((-R,R);\mathbb{R}^n)$,*

$$\begin{aligned}\partial_t u(t) = v_0 &+ \int_0^t \left[\partial_{xx}u(s) - A_{u(s)}(\partial_x u(s),\partial_x u(s)) + A_{u(s)}(\partial_t u(s),\partial_t u(s))\right] ds \\ &+ \int_0^t Y_{u(s)}(\partial_t u(s),\partial_x u(s))\dot{h}(s)\, ds. \end{aligned} \tag{4.2}$$

Moreover, if there exists another map $U:[0,T)\times\mathbb{R}\to M$ which also satisfies the above properties then

$$U(t,x) = u(t,x) \quad \text{for every} \quad |x| \le R-t \quad \text{and} \quad t \in [0,T).$$

Proof of Theorem 4.1. The method of proof is motivated by Sections 7-11 of [13]. We will seek solutions that take values in the Fréchet space $H^2_{\rm loc}(\mathbb{R};\mathbb{R}^n)\times H^1_{\rm loc}(\mathbb{R};\mathbb{R}^n)$. To this end we will localize the problem using a sequence of non-linear wave equations.

For a given $R>0$, fix $r>R+T$, and $k\in\mathbb{N}$. Let $\varphi:\mathbb{R}\to\mathbb{R}$ be a smooth function such that $\varphi(x)=1$ for $x\in(-r,r)$ and $\varphi(x)=0$ for $x\notin(-2r,2r)$. Next, with the convention $z=(u,v)\in\mathcal{H}$, we define the following maps

$$\begin{aligned}
\mathbf{F}_r &: [0,T]\times\mathcal{H} \ni (t,z) \mapsto \begin{pmatrix} 0 \\ E^1_{r-t}[\mathcal{A}_u(v,v) - \mathcal{A}_u(u_x,u_x)] \end{pmatrix} \in \mathcal{H},\\
\mathbf{F}_{r,k} &: [0,T]\times\mathcal{H} \ni (t,z) \mapsto \begin{cases} \mathbf{F}_r(t,z), & \text{if } |z|_{\mathcal{H}_{r-t}} \le k \\ \left(2-\frac{1}{k}|z|_{\mathcal{H}_{r-t}}\right)\mathbf{F}_r(t,z), & \text{if } k \le |z|_{\mathcal{H}_{r-t}} \le 2k \\ 0, & \text{if } 2k \le |z|_{\mathcal{H}_{r-t}} \end{cases} \in \mathcal{H},\\
\mathbf{G}_r &: [0,T]\times\mathcal{H} \ni (t,z) \mapsto \begin{pmatrix} 0 \\ (E^1_{r-t}Y_u(v,u_x))\cdot \end{pmatrix} \in \mathscr{L}_2(H_\mu,\mathcal{H}),\\
\mathbf{G}_{r,k} &: [0,T]\times\mathcal{H} \ni (t,z) \mapsto \begin{cases} \mathbf{G}_r(t,z), & \text{if } |z|_{\mathcal{H}_{r-t}} \le k \\ \left(2-\frac{1}{k}|z|_{\mathcal{H}_{r-t}}\right)\mathbf{G}_r(t,z), & \text{if } k \le |z|_{\mathcal{H}_{r-t}} \le 2k \\ 0, & \text{if } 2k \le |z|_{\mathcal{H}_{r-t}} \end{cases} \in \mathscr{L}_2(H_\mu,\mathcal{H}),\\
\mathbf{Q}_r &: \mathcal{H} \ni z \mapsto \begin{pmatrix} \varphi\cdot\Upsilon(u) \\ \varphi\cdot\Upsilon'(u)v \end{pmatrix} \in \mathcal{H},
\end{aligned}$$

where $(E^1_{r-t}Y_u(v,u_x))\cdot$ means that, for every $(u,v)\in\mathcal{H}$, $E^1_{r-t}Y_u(v,u_x)\in H^1_{\rm loc}(\mathbb{R};\mathbb{R}^n)$ and the multiplication operator defined as

$$(E^1_{r-t}Y_u(v,u_x))\cdot : H_\mu \ni \xi \mapsto (E^1_{r-t}Y_u(v,u_x))\cdot\xi \in H^1_{\rm loc}(\mathbb{R};\mathbb{R}^n),$$

satisfy Lemma 3.2.

The following two properties of $\mathbf{Q}_r$ are taken from Section 7 of [13].

Lemma 4.2. *If $z = (u, v) \in \mathcal{H}$ is such that $u(x) \in M$ and $v(x) \in T_{u(x)}M$ for $|x| < r$, then $\mathbf{Q}_r(z) = z$ on $(-r, r)$.*

Lemma 4.3. *The mapping $\mathbf{Q}_r$ is of $\mathcal{C}^1$-class and its derivative, with $z = (u, v) \in \mathcal{H}$, satisfies*

$$\mathbf{Q}_r'(z)w = \begin{pmatrix} \varphi \cdot \Upsilon'(u)w^1 \\ \varphi \cdot [\Upsilon''(u)(v, w^1) + \Upsilon'(u)w^2] \end{pmatrix}, \quad w = (w^1, w^2) \in \mathcal{H}.$$

The next lemma is about the locally Lipschitz properties of the localized maps defined above.

Lemma 4.4. *For each $k \in \mathbb{N}$ the functions $\mathbf{F}_r$, $\mathbf{F}_{r,k}$, $\mathbf{G}_r$, $\mathbf{G}_{r,k}$ are continuous and there exists a constant $C_{r,k} > 0$ such that*

$$\|\mathbf{F}_{r,k}(t, z) - \mathbf{F}_{r,k}(t, w)\|_{\mathcal{H}} + \|\mathbf{G}_{r,k}(t, z) - \mathbf{G}_{r,k}(t, w)\|_{\mathscr{L}_2(H_\mu, \mathcal{H})} \leq C_{r,k}\|z - w\|_{\mathcal{H}_{r-t}}, \tag{4.3}$$

holds for every $t \in [0, T]$ and every $z, w \in \mathcal{H}$.

Proof of Lemma 4.4. Let us fix $t \in [0, T]$ and $z = (u, v), w = (\tilde{u}, \tilde{v}) \in \mathcal{H}$. Note that due to the definitions of $\mathbf{F}_{r,k}$ and $\mathbf{G}_{r,k}$, it is sufficient to prove (4.3) in the case $\|z\|_{\mathcal{H}_{r-t}}, \|w\|_{\mathcal{H}_{r-t}} \leq k$.

Let us set $I_{rt} := (t - r, r - t)$. Since in the chosen case $\mathbf{F}_{r,k}(t, z) = \mathbf{F}_r(t, z)$ and $\mathbf{F}_{r,k}(t, w) = \mathbf{F}_r(t, w)$, by Proposition 3.9 and Remark 3.11, there exists $C_E(r, t) > 0$ such that

$$\begin{aligned} \|\mathbf{F}_{r,k}(t, z) - \mathbf{F}_{r,k}(t, w)\|_{\mathcal{H}} \leq C_E(r, t) \big[&\|\mathcal{A}_u(v, v) - \mathcal{A}_{\tilde{u}}(\tilde{v}, \tilde{v})\|_{H^1(I_{rt})} \\ &+ \|\mathcal{A}_u(u_x, u_x) - \mathcal{A}_{\tilde{u}}(\tilde{u}_x, \tilde{u}_x)\|_{H^1(I_{rt})} \big]. \end{aligned} \tag{4.4}$$

Since Υ is smooth and has compact support, see Lemma 3.5, from (3.3) observe that

$$\mathcal{A} : \mathbb{R}^n \ni q \mapsto \mathcal{A}_q \in \mathcal{L}(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n),$$

is smooth, compactly supported (in particular bounded) and globally Lipschitz. Recall the following well-known interpolation inequality, refer [9, (2.12)],

$$\|u\|_{L^\infty(I)}^2 \leq k_e^2 \|u\|_{L^2(I)} \|u\|_{H^1(I)}, \quad u \in H^1(I), \tag{4.5}$$

where I is any open interval in $\mathbb{R}$ and $k_e = 2\max\left\{1, \frac{1}{\sqrt{|I|}}\right\}$. Note that since $r > R+T$ and $t \in [0, T]$, $|I_{rt}| = 2(r - t) > 2R$ and we can choose $k_e = 2\max\left\{1, \frac{1}{\sqrt{|R|}}\right\}$. Then, using the above mentioned properties of $\mathcal{A}$ and the interpolation inequality (4.5) we find that

$$\begin{aligned} \|\mathcal{A}_u(v, v) - \mathcal{A}_{\tilde{u}}(\tilde{v}, \tilde{v})\|_{L^2(I_{rt})} &\leq \|\mathcal{A}_u(v, v) - \mathcal{A}_{\tilde{u}}(v, v)\|_{L^2(I_{rt})} \\ &\quad + \|\mathcal{A}_{\tilde{u}}(v, v) - \mathcal{A}_{\tilde{u}}(\tilde{v}, v)\|_{L^2(I_{rt})} \\ &\quad + \|\mathcal{A}_{\tilde{u}}(\tilde{v}, v) - \mathcal{A}_{\tilde{u}}(\tilde{v}, \tilde{v})\|_{L^2(I_{rt})} \\ &\leq L_{\mathcal{A}} \|v\|_{L^\infty(I_{rt})}^2 \|u - \tilde{u}\|_{L^2(I_{rt})} \\ &\quad + B_{\mathcal{A}} \left[\|v\|_{L^\infty(I_{rt})} + \|\tilde{v}\|_{L^\infty(I_{rt})} \right] \|v - \tilde{v}\|_{L^2(I_{rt})} \end{aligned}$$

$$\leq C(L_{\mathcal{A}}, B_{\mathcal{A}}, R, k, k_e)\|z - w\|_{\mathcal{H}_{r-t}}, \tag{4.6}$$

where $L_{\mathcal{A}}$ and $B_{\mathcal{A}}$ are the Lipschitz constants and bound of $\mathcal{A}$, respectively. Next, since $\mathcal{A}$ is smooth and have compact support, if we set $L_{\mathcal{A}'}$ and $B_{\mathcal{A}'}$ are the Lipschitz constants and bound of

$$\mathcal{A}' : \mathbb{R}^n \ni q \mapsto d_q\mathcal{A} \in \mathcal{L}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n),$$

then by adding and subtracting the terms as we did to get (4.6) followed by the properties of $\mathcal{A}'$ and the interpolation inequality (4.5) we have

$$\begin{aligned}
&\|d_x [\mathcal{A}_u(v, v) - \mathcal{A}_{\tilde{u}}(\tilde{v}, \tilde{v})]\|_{L^2(I_{rt})} \\
&\quad \leq \|d_u\mathcal{A}(v, v)(u_x) - d_{\tilde{u}}\mathcal{A}(\tilde{v}, \tilde{v})(\tilde{u}_x)\|_{L^2(I_{rt})} + 2\|\mathcal{A}_u(v_x, v) - \mathcal{A}_{\tilde{u}}(\tilde{v}_x, \tilde{v})\|_{L^2(I_{rt})} \\
&\leq L_{\mathcal{A}'}\|u_x\|_{L^\infty(I_{rt})}\|v\|^2_{L^\infty(I_{rt})}\|u - \tilde{u}\|_{L^2(I_{rt})} + B_{\mathcal{A}'}\|v\|^2_{L^\infty(I_{rt})}\|u_x - \tilde{u}_x\|_{L^2(I_{rt})} \\
&\quad + B_{\mathcal{A}'}\left[\|v\|_{L^\infty(I_{rt})} + \|\tilde{v}\|_{L^\infty(I_{rt})}\right]\|v - \tilde{v}\|_{L^2(I_{rt})}\|\tilde{u}_x\|_{L^\infty(I_{rt})} \\
&\quad + 2\left[L_{\mathcal{A}}\|u - \tilde{u}\|_{L^\infty(I_{rt})}\|v\|_{L^\infty(I_{rt})}\|v_x\|_{L^2(I_{rt})} + B_{\mathcal{A}}\|v_x - \tilde{v}_x\|_{L^2(I_{rt})}\|v\|_{L^\infty(I_{rt})}\right. \\
&\quad \left. + B_{\mathcal{A}}\|v - \tilde{v}\|_{L^\infty(I_{rt})}\|\tilde{v}_x\|_{L^2(I_{rt})}\right] \\
&\lesssim_{L_{\mathcal{A}}, B_{\mathcal{A}}, L_{\mathcal{A}'}, B_{\mathcal{A}'}, k_e} \left[\|u - \tilde{u}\|_{H^2(I_{rt})}\|u\|_{H^2(I_{rt})}\|v\|^2_{H^1(I_{rt})} + \|u - \tilde{u}\|_{H^2(I_{rt})}\|v\|^2_{H^1(I_{rt})}\right. \\
&\quad + \|v - \tilde{v}\|_{H^1(I_{rt})}\left[\|v\|_{H^1(I_{rt})} + \|\tilde{v}\|_{H^1(I_{rt})}\right]\|\tilde{u}\|_{H^2(I_{rt})} + \|u - \tilde{u}\|_{H^2(I_{rt})}\|v\|^2_{H^1(I_{rt})} \\
&\quad \left. + \|v - \tilde{v}\|_{H^1(I_{rt})}\left(\|v\|_{H^1(I_{rt})} + \|\tilde{v}\|_{H^1(I_{rt})}\right)\right] \\
&\lesssim_k \|z - w\|_{\mathcal{H}_{r-t}},
\end{aligned} \tag{4.7}$$

where the last step is due to the case $\|z\|_{\mathcal{H}_{r-t}}, \|w\|_{\mathcal{H}_{r-t}} \leq k$. By following similar procedure of (4.6) and (4.7) we also get

$$\|\mathcal{A}_u(u_x, u_x) - \mathcal{A}_{\tilde{u}}(\tilde{u}_x, \tilde{u}_x)\|_{H^1(I_{rt})} \lesssim_{L_{\mathcal{A}}, B_{\mathcal{A}}, L_{\mathcal{A}'}, B_{\mathcal{A}'}, k_e, k} \|z - w\|_{\mathcal{H}_{r-t}}.$$

Hence by substituting the estimates back in (4.4) we are done with (4.3) for $F_{r,k}$-term.

Next, we move to the terms of $G_{r,k}$. As for $F_{r,k}$ we only show the calculations in the case $\|z\|_{\mathcal{H}_{r-t}}, \|w\|_{\mathcal{H}_{r-t}} \leq k$. By invoking Lemma 3.2 followed by Remark 3.11 we have

$$\begin{aligned}
\|\mathbf{G}_{r,k}(t, z) - \mathbf{G}_{r,k}(t, w)\|^2_{\mathscr{L}_2(H_\mu, \mathcal{H})} &\leq \|(E^1_{r-t}Y_u(v, u_x)) \cdot - (E^1_{r-t}Y_{\tilde{u}}(\tilde{v}, \tilde{u}_x)) \cdot\|^2_{\mathscr{L}_2(H_\mu, H^1(\mathbb{R}))} \\
&\leq c_{r,t}\, C_E(r, t)\, \|Y_u(v, u_x) - Y_{\tilde{u}}(\tilde{v}, \tilde{u}_x)\|^2_{H^1(I_{rt})}.
\end{aligned}$$

Recall that the 1-D Sobolev embedding gives $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$. Consequently, by Taylor's formula [22, Theorem 5.6.1] and (3.9)-(3.10) we have

$$\begin{aligned}
\|Y_u(v, \partial_x u) - Y_{\tilde{u}}(\tilde{v}, \tilde{u}_x)\|^2_{L^2(I_{rt})} &\leq \int_{I_{rt}} |Y_{u(x)}(v(x), u_x(x)) - Y_{\tilde{u}(x)}(v(x), u_x(x))|^2\, dx \\
&\quad + \int_{I_{rt}} |Y_{\tilde{u}(x)}(v(x), u_x(x)) - Y_{\tilde{u}(x)}(v(x), \tilde{u}_x(x))|^2\, dx \\
&\quad + \int_{I_{rt}} |Y_{\tilde{u}(x)}(v(x), \tilde{u}_x(x)) - Y_{\tilde{u}(x)}(\tilde{v}(x), \tilde{u}_x(x))|^2\, dx \\
&\leq C_Y^2\left[1 + \|v\|^2_{H^1(I_{rt})} + \|u\|^2_{H^1(I_{rt})}\right]\|u - \tilde{u}\|^2_{H^2(I_{rt})}
\end{aligned}$$

$$
\begin{aligned}
&+ C_{Y_2}^2 \left[\|u_x - \tilde{u}_x\|^2_{H^1(I_{rt})} + \|v - \tilde{v}\|^2_{H^1(I_{rt})}\right] \\
&\lesssim_{k, C_Y, C_{Y_2}} \|z - w\|^2_{\mathcal{H}_{r-t}}. \qquad (4.8)
\end{aligned}
$$

For homogeneous part of the norm, that is L^2-norm of the derivative, we have

$$
\begin{aligned}
&\|d_x \left[Y_u(v, u_x) - Y_{\tilde{u}}(\tilde{v}, \tilde{u}_x)\right]\|^2_{L^2(I_{rt})} \\
&\lesssim \int_{I_{rt}} \sum_{i=1}^n \Bigg\{ \left| \frac{\partial Y}{\partial p_i}(u(x), v(x), u_x(x)) \frac{du^i}{dx}(x) - \frac{\partial Y}{\partial p_i}(\tilde{u}(x), \tilde{v}(x), \tilde{u}_x(x)) \frac{d\tilde{u}^i}{dx}(x) \right|^2 \\
&\quad + \left| \frac{\partial Y}{\partial a_i}(u(x), v(x), u_x(x)) \frac{dv^i}{dx}(x) - \frac{\partial Y}{\partial a_i}(\tilde{u}(x), \tilde{v}(x), \tilde{u}_x(x)) \frac{d\tilde{v}^i}{dx}(x) \right|^2 \\
&\quad + \left| \frac{\partial Y}{\partial b_i}(u(x), v(x), u_x(x)) \frac{du_x^i}{dx}(x) - \frac{\partial Y}{\partial b_i}(\tilde{u}(x), \tilde{v}(x), \tilde{u}_x(x)) \frac{d\partial_x \tilde{u}^i}{dx}(x) \right|^2 \Bigg\} \, dx \\
&=: Y_1 + Y_2 + Y_3. \qquad (4.9)
\end{aligned}
$$

We will estimate each term separately by using 1-D Sobolev embedding, Taylor's formula and (3.9)-(3.11) as follows:

$$
\begin{aligned}
Y_1 &\lesssim \int_{I_{rt}} \sum_{i=1}^n \left| \frac{\partial Y}{\partial p_i}(u(x), v(x), u_x(x)) \frac{du^i}{dx}(x) - \frac{\partial Y}{\partial p_i}(\tilde{u}(x), \tilde{v}(x), \tilde{u}_x(x)) \frac{d\tilde{u}^i}{dx}(x) \right|^2 dx \\
&\lesssim \int_{I_{rt}} \sum_{i=1}^n \Bigg\{ \left| \frac{\partial Y}{\partial p_i}(u(x), v(x), \; u_x(x)) \frac{du^i}{dx}(x) - \frac{\partial Y}{\partial p_i}(\tilde{u}(x), v(x), u_x(x)) \frac{du^i}{dx}(x) \right|^2 \\
&\quad + \left| \frac{\partial Y}{\partial p_i}(\tilde{u}(x), v(x), u_x(x)) \frac{du^i}{dx}(x) - \frac{\partial Y}{\partial p_i}(\tilde{u}(x), v(x), u_x(x)) \frac{d\tilde{u}^i}{dx}(x) \right|^2 \\
&\quad + \left| \frac{\partial Y}{\partial p_i}(\tilde{u}(x), v(x), u_x(x)) \frac{d\tilde{u}^i}{dx}(x) - \frac{\partial Y}{\partial p_i}(\tilde{u}(x), \tilde{v}(x), u_x(x)) \frac{d\tilde{u}^i}{dx}(x) \right|^2 \\
&\quad + \left| \frac{\partial Y}{\partial p_i}(\tilde{u}(x), \tilde{v}(x), u_x(x)) \frac{d\tilde{u}^i}{dx}(x) - \frac{\partial Y}{\partial p_i}(\tilde{u}(x), \tilde{v}(x), \tilde{u}_x(x)) \frac{d\tilde{u}^i}{dx}(x) \right|^2 \Bigg\} \, dx \\
&\lesssim C_{Y_3}^2 \|u - \tilde{u}\|^2_{L^2(I_{rt})} \|u_x\|^2_{H^1(I_{rt})} + C_{Y_1}^2 \left[1 + \|v\|^2_{H^1(I_{rt})} + \|u_x\|^2_{H^1(I_{rt})}\right] \|u_x - \tilde{u}_x\|^2_{L^2(I_{rt})} \\
&\quad + C_{Y_3}^2 \|v - \tilde{v}\|^2_{L^2(I_{rt})} \|\tilde{u}_x\|^2_{H^1(I_{rt})} + C_{Y_3}^2 \|u_x - \tilde{u}_x\|^2_{L^2(I_{rt})} \|\tilde{u}_x\|^2_{H^1(I_{rt})} \\
&\lesssim_{k, C_{Y_2}, C_{Y_3}, C_{Y_1}} \|z - w\|^2_{\mathcal{H}_{r-t}}. \qquad (4.10)
\end{aligned}
$$

Terms Y_2 and Y_3 are quite similar so it is enough to estimate only one. We do the calculation for Y_2.

$$
\begin{aligned}
Y_2 &= \int_{I_{rt}} \sum_{i=1}^n \left| \frac{\partial Y}{\partial a_i}(u(x), v(x), u_x(x)) \frac{dv^i}{dx}(x) - \frac{\partial Y}{\partial a_i}(\tilde{u}(x), \tilde{v}(x), \tilde{u}_x(x)) \frac{d\tilde{v}^i}{dx}(x) \right|^2 dx \\
&\lesssim \int_{I_{rt}} \sum_{i=1}^n \Bigg\{ \left| \frac{\partial Y}{\partial a_i}(u(x), v(x), u_x(x)) \frac{dv^i}{dx}(x) - \frac{\partial Y}{\partial a_i}(\tilde{u}(x), v(x), u_x(x)) \frac{dv^i}{dx}(x) \right|^2 dx
\end{aligned}
$$

$$
\begin{aligned}
&+\left|\frac{\partial Y}{\partial a_i}(\tilde u(x),v(x),u_x(x))\frac{dv^i}{dx}(x)-\frac{\partial Y}{\partial a_i}(\tilde u(x),\tilde v(x),u_x(x))\frac{dv^i}{dx}(x)\right|^2 dx\\
&+\left|\frac{\partial Y}{\partial a_i}(\tilde u(x),\tilde v(x),u_x(x))\frac{dv^i}{dx}(x)-\frac{\partial Y}{\partial a_i}(\tilde u(x),\tilde v(x),\tilde u_x(x))\frac{dv^i}{dx}(x)\right|^2 dx\\
&+\left|\frac{\partial Y}{\partial a_i}(\tilde u(x),\tilde v(x),\tilde u_x(x))\frac{dv^i}{dx}(x)-\frac{\partial Y}{\partial a_i}(\tilde u(x),\tilde v(x),\tilde u_x(x))\frac{d\tilde v^i}{dx}(x)\right|^2 dx\Bigg\}\\
&\lesssim C_{Y_3}^2\|u-\tilde u\|^2_{H^1(I_{rt})}\|v_x\|^2_{L^2(I_{rt})}+C_{Y_3}^2\|v-\tilde v\|^2_{H^1(I_{rt})}\|v_x\|^2_{L^2(I_{rt})}\\
&\quad+C_{Y_3}^2\|u_x-\tilde u_x\|^2_{H^1(I_{rt})}\|v_x\|^2_{L^2(I_{rt})}+C_{Y_3}^2C_{r,t}\|v_x-\tilde v_x\|^2_{L^2(I_{rt})}\\
&\lesssim_{k,C_{r,t}C_{Y_3}}\|z-w\|^2_{\mathcal{H}_{r-t}}.
\end{aligned}\tag{4.11}
$$

Hence by substituting (4.10)-(4.11) into (4.9) we get

$$
\|d_x\left[Y_u(v,u_x)-Y_{\tilde u}(\tilde v,\tilde u_x)\right]\|^2_{L^2(I_{rt})}\lesssim_{k,C_{r,t},C_{Y_2},C_{Y_3},C_{Y_1}}\|z-w\|^2_{\mathcal{H}_{r-t}}.
$$

which together with (4.8) gives $G_{r,k}$ part of (4.3). Hence the Lipschitz property Lemma 4.4. □

The following result follows directly from Lemma 4.4 and the standard theory of PDE via semigroup approach, refer [1] and [39] for detailed proof.

Corollary 4.5. *Given any $\xi\in\mathcal{H}$ and $h\in {}_0H^{1,2}(0,T;H_\mu)$, there exists a unique z in $\mathcal{C}([0,T];\mathcal{H})$ such that for all $t\in[0,T]$*

$$
z(t)=S_t\xi+\int_0^t S_{t-s}\mathbf{F}_{r,k}(s,z(s))\,ds+\int_0^t S_{t-s}(\mathbf{G}_{r,k}(s,z(s))\dot h(s))\,ds.
$$

Remark 4.6. *Here by $\mathbf{G}_{r,k}(s,z(s))\dot h(s)$ we mean that both the components of $\mathbf{G}_{r,k}(s,z(s))$ are acting on $\dot h(s)$.*

From now on, for each $r>2T$ and $k\in\mathbb{N}$, the solution from Corollary 4.5 will be denoted by $z_{r,k}$ and called the *approximate solution*. To proceed further we define the following two auxiliary functions

$$
\begin{aligned}
\widetilde F_{r,k}\ &:\ [0,T]\times\mathcal{H}\ni(t,z)\mapsto\begin{pmatrix}0\\ \varphi\cdot\Upsilon'(u)\mathbf{F}^2_{r,k}(t,z)+\varphi B_u(v,v)-\varphi B_u(u_x,u_x)\end{pmatrix}\\
&\qquad\qquad-\begin{pmatrix}0\\ \Delta\varphi\cdot h(u)+2\varphi_x\cdot h'(u)u_x\end{pmatrix}\in\mathcal{H},
\end{aligned}
$$

and

$$
\widetilde G_{r,k}\ :\ [0,T]\times\mathcal{H}\ni(t,z)\mapsto\begin{pmatrix}0\\ \varphi\cdot\Upsilon'(u)\mathbf{G}^2_{r,k}(t,z)\end{pmatrix}\in\mathcal{H}.
$$

Here $\mathbf{F}^2_{r,k}(s,z_{r,k}(s))$ and $\mathbf{G}^2_{r,k}(s,z_{r,k}(s))$ denote the second components of the vectors $\mathbf{F}_{r,k}(s,z_{r,k}(s))$ and $\mathbf{G}_{r,k}(s,z_{r,k}(s))$, respectively. The following corollary relates the solution $z_{r,k}$ with its transformation under the map $\mathbf{Q}_r$ and allow to understand the need of the functions $\widetilde F_{r,k}$ and $\widetilde G_{r,k}$.

Corollary 4.7. *Let us assume that* $\xi := (E_r^2 u_0, E_r^1 v_0)$ *and that* $z_{r,k} \in \mathcal{C}([0,T];\mathcal{H})$ *satisfies*

$$z_{r,k}(t) = S_t\xi + \int_0^t S_{t-s}\mathbf{F}_{r,k}(s, z_{r,k}(s))\, ds + \int_0^t S_{t-s}(\mathbf{G}_{r,k}(s, z_{r,k}(s))\dot{h}(s))\, ds, \qquad t \in [0,T]. \tag{4.12}$$

Then $\widetilde{z}_{r,k} = \mathbf{Q}_r(z_{r,k})$ *satisfies, for each* $t \in [0,T]$,

$$\widetilde{z}_{r,k}(t) = S_t\mathbf{Q}_r(\xi) + \int_0^t S_{t-s}\widetilde{F}_{r,k}(s, z_{r,k}(s))\, ds + \int_0^t S_{t-s}(\widetilde{G}_{r,k}(s, z_{r,k}(s))\dot{h}(s))\, ds.$$

Proof of Corollary 4.7. First observe that by the action of $\mathbf{Q}_r'$ and $\mathcal{G}$ on the elements of $\mathcal{H}$ from Lemma 4.3 and (3.8), respectively, we get

$$\begin{aligned}
&\mathbf{Q}_r'(z_{r,k}(s)) \left(\mathbf{F}_{r,k}(s, z_{r,k}(s)) + \mathbf{G}_{r,k}(s, z_{r,k}(s))\dot{h}(s)\right) \\
&\qquad = \begin{pmatrix} 0 \\ \varphi \cdot \left\{[\Upsilon'(u_{r,k}(s))](\mathbf{F}^2_{r,k}(s, z_{r,k}(s))) + [\Upsilon'(u_{r,k}(s))](\mathbf{G}^2_{r,k}(s, z_{r,k}(s))\dot{h}(s))\right\} \end{pmatrix}.
\end{aligned} \tag{4.13}$$

Moreover, since by applying Lemma 4.3 and (3.8) to $z = (u,v) \in \mathcal{H}$ we have

$$\begin{aligned}
F(z) &:= \mathbf{Q}_r'\mathcal{G}z - \mathcal{G}\mathbf{Q}_r z = \begin{pmatrix} \varphi \cdot [\Upsilon'(u)](v) \\ \varphi \cdot \{[\Upsilon''(u)](v,v) + [\Upsilon'(u)](u'')\} \end{pmatrix} \\
&\quad - \begin{pmatrix} \varphi \cdot [\Upsilon'(u)](v) \\ \varphi'' \cdot \Upsilon(u) + 2\varphi' \cdot [\Upsilon'(u)](u') + \varphi \cdot [\Upsilon'(u)](u'') + \varphi \cdot [\Upsilon''(u)](u',u') \end{pmatrix},
\end{aligned} \tag{4.14}$$

substitution $z = z_{r,k}(s) = (u_{r,k}(s), v_{r,k}(s)) \in \mathcal{H}$ in (4.14) with (4.13) followed by definition (3.2) gives, for $s \in [0,T]$,

$$\begin{aligned}
&\mathbf{Q}_r'(z_{r,k}(s))\left(\mathbf{F}_{r,k}(s, z_{r,k}(s)) + \mathbf{G}_{r,k}(s, z_{r,k}(s))\right) + F(z_{r,k}(s)) \\
&= \begin{pmatrix} 0 \\ \varphi \cdot [\Upsilon'(u_{r,k}(s))](\mathbf{F}^2_{r,k}(s, z_{r,k}(s))) + \varphi \cdot [\Upsilon''(u_{r,k}(s))](v_{r,k}(s), v_{r,k}(s)) \\ -\varphi \cdot [\Upsilon''(u_{r,k}(s))](\partial_x u_{r,k}(s), \partial_x u_{r,k}(s)) \end{pmatrix} \\
&- \begin{pmatrix} 0 \\ -\varphi'' \cdot \Upsilon(u_{r,k}(s)) + 2\varphi' \cdot [\Upsilon'(u_{r,k}(s))](\partial_x u_{r,k}(s)) + \varphi \cdot [\Upsilon'(u_{r,k}(s))](\mathbf{G}^2_{r,k}(s, z_{r,k}(s))) \end{pmatrix} \\
&= \widetilde{F}_{r,k}(s, z_{r,k}(s)) + \widetilde{G}_{r,k}(s, z_{r,k}(s)).
\end{aligned}$$

Hence, if we have

$$\int_0^T \left[\|\mathbf{F}_{r,k}(s, z_{r,k}(s))\|_{\mathcal{H}} + \|\mathbf{G}_{r,k}(s, z_{r,k}(s))\dot{h}(s)\|_{\mathcal{H}}\right] ds < \infty, \tag{4.15}$$

then by invoking [13, Lemma 6.4] with

$$L = \mathbf{Q}_r, K = U = \mathcal{H}, A = B = \mathcal{G}, g(s) = 0, f(s) = \mathbf{F}_{r,k}(s, z_{r,k}(s)) + \mathbf{G}_{r,k}(s, z_{r,k}(s))\dot{h}(s),$$

we are done with the proof here. But (4.15) follows by Lemma 4.4, because $h \in {}_0H^{1,2}(0,T;H_\mu)$ and the following holds, due to the Hölder inequality with the abuse of notation as mentioned in Remark 4.6,

$$\int_0^T \|\mathbf{G}_{r,k}(s, z_{r,k}(s))\dot{h}(s)\|_{\mathcal{H}}\, ds = \int_0^T \|\mathbf{G}^2_{r,k}(s, z_{r,k}(s))\dot{h}(s)\|_{H^1(\mathbb{R})}\, ds$$
$$\leq \left(\int_0^T \|(\mathbf{G}^2_{r,k}(s, z_{r,k}(s)))\cdot\|^2_{\mathscr{L}_2(H_\mu, H^1(\mathbb{R}))}\, ds\right)^{\frac{1}{2}} \left(\int_0^T \|\dot{h}(s)\|^2_{H_\mu}\, ds\right)^{\frac{1}{2}}.$$

□

Next we prove that the approximate solution $z_{r,k}$ stays on the manifold. Define the following three positive reals: for each $r > R + T$ and $k \in \mathbb{N}$,

$$\begin{cases} \tau_k^1 := \inf\{t \in [0,T] : \|z_{r,k}(t)\|_{\mathcal{H}_{r-t}} \geq k\}, \\ \tau_k^2 := \inf\{t \in [0,T] : \|\widetilde{z}_{r,k}(t)\|_{\mathcal{H}_{r-t}} \geq k\}, \\ \tau_k^3 := \inf\{t \in [0,T] : \exists x,\, |x| \leq r-t,\, u_{r,k}(t,x) \notin O\}, \\ \tau_k := \tau_k^1 \wedge \tau_k^2 \wedge \tau_k^3. \end{cases} \tag{4.16}$$

Also, define the following $\mathcal{H}$-valued functions of time $t \in [0,T]$

$$a_k(t) = S_t\xi + \int_0^t S_{t-s}\mathbb{1}_{[0,\tau_k)}(s)\mathbf{F}_{r,k}(s, z_{r,k}(s))\, ds + \int_0^t S_{t-s}(\mathbb{1}_{[0,\tau_k)}(s)\mathbf{G}_{r,k}(s, z_{r,k}(s))\dot{h}(s))\, ds$$
$$\widetilde{a}_k(t) = S_t\mathbf{Q}_r(\xi) + \int_0^t S_{t-s}\mathbb{1}_{[0,\tau_k)}(s)\widetilde{F}_{r,k}(s, z_{r,k}(s))\, ds + \int_0^t S_{t-s}(\mathbb{1}_{[0,\tau_k)}(s)\widetilde{G}_{r,k}(s, z_{r,k}(s))\dot{h}(s))\, ds. \tag{4.17}$$

Proposition 4.8. *For each $k \in \mathbb{N}$ and $\xi := (E_r^2 u_0, E_r^1 v_0)$, the functions a_k, $\widetilde{a}_k$, $z_{r,k}$ and $\widetilde{z}_{r,k}$ coincide on $[0,\tau_k)$. In particular, $u_{r,k}(t,x) \in M$ for $|x| \leq r-t$ and $t \leq \tau_k$. Consequently, $\tau_k = \tau_k^1 = \tau_k^2 \leq \tau_k^3$.*

Proof of Proposition 4.8. Let us fix k. First note that, due to indicator function,

$$a_k = z_{r,k} \qquad \text{and} \qquad \widetilde{a}_k = \widetilde{z}_{r,k} \text{ on } [0,\tau_k). \tag{4.18}$$

Next, since $E^1_{r-s}f = f$ on $|x| \leq r-s$, see Proposition 3.9, and $\varphi = 1$ on $(-r,r)$, by Lemma 4.2 followed by (3.4) we infer that

$$\begin{cases} \mathbb{1}_{[0,\tau_k)}(s)[\widetilde{F}_{r,k}(s, z_{r,k}(s))](x) = \mathbb{1}_{[0,\tau_k)}(s)[\mathbf{F}_{r,k}(s, \widetilde{z}_{r,k}(s))](x), \\ \mathbb{1}_{[0,\tau_k)}(s)[\widetilde{G}_{r,k}(s, z_{r,k}(s))e](x) = \mathbb{1}_{[0,\tau_k)}(s)[\mathbf{G}_{r,k}(s, \widetilde{z}_{r,k}(s))e](x), \quad e \in K, \end{cases} \tag{4.19}$$

holds for every $|x| \leq r-s$, $0 \leq s \leq T$. Now we claim that if we denote

$$p(t) := \frac{1}{2}\|a_k(t) - \widetilde{a}_k(t)\|^2_{\mathcal{H}_{r-t}},$$

then the map $s \mapsto p(s \wedge \tau_k)$ is continuous and uniformly bounded. Indeed, since, by Proposition 3.9, $\xi(x) = (u_0(x), v_0(x)) \in TM$ for $|x| \leq r$, the uniform boundedness is an easy consequence of bound property of C_0-group, Lemmata 4.2 and 4.4. Continuity of $s \mapsto p(s \wedge \tau_k)$ follows from the following:

(1) for every $z \in \mathcal{H}$, the map $t \mapsto \|z\|^2_{\mathcal{H}_{r-t}}$ is continuous;

(2) for each t, the map

$$L^2(\mathbb{R}) \ni u \mapsto \int_0^t |u(s)|^2\,ds \in \mathbb{R},$$

is locally Lipschitz.

Now observe that by applying Proposition B.1 for

$$k = 1, L = I, T = r, x = 0 \quad \text{and} \quad z(t) = (u(t), v(t)) := a_k(t) - \widetilde{a}_k(t),$$

we get $\mathbf{e}(t, z(t)) = p(t)$, and the following

$$\mathbf{e}(t, z(t)) \le \mathbf{e}(0, z_0) + \int_0^t V(r, z(r))\,dr. \tag{4.20}$$

Here

$$\begin{aligned} V(t, z(t)) := {}& \langle u(t), v(t)\rangle_{L^2(B_{r-t})} + \langle v(t), f(t)\rangle_{L^2(B_{r-t})} + \langle \partial_x v(t), \partial_x f(t)\rangle_{L^2(B_{r-t})} \\ &+ \langle v(t), g(t)\rangle_{L^2(B_{r-t})} + \langle \partial_x v(t), \partial_x g(t)\rangle_{L^2(B_{r-t})}, \end{aligned}$$

and

$$\begin{aligned} \begin{pmatrix} 0 \\ f(t) \end{pmatrix} &:= \mathbb{1}_{[0,\tau_k)}(t)[\mathbf{F}_{r,k}(s, z_{r,k}(t)) - \widetilde{F}_{r,k}(s, z_{r,k}(t))], \\ \begin{pmatrix} 0 \\ g(t) \end{pmatrix} &:= \mathbb{1}_{[0,\tau_k)}(t)[\mathbf{G}_{r,k}(s, z_{r,k}(t))\dot{h}(t) - \widetilde{G}_{r,k}(s, z_{r,k}(t))\dot{h}(t)]. \end{aligned}$$

Due to the extension operators E_r^2 and E_r^1 the initial data ξ in the definition (4.17) satisfies the assumption of Lemma 4.2, $S_t\mathbf{Q}_r(\xi) = S_t\xi$, and so $\mathbf{e}(0, z(0)) = p(0) = 0$. Next observe that by the Cauchy-Schwarz inequality we have

$$\begin{aligned} V(t, z(t)) &\le \frac{1}{2}\|u(t)\|^2_{L^2(B_{r-t})} + \frac{3}{2}\|v(t)\|^2_{L^2(B_{r-t})} + \frac{1}{2}\|f(t)\|^2_{L^2(B_{r-t})} + \|\partial_x v(t)\|^2_{L^2(B_{r-t})} \\ &\quad + \frac{1}{2}\|\partial_x f(t)\|^2_{L^2(B_{r-t})} + \frac{1}{2}\|g(t)\|^2_{L^2(B_{r-t})} + \frac{1}{2}\|\partial_x g(t)\|^2_{L^2(B_{r-t})} \\ &\le 3p(t) + \frac{1}{2}\|f(t)\|^2_{H^1(B_{r-t})} + \frac{1}{2}\|g(t)\|^2_{H^1(B_{r-t})}. \end{aligned}$$

Using above into (4.20) and, then, invoking equalities (4.19) and (4.18), definition (4.16), Lemma 3.2 and Lemma 4.4 we have the following calculation, for every $t \in [0, T]$,

$$\begin{aligned} p(t) &\le \int_0^t 3p(s)\,ds + \frac{1}{2}\int_0^t \mathbb{1}_{[0,\tau_k)}(s)\|\mathbf{F}^2_{r,k}(s, z_{r,k}(s)) - \mathbf{F}^2_{r,k}(s, \tilde{z}_{r,k}(s))\|^2_{H^1(B_{r-s})}\,ds \\ &\quad + \frac{1}{2}\int_0^t \mathbb{1}_{[0,\tau_k)}(s)\|\mathbf{G}^2_{r,k}(s, z_{r,k}(s)) - \mathbf{G}^2_{r,k}(s, \tilde{z}_{r,k}(s))\|^2_{\mathscr{L}_2(H_\mu, H^1(B_{r-s}))}\|\dot{h}(s)\|^2_{H_\mu}\,ds \\ &\le 3\int_0^t p(s)\,ds + \frac{1}{2}C^2_{r,k}\int_0^t \mathbb{1}_{[0,\tau_k)}(s)\|z_{r,k}(s) - \widetilde{z}_{r,k}(s)\|^2_{\mathcal{H}_{r-s}}\,ds \\ &\quad + \frac{1}{2}C^2_{r,k}\int_0^t \mathbb{1}_{[0,\tau_k)}(s)\|z_{r,k}(s) - \widetilde{z}_{r,k}(s)\|^2_{\mathcal{H}_{r-s}}\|\dot{h}(s)\|^2_{H_\mu}\,ds \end{aligned}$$

$$\leq (3 + C_{r,k}^2) \int_0^t p(s)(1 + \|\dot{h}(s)\|_{H_\mu}^2)\, ds. \tag{4.21}$$

Consequently by the Gronwall Lemma, for $t \in [0, \tau_k]$,

$$p(t) \lesssim_{C_{r,k}} p(0) \exp\left[\int_0^t (1 + \|\dot{h}(s)\|_{H_\mu}^2)\, ds\right]. \tag{4.22}$$

Note that the right hand side in (4.22) is finite because $h \in {}_0H^{1,2}(0, T; H_\mu)$. Since we know that $p(0) = 0$ we arrive to $p(t) = 0$ on $t \in [0, \tau_k]$. This further implies that $a_k(t, x) = \widetilde{a}_k(t, x)$ hold for $|x| \leq r - t$ and $t \leq \tau_k$. Consequently, $z_{r,k}(t, x) = \widetilde{z}_{r,k}(t, x)$ hold for $|x| \leq r - t$ and $t \leq \tau_k$. So, because $\widetilde{z}_{r,k}(t, x) = \mathbf{Q}_r(z_{r,k}(t))$ and $\varphi = 1$ on $(-r, r)$,

$$u_{r,k}(t, x) = \Upsilon(u_{r,k}(t, x)), \qquad \text{for } |x| \leq r - t, \quad t \leq \tau_k. \tag{4.23}$$

Since, by definition (4.16) of τ_k, $u_{r,k}(t, x) \in O$, equality (4.23) and Lemma 3.5, gives $u_{r,k}(t, x) \in M$ for $|x| \leq r - t$ and $t \leq \tau_k$. This suggests that $\tau_k \leq \tau_k^3$ and hence $\tau_k = \tau_k^1 \wedge \tau_k^2$. It remains to show that $\tau_k^1 = \tau_k^2$. But suppose it does not hold and without loss of generality we assume that $\tau_k^1 > \tau_k^2$. Then by definition (4.16) and the continuity of $z_{r,k}$ and $\tilde{z}_{r,k}$ in time we have

$$\|z_{r,k}(\tau_k^2, \cdot)\|_{\mathcal{H}_{r-\tau_k^2}} < k \quad \text{ but } \quad \|\widetilde{z}_{r,k}(\tau_k^2, \cdot)\|_{\mathcal{H}_{r-\tau_k^2}} \geq k,$$

which contradicts the above mentioned consequence of $p = 0$ on $[0, \tau_k]$. Hence we conclude that $\tau_k^1 = \tau_k^2$ and this finishes the proof of Proposition 4.8. □

Next in the ongoing proof of Theorem 4.1 we show that the approximate solutions extend each other. Recall that $r > R + T$ is fixed for given $T > 0$.

Lemma 4.9. *Let $k \in \mathbb{N}$ and $\xi = (E_r^2 u_0, E_r^1 v_0)$. Then $z_{r,k+1}(t, x) = z_{r,k}(t, x)$ on $|x| \leq r - t$, $t \leq \tau_k$, and $\tau_k \leq \tau_{k+1}$.*

Proof of Lemma 4.9. Define

$$p(t) := \frac{1}{2}\|a_{k+1}(t) - a_k(t)\|_{H^1(B_{r-t}) \times L^2(B_{r-t})}^2.$$

As an application of Proposition B.1, by performing the computation based on (4.20) - (4.21), with $k = 0$ and rest the same, we obtain

$$\begin{aligned} p(t) \leq 2 \int_0^t p(s)\, ds &+ \frac{1}{2}\int_0^t \|\mathbb{1}_{[0,\tau_{k+1})}(s)\mathbf{F}_r^2(s, z_{r,k+1}(s)) - \mathbb{1}_{[0,\tau_k)}(s)\mathbf{F}_r^2(s, z_{r,k}(s))\|_{L^2(B_{r-s})}^2\, ds \\ &+ \frac{1}{2}\int_0^t \|\mathbb{1}_{[0,\tau_{k+1})}(s)\mathbf{G}_r^2(s, z_{r,k+1}(s))\dot{h}(s) - \mathbb{1}_{[0,\tau_k)}(s)\mathbf{G}_r^2(s, z_{r,k}(s))\dot{h}(s)\|_{L^2(B_{r-s})}^2\, ds. \end{aligned} \tag{4.24}$$

Then, since F_r and G_r depends on $u_{r,k}(s)$, $u_{r,k+1}(s)$ and their first partial derivatives, with respect to time t and space x, which are actually bounded on the interval $(-(r - s), r - s)$ by some constant C_r for every $s < \tau_{k+1} \wedge \tau_k$, by evaluating (4.24) on $t \wedge \tau_{k+1} \wedge \tau_k$ following the use of Lemmata 4.4 and 3.2 we get

$$p(t \wedge \tau_{k+1} \wedge \tau_k) \leq 2\int_0^t p(s \wedge \tau_{k+1} \wedge \tau_k)\, ds$$

$$
\begin{aligned}
&+\frac{1}{2}\int_0^{t\wedge\tau_{k+1}\wedge\tau_k} \|\mathbf{F}_r^2(s,z_{r,k+1}(s))-\mathbf{F}_r^2(s,z_{r,k}(s))\|_{L^2(B_{r-s})}^2\,ds\\
&+\frac{1}{2}\int_0^{t\wedge\tau_{k+1}\wedge\tau_k} \|\mathbf{G}_r^2(s,z_{r,k+1}(s))\zeta(s)-\mathbf{G}_r^2(s,z_{r,k}(s))\dot{h}(s)\|_{L^2(B_{r-s})}^2\,ds\\
\lesssim_k &\int_0^t p(s\wedge\tau_{k+1}\wedge\tau_k)(1+\|\dot{h}(s)\|_{H_\mu}^2)\,ds.
\end{aligned}
$$

Hence by the Gronwall Lemma we infer that $p=0$ on $[0,\tau_{k+1}\wedge\tau_k]$.

Consequently, we claim that $\tau_k\le\tau_{k+1}$. We divide the proof of our claim in the following three exhaustive subcases. Due to (4.16), the subcases when $\|\xi\|_{\mathcal{H}_r}>k+1$ and $k<\|\xi\|_{\mathcal{H}_r}\le k+1$ are trivial. In the last subcase when $\|\xi\|_{\mathcal{H}_r}\le k$ we prove the claim $\tau_k\le\tau_{k+1}$ by the method of contradiction, and so assume that $\tau_k>\tau_{k+1}$ is true. Then, because of continuity in time of $z_{r,k}$ and $z_{r,k+1}$, by (4.16) we have

$$
\|z_{r,k}(\tau_{k+1})\|_{\mathcal{H}_{r-\tau_{k+1}}}<k \quad \text{and} \quad \|z_{r,k+1}(\tau_{k+1})\|_{\mathcal{H}_{r-\tau_{k+1}}}\ge k. \tag{4.25}
$$

However, since $p(t)=0$ for $t\in[0,\tau_{k+1}\wedge\tau_k]$ and $(u_0(x),v_0(x))\in TM$ for $|x|<r$, by argument based on the one made after (4.22), in the Proposition 4.8, we get $z_{r,k}(t,x)=z_{r,k+1}(t,x)$ for every $t\in[0,\tau_{k+1}]$ and $|x|\le r-t$. But this contradicts (4.25) and we finish the proof of our claim and, in result, the proof of Lemma 4.9. ☐

Since by definition (4.16) and Lemma 4.9 the sequence of stopping times $\{\tau_k\}_{k\ge1}$ is bounded and non-decreasing, it makes sense to denote by τ the limit of $\{\tau_k\}_{k\ge1}$. Now with the help of [13, Lemma 10.1], we prove that the approximate solutions do not explode which is same as the following in terms of τ.

Proposition 4.10. *For τ_k defined in (4.16), $\tau:=\lim_{k\to\infty}\tau_k=T$.*

Proof of Proposition 4.10. We first notice that by considering a particular case of the Chojnowska-Michalik Theorem [24], when the diffusion coefficient is absent, we have that for each k the approximate solution $z_{r,k}$, as a function of time t, is $H^1(\mathbb{R};\mathbb{R}^n)\times L^2(\mathbb{R};\mathbb{R}^n)$-valued and satisfies

$$
z_{r,k}(t)=\xi+\int_0^t\mathcal{G}z_{r,k}(s)\,ds+\int_0^t\mathbf{F}_{r,k}(s,z_{r,k}(s))\,ds+\int_0^t\mathbf{G}_{r,k}(s,z_{r,k}(s))\dot{h}(s)\,ds, \tag{4.26}
$$

for $t\le T$. In particular,

$$
u_{r,k}(t)=\xi_1+\int_0^t v_{r,k}(s)\,ds,
$$

for $t\le T$, where $\xi_1=E_r^2u_0$ and the integral converges in $H^1(\mathbb{R};\mathbb{R}^n)$. Hence

$$
\partial_t u_{r,k}(s,x)=v_{r,k}(s,x), \qquad \text{for all} \quad s\in[0,T], x\in\mathbb{R}.
$$

Next, by keeping in mind the Proposition 4.8, we set

$$
l(t):=\|a_k(t)\|_{H^1(B_{r-t})\times L^2(B_{r-t})}^2 \quad \text{and} \quad q(t):=\log(1+\|a_k(t)\|_{\mathcal{H}_{r-t}}^2).
$$

By applying Proposition B.1, respectively, with $k=0,1$ and $L(x)=x,\log(1+x)$, followed by the use of Lemma 4.4 we get

$$
l(t)\le l(0)+\int_0^t l(s)\,ds+\int_0^t \mathbb{1}_{[0,\tau_k]}(s)\langle v_{r,k}(s),\varphi(s)\rangle_{L^2(B_{r-s})}\,ds
$$

$$+\int_0^t \mathbb{1}_{[0,\tau_k]}(s)\langle v_{r,k}(s),\psi(s)\rangle_{L^2(B_{r-s})}\,ds, \tag{4.27}$$

and

$$\begin{aligned}
q(t) &\le q(0)+\int_0^t \frac{\|a_k(s)\|^2_{\mathcal{H}_{r-s}}}{1+\|a_k(s)\|^2_{\mathcal{H}_{r-s}}}\,ds\\
&\quad+\int_0^t \mathbb{1}_{[0,\tau_k]}(s)\frac{\langle v_{r,k}(s),\varphi(s)\rangle_{L^2(B_{r-s})}}{1+\|a_k(s)\|^2_{\mathcal{H}_{r-s}}}\,ds+\int_0^t \mathbb{1}_{[0,\tau_k]}(s)\frac{\langle \partial_x v_{r,k}(s),\partial_x[\varphi(s)]\rangle_{L^2(B_{r-s})}}{1+\|a_k(s)\|^2_{\mathcal{H}_{r-s}}}\,ds\\
&\quad+\int_0^t \mathbb{1}_{[0,\tau_k]}(s)\frac{\langle v_{r,k}(s),\psi(s)\rangle_{L^2(B_{r-s})}}{1+\|a_k(s)\|^2_{\mathcal{H}_{r-s}}}\,ds+\int_0^t \mathbb{1}_{[0,\tau_k]}(s)\frac{\langle \partial_x v_{r,k}(s),\partial_x[\psi(s)]\rangle_{L^2(B_{r-s})}}{1+\|a_k(s)\|^2_{\mathcal{H}_{r-s}}}\,ds.
\end{aligned} \tag{4.28}$$

Here

$$\begin{aligned}
\varphi(s) &:= \mathcal{A}_{u_{r,k}(s)}(v_{r,k}(s),v_{r,k}(s))-\mathcal{A}_{u_{r,k}(s)}(\partial_x u_{r,k}(s),\partial_x u_{r,k}(s)),\\
\psi(s) &:= Y_{u_{r,k}(s)}(\partial_t u_{r,k}(s),\partial_x u_{r,k}(s))\dot{h}(s).
\end{aligned}$$

Since by Proposition 4.8 $u_{r,k}(s,x)\in M$ for $|x|\le r-s$ and $s\le\tau_k$, we have

$$u_{r,k}(s,x)\in M \quad\text{and}\quad \partial_t u_{r,k}(s,x)=v_{r,k}(s,x)\in T_{u_{r,k}(s,x)}M,$$

on the mentioned domain of s and x. Consequently, by Proposition 3.6, we get

$$\begin{aligned}
\mathcal{A}_{u_{r,k}(s,x)}(v_{r,k}(s,x),v_{r,k}(s,x)) &= A_{u_{r,k}(s,x)}(v_{r,k}(s,x),v_{r,k}(s,x)), \qquad (4.29)\\
\mathcal{A}_{u_{r,k}(s,x)}(\partial_x u_{r,k}(s,x),\partial_x u_{r,k}(s,x)) &= A_{u_{r,k}(s,x)}(\partial_x u_{r,k}(s,x),\partial_x u_{r,k}(s,x)),
\end{aligned}$$

on $|x|\le r-s$ and $s\le\tau_k$. Hence, since $v_{r,k}(s,x)\in T_{u_{r,k}(s,x)}M$, and by definition, $A_{u_{r,k}(s,x)}\in N_{u_{r,k}(s,x)}M$, the L^2-inner product on domain B_{r-s} vanishes and, in result, the second integrals in (4.27) and (4.28) are equal to zero.

Next, to deal with the integral containing terms ψ, we follow Lemma 4.4 and we invoke Lemma 3.2, estimate (3.8), and Proposition 4.8 to get

$$\begin{aligned}
&\langle v_{r,k}(s),Y_{u_{r,k}(s)}(\partial_t u_{r,k}(s),\partial_x u_{r,k}(s))\dot{h}(s)\rangle_{L^2(B_{r-s})}\\
&\lesssim \|v_{r,k}(s)\|^2_{L^2(B_{r-s})}+\|Y_{u_{r,k}(s)}(\partial_t u_{r,k}(s),\partial_x u_{r,k}(s))\dot{h}(s)\|^2_{L^2(B_{r-s})}\\
&\le \|v_{r,k}(s)\|^2_{L^2(B_{r-s})}+C^2_{Y_0}C_r^2\left(1+\|v_{r,k}(s)\|^2_{L^2(B_{r-s})}+\|\partial_x u_{r,k}(s)\|^2_{L^2(B_{r-s})}\right)\|\dot{h}(s)\|^2_{H_\mu}\\
&\lesssim (1+l(s))(1+\|\dot{h}(s)\|^2_{H_\mu}),
\end{aligned} \tag{4.30}$$

for some $C_r>0$, and estimates (3.9)-(3.10) yields

$$\begin{aligned}
&\langle v_{r,k}(s),Y_{u_{r,k}(s)}(\partial_t u_{r,k}(s),\partial_x u_{r,k}(s))\dot{h}(s)\rangle_{L^2(B_{r-s})}\\
&\qquad+\langle \partial_x v_{r,k}(s),\partial_x[Y_{u_{r,k}(s)}(\partial_t u_{r,k}(s),\partial_x u_{r,k}(s))\dot{h}(s)]\rangle_{L^2(B_{r-s})}\\
&\lesssim \|v_{r,k}(s)\|^2_{H^1(B_{r-s})}+\|Y_{u_{r,k}(s)}(\partial_t u_{r,k}(s),\partial_x u_{r,k}(s))\dot{h}(s)\|^2_{H^1(B_{r-s})}\\
&\le \|v_{r,k}(s)\|^2_{H^1(B_{r-s})}+\|\dot{h}(s)\|^2_{H_\mu}\Big[C^2_{Y_0}C_r^2\left(1+\|v_{r,k}(s)\|^2_{L^2(B_{r-s})}+\|\partial_x u_{r,k}(s)\|^2_{L^2(B_{r-s})}\right)\\
&\quad+C^2_{Y_1}\left(1+\|v_{r,k}(s)\|^2_{H^1(B_{r-s})}+\|\partial_x u_{r,k}(s)\|^2_{H^1(B_{r-s})}\right)\|u_{r,k}(s)\|^2_{H^1(B_{r-s})}
\end{aligned}$$

$$
\begin{aligned}
&+C_{Y_2}^2\left(\|v_{r,k}(s)\|_{L^2(B_{r-s})}^2+\|\partial_x u_{r,k}(s)\|_{L^2(B_{r-s})}^2\right)\Big] \\
&\lesssim_{C_r,C_{Y_i}} (1+l(s))\ (1+\|a_k(s)\|_{\mathcal{H}_{r-s}}^2)(1+\|\dot h(s)\|_{H_\mu}^2), \quad i=0,1,2. \qquad (4.31)
\end{aligned}
$$

By substituting the estimates (4.29) and (4.30) in the inequality (4.27) we get

$$
l(t)\lesssim l(0)+\int_0^t \mathbb{1}_{[0,\tau_k]}(s)(1+l(s))\ (1+\|\dot h(s)\|_{H_\mu}^2)\,ds. \qquad (4.32)
$$

Now we define S_j as the set of initial data whose norm under extension is bounded by j, in precise,

$$
S_j:=\{(u_0,v_0)\in\mathcal{H}_{\rm loc}:\ \|\xi\|_{\mathcal{H}_r}\le j \text{ where } \xi:=(E_r^2u_0,E_r^1v_0)\}.
$$

Then, for the initial data belonging to S_j, the Gronwall Lemma on (4.32) yields

$$
1+l_j(t\wedge\tau_k)\le K_{r,j}, \qquad t\le T, \quad j\in\mathbb{N}, \qquad (4.33)
$$

where the constant $K_{r,j}$ also depends on $\|\dot h\|_{L^2(0,T;H_\mu)}$ and l_j stands to show that (4.33) holds under S_j only.

Next to deal with the third integral in (4.28), denote by O its integrand, we recall the following celebrated Gagliardo-Nirenberg inequalities, see e.g. [32],

$$
|\psi|_{L^\infty(r-s)}^2\le|\psi|_{L^2(B_{r-s})}^2+2|\psi|_{L^2(B_{r-s})}|\dot\psi|_{L^2(B_{r-s})}, \qquad \psi\in H^1(B_{r-s}). \qquad (4.34)
$$

Then by applying [13, Lemma 10.1] followed by the generalized Hölder inequality and (4.34) we infer

$$
\begin{aligned}
|O(s)|&\lesssim \mathbb{1}_{[0,\tau_k)}(s)\frac{\int_{B_{r-s}}\{|\partial_x v_{r,k}||\partial_x u_{r,k}||v_{r,k}|^2+|\partial_{xx}u_{r,k}||\partial_x u_{r,k}|^2|v_{r,k}|+|\partial_x v_{r,k}||\partial_x u_{r,k}|^3\}\,dx}{1+\|a_k(s)\|_{\mathcal{H}_{r-s}}^2} \\
&\lesssim \mathbb{1}_{[0,\tau_k)}(s)\frac{l(s)\|a_k(s)\|_{\mathcal{H}_{r-s}}^2}{1+\|a_k(s)\|_{\mathcal{H}_{r-s}}^2}\le \mathbb{1}_{[0,\tau_k)}(s)(1+l(s)). \qquad (4.35)
\end{aligned}
$$

So, by substituting (4.29), (4.30) and (4.35) in (4.28) we have

$$
q(t)\lesssim 1+q(0)+\int_0^t \mathbb{1}_{[0,\tau_k)}(s)(1+l(s))\ (1+\|\dot h(s)\|_{H_\mu}^2)\,ds
$$

Consequently, by applying (4.33), we obtain on S_j,

$$
\begin{aligned}
q_j(t\wedge\tau_k)&\lesssim 1+q_j(0)+\int_0^t[1+l_j(s\wedge\tau_k)]\ (1+\|\dot h(s)\|_{H_\mu}^2)\,ds \\
&\le C_{r,j}\ \|\dot h\|_{L^2(0,T;H_\mu)}, \qquad j\in\mathbb{N}, t\in[0,T], \qquad (4.36)
\end{aligned}
$$

for some $C_{r,j}>0$, where in the last step we have used that $r>T$ and on set S_j the quantity $q_j(0)$ is bounded by j.

To complete the proof let us fix $t<T$. Then, by Proposition 4.8,

$$
|a_k(\tau_k)|_{\mathcal{H}_{r-\tau_k}}=|z_{r,k}(\tau_k)|_{\mathcal{H}_{r-\tau_k}}\ge k \quad \text{whenever} \quad \tau_k\le t.
$$

So for every k such that $\tau_k\le t$ we have

$$
\log(1+k^2)\le q(\tau_k)=q(t\wedge\tau_k).
$$

Then by restricting us to S_j and using inequality (4.36), we obtain

$$\log(1+k^2) \le q_j(t\wedge\tau_k) \lesssim C_{r,j}\|\dot{h}\|_{L^2(0,T;H_\mu)}. \tag{4.37}$$

In this way, if $\lim\limits_{k\to\infty}\tau_k = t_0$ for any $t_0 < T$, then by taking $k\to\infty$ in (4.37) we get $C_{r,j}\|\dot{h}\|_{L^2(0,T;H_\mu)} \ge \infty$ which is absurd. Since this holds for every $j\in\mathbb{N}$ and $t_0 < T$, we infer that $\tau = T$. Hence, the proof of Proposition 4.10 is complete. □

Now we have all the machinery required to finish the proof of Theorem 4.1 which is for the skeleton Cauchy problem (4.1). Define

$$w_{r,k}(t) := \begin{pmatrix} E^2_{r-t}u_{r,k}(t) \\ E^1_{r-t}v_{r,k}(t) \end{pmatrix},$$

and observe that $w_{r,k} : [0,T) \to \mathcal{H}$ is continuous. If we set

$$z_r(t) := \lim_{k\to\infty} w_{r,k}(t), \qquad t<T, \tag{4.38}$$

then by Lemma 4.9 and Proposition 4.10 it is straightforward to verify that, for every $t<T$, the sequence $\{w_{r,k}(t)\}_{k\in\mathbb{N}}$ is Cauchy in $\mathcal{H}$. But since $\mathcal{H}$ is complete, the limit in (4.38) converges in $\mathcal{H}$. Moreover, since by Proposition 4.10 $z_{r,k}(t) = z_{r,k_1}(t)$ for every $k_1 \ge k$ and $t\le\tau_k$, we have that $z_r(t) = w_{r,k}(t)$ for $t\le\tau_k$. In particular, $[0,T)\ni t\mapsto z_r(t)\in\mathcal{H}$ is continuous and $z_r(t,x) = z_{r,k}(t,x)$ for $|x|\le r-t$ if $t\le\tau_k$.

Hence, if we write $z_r(t) = (u_r(t), v_r(t))$, then we have shown that u_r satisfy the first conclusion of the Theorem A.1. In the remaining proof of the existence part we will show that the z_r, defined in (4.38), will satisfy all the remaining conclusions. Evaluation of (4.26) at $t\wedge\tau_k$ together applying the result from previous paragraph gives

$$z_{r,k}(t\wedge\tau_k) = \xi + \int_0^{t\wedge\tau_k}\mathcal{G}z_{r,k}(s)\,ds + \int_0^{t\wedge\tau_k}\mathbf{F}_r(s,z_{r,k}(s))\,ds + \int_0^{t\wedge\tau_k}\mathbf{G}_r(s,z_{r,k}(s))\dot{h}(s)\,ds, \tag{4.39}$$

and this equality holds in $H^1(\mathbb{R};\mathbb{R}^n)\times L^2(\mathbb{R};\mathbb{R}^n)$. Restricting to the interval $(-R,R)$, (4.39) becomes

$$z_r(t\wedge\tau_k) = \xi + \int_0^{t\wedge\tau_k}\mathcal{G}z_r(s)\,ds + \int_0^{t\wedge\tau_k}\mathbf{F}_r(s,z_r(s))\,ds + \int_0^{t\wedge\tau_k}\mathbf{G}_r(s,z_r(s))\dot{h}(s)\,ds,$$

under the action of natural projection from $H^1(\mathbb{R};\mathbb{R}^n)\times L^2(\mathbb{R};\mathbb{R}^n)$ to $H^1((-R,R);\mathbb{R}^n)\times L^2((-R,R);\mathbb{R}^n)$. Here the integrals converge in $H^1((-R,R);\mathbb{R}^n)\times L^2((-R,R);\mathbb{R}^n)$. Taking the limit $k\to\infty$ on both the sides, the dominated convergence theorem yields

$$z_r(t) = \xi + \int_0^t \mathcal{G}z_r(s)\,ds + \int_0^t \mathbf{F}_r(s,z_r(s))\,ds + \int_0^t \mathbf{G}_r(s,z_r(s))\dot{h}(s)\,ds, \qquad t<T,$$

in $H^1((-R,R);\mathbb{R}^n)\times L^2((-R,R);\mathbb{R}^n)$. In particular, by looking to each component separately we have, for every $t<T$,

$$u_r(t) = u_0 + \int_0^t v_r(s)\,ds, \tag{4.40}$$

in $H^1((-R,R);\mathbb{R}^n)$, and

$$v_r(t) = v_0 + \int_0^t \left[\partial_{xx} u_r(s) + A_{u_r(s)}(v_r(s), v_r(s)) - A_{u_r(s)}(\partial_x u_r(s), \partial_x u_r(s))\right] ds \\ + \int_0^t Y_{u_r(s)}(v_r(s), \partial_x u_r(s))\dot{h}(s)\, ds, \tag{4.41}$$

holds in $L^2((-R,R);\mathbb{R}^n)$. It is relevant to note that in the formula above, we have replaced $\mathcal{A}$ by A which make sense because due to Proposition 4.8 and Proposition 4.10, $u_r(t,x) = u_{r,k}(t,x) \in M$ for $|x| \leq r-t$ and $t < T$. Hence we are done with the proof of existence part.

Concerning the uniqueness, define

$$Z(t) := \begin{pmatrix} E_R^2 U(t) \\ E_R^1 \partial_t U(t) \end{pmatrix}, \qquad t < T,$$

and observe that it is a $\mathcal{H}$-valued continuous function of $t \in [0,T)$. Define also

$$\sigma_k := \tau_k \wedge \inf\{t < T : \|Z(t)\|_{\mathcal{H}_{r-t}} \geq k\},$$

and the $\mathcal{H}$-valued function, for $t < T$,

$$\beta(t) := S_t\xi + \int_0^t S_{t-s}\mathbb{1}_{[0,\sigma_k)}(s)\mathbf{F}_{r,k}(s, Z(s))\, ds + \int_0^t S_{t-s}\mathbb{1}_{[0,\sigma_k)}(s)\mathbf{G}_{r,k}(s, Z(s))\dot{h}(s)\, ds.$$

In the same vein as in the existence part of the proof, as an application of the Chojnowska-Michalik Theorem and projection operator, the restriction of β on $\mathcal{H}_R$, which we denote by b, satisfies

$$b(t) = \xi + \int_0^t \mathcal{G}b(s)\, ds + \int_0^t \begin{pmatrix} 0 \\ \mathcal{A}_{U(s)}(\partial_t U(s), \partial_t U(s)) - \mathcal{A}_{U(s)}(\partial_x U(s), \partial_x U(s)) \end{pmatrix} ds \\ + \int_0^t \begin{pmatrix} 0 \\ Y_{U(s)}(\partial_t U(s), \partial_x U(s))\dot{h}(s) \end{pmatrix} ds, \qquad t \leq \sigma_k,$$

where the integrals converge in $H^1((-R,R);\mathbb{R}^n) \times L^2((-R,R);\mathbb{R}^n)$. Then since $U(t)$ and $\partial_t U(t)$ have similar form, respectively to (4.40) and (4.41), by direct computation we deduce that function p defined as

$$p(t) := b(t) - \begin{pmatrix} U(t) \\ \partial_t U(t) \end{pmatrix},$$

satisfies

$$p(t) = \int_0^t \mathcal{G}p(s)\, ds, \qquad t \leq \sigma_k.$$

Since above implies that p satisfies the linear homogeneous wave equation with null initial data, by [13, Remark 6.2], $p(t,x) = 0$ for $|x| \leq R-t$, $t \leq \sigma_k$. Next we set

$$q(t) := \|\beta(t) - a_k(t)\|^2_{\mathcal{H}_{R-t}},$$

and apply Proposition B.1, with $k=1, T=r, L=I$, to obtain

$$q(t\wedge\sigma_k) \leq 2\int_0^{t\wedge\sigma_k} q(s)\, ds + \int_0^t \|\mathbf{F}_{r,k}(s, Z(s)) - \mathbf{F}_{r,k}(s, a_k(s))\|^2_{\mathcal{H}}\, ds$$

$$+\int_0^{t\wedge\sigma_k} \|\mathbf{G}_{r,k}(s,Z(s))\dot{h}(s)-\mathbf{G}_{r,k}(s,a_k(s))\dot{h}(s)\|^2_{\mathcal{H}}\,ds. \tag{4.42}$$

But we know that $r-t>R-t$, and by definition $\sigma_k\le\tau_k$ which implies

$$F_{r,k}(t,z)=F_{R,k}(t,z),\qquad G_{r,k}(t,z)=G_{R,k}(t,z)\quad\text{on}\quad(t-R,R-t),$$

whenever $\|z\|_{\mathcal{H}_{r-t}}\le k$. Consequently, the estimate (4.42) becomes

$$\begin{aligned}q(t\wedge\sigma_k)\le 2\int_0^{t\wedge\sigma_k} q(s)\,ds+\int_0^{t\wedge\sigma_k}\|\mathbf{F}_{R,k}(s,Z(s))-\mathbf{F}_{R,k}(s,a_k(s))\|^2_{\mathcal{H}}]\,ds\\ +\int_0^{t\wedge\sigma_k}\|\mathbf{G}_{R,k}(s,Z(s))\dot{h}(s)-\mathbf{G}_{R,k}(s,a_k(s))\dot{h}(s)\|^2_{\mathcal{H}}\,ds.\end{aligned}$$

Invoking Lemmata 4.4 and 3.2 yields

$$q(t\wedge\sigma_k)\le C_R\int_0^{t\wedge\sigma_k} q(s)(1+\|\dot{h}(s)\|^2_{H_\mu})\,ds.$$

Therefore, we get $q=0$ on $[0,\sigma_k)$ by the Gronwall Lemma. Since in the limit $k\to\infty$, σ_k goes to T as τ_k, by taking k to infinity, by Proposition 4.8 we obtain that $u_r(t,x)=U(t,x)$ for each $t<T$ and $|x|\le R-t$. The proof of Theorem 4.1 completes here. □

5. Large deviation principle

In this section we establish a large deviation principle (LDP) for system (1.2) via a weak convergence approach developed in [19] and [20] which is based on variational representations of infinite-dimensional Wiener processes.

First, let us recall the general criteria for LDP obtained in [19]. Let $(\Omega,\mathfrak{F},\mathbb{P})$ be a probability space with an increasing family $\mathbb{F}:=\{\mathfrak{F}_t, 0\le t\le T\}$ of the sub-σ-fields of $\mathfrak{F}$ satisfying the usual conditions. Let $\mathscr{B}(E)$ denote the Borel σ-field of the Polish space E (i.e. complete separable metric space). Since we are interested in the large deviations of continuous stochastic processes, we follow [23] and consider the following definition of large deviations principle given in terms of random variables.

Definition 5.1. *The $(E,\mathscr{B}(E))$-valued random family $\{X^\varepsilon\}_{\varepsilon>0}$, defined on $(\Omega,\mathfrak{F},\mathbb{P})$, is said to satisfy a large deviation principle on E with the good rate function $\mathcal{I}$ if the following conditions hold:*

(1) $\mathcal{I}$ ***is a good rate function****: The function $\mathcal{I}:E\to[0,\infty]$ is such that for each $M\in[0,\infty)$ the level set $\{\phi\in E:\mathcal{I}(\phi)\le M\}$ is a compact subset of E.*
(2) ***Large deviation upper bound****: For each closed subset F of E*

$$\limsup_{\varepsilon\to0}\ \varepsilon\log\mathbb{P}[X^\varepsilon\in F]\le-\inf_{u\in F}\mathcal{I}(u).$$

(3) ***Large deviation lower bound****: For each open subset G of E*

$$\liminf_{\varepsilon\to0}\ \varepsilon\log\mathbb{P}[X^\varepsilon\in G]\ge-\inf_{u\in G}\mathcal{I}(u),$$

where by convention the infimum over an empty set is $+\infty$.

Assume that K, H are separable Hilbert spaces such that the embedding $K \hookrightarrow H$ is Hilbert-Schmidt. Let $W := \{W(t), t \in [0,T]\}$ be a cylindrical Wiener process on K defined on $(\Omega, \mathfrak{F}, \mathbb{F}, \mathbb{P})$. Hence the paths of W take values in $\mathcal{C}([0,T]; H)$. Note that the RKHS linked to W is precisely ${}_0H^{1,2}(0,T;K)$. Let $\mathscr{S}$ be the class of K-valued $\mathfrak{F}_t$-predictable processes ϕ belonging to ${}_0H^{1,2}(0,T;K)$, $\mathbb{P}$-almost surely. For $M > 0$, we set

$$S_M := \left\{ h \in {}_0H^{1,2}(0,T;K) : \int_0^T \|\dot{h}(s)\|_K^2 \, ds \le M \right\}.$$

The set S_M endowed with the weak topology is the Polish space, see [20], with the metric

$$d_1(h,k) := \sum_{i=1}^{\infty} \frac{1}{2^i} \left| \int_0^T \langle \dot{h}(s) - \dot{k}(s), e_i \rangle_K \, ds \right|,$$

where $\{e_i\}_{i\in\mathbb{N}}$ is a complete orthonormal basis for $L^2(0,T;K)$. Define $\mathscr{S}_M$ as the set of bounded stochastic controls by

$$\mathscr{S}_M := \{\phi \in \mathscr{S} : \phi(\omega) \in S_M, \mathbb{P}\text{-a.s.}\}.$$

Note that $\cup_{M>0}\mathscr{S}_M$ is a proper subset of $\mathscr{S}$. Next, consider a family indexed by $\varepsilon \in (0,1]$ of Borel measurable maps

$$J^\varepsilon : {}_0\mathcal{C}([0,T], H) \to E.$$

We denote by μ^ε the " image" measure on E of $\mathbb{P}$ by J^ε, that is,

$$\mu^\varepsilon = J^\varepsilon(\mathbb{P}), \quad i.e. \quad \mu^\varepsilon(A) = \mathbb{P}\left((J^\varepsilon)^{-1}(A)\right), \quad A \in \mathscr{B}(E).$$

We have the following result.

Theorem 5.2. [19, Theorem 4.4] *Suppose that there exists a measurable map $J^0 : {}_0\mathcal{C}([0,T], H) \to E$ such that*

BD1 : *if $M > 0$ and a family $\{h_\varepsilon\} \subset \mathscr{S}_M$ converges in law as S_M-valued random elements to $h \in \mathscr{S}_M$ as $\varepsilon \to 0$, then the processes*

$${}_0\mathcal{C}([0,T], H) \ni \omega \mapsto J^\varepsilon\left(\omega + \frac{1}{\sqrt{\varepsilon}} \int_0^\cdot \dot{h}_\varepsilon(s) \, ds\right) \in E,$$

converges in law, as $\varepsilon \searrow 0$, to the process $J^0\left(\int_0^\cdot \dot{h}_\varepsilon(s)\, ds\right)$,

BD2 : *for every $M > 0$, the set*

$$\left\{ J^0\left(\int_0^\cdot \dot{h}(s)\, ds\right) : h \in S_M \right\},$$

is a compact subset of E.

Then the family of measures μ^ε satisfies the large deviation principle (LDP) with the rate function defined by

$$\mathcal{I}(u) := \inf\left\{ \frac{1}{2}\int_0^T \|\dot{h}(s)\|_K^2 \, ds : {}_0H^{1,2}(0,T;K) \text{ and } u = J^0\left(\int_0^\cdot \dot{h}(s)\, ds\right) \right\}, \quad (5.1)$$

with the convention $\inf\{\emptyset\} = +\infty$.

5.1. **Main result.** In is important to note that in transferring the general theory argument from Theorem 5.2 in our setting we require some information about the difference of solutions at two different times, hence we need to strengthen the assumptions on diffusion coefficient. In the remaining part of this paper, we assume $Y : M \ni p \mapsto Y(p) \in T_pM$ is a smooth vector field on M such that its extension, denote again by Y, on the ambient space $\mathbb{R}^n$, defined using [13, Propositon 3.9], is smooth and satisfies

Y.4 there exists a compact set $K_Y \subset \mathbb{R}^n$ such that $Y(p) = 0$ if $p \notin K_Y$,
Y.5 for $q \in O$, $Y(\Upsilon(q)) = \Upsilon'(q)Y(q)$,
Y.6 for some $C_Y > 0$

$$|Y(p)| \le C_Y(1 + |p|), \quad \left|\frac{\partial Y}{\partial p_i}(p)\right| \le C_Y, \text{ and } \left|\frac{\partial^2 Y}{\partial p_i \partial p_j}(p)\right| \le C_Y,$$

for $p \in K_Y, i, j = 1, \ldots, n$.

Remark 5.3. (1) *Since K_Y is compact, there exists a C_K such that $|Y(p)| \le C_K$ for $p \in \mathbb{R}^n$.*
(2) *For $M = \mathbb{S}^2$ case, $Y(p) = p \times e, p \in M$, for some fixed vector $e \in \mathbb{R}^3$ satisfies above assumptions.*

Since, due to the above assumptions, Y and its first order partial derivatives are Lipschitz, by 1-D Sobolev embedding we easily get the next result.

Lemma 5.4. *There exists $C_{Y,R} > 0$ such that the extension Y defined above satisfy*
(1) $\|Y(u)\|_{H^j(B_R)} \le C_{Y,R}(1 + \|u\|_{H^j(B_R)}), \quad j = 0, 1, 2,$
(2) $\|Y(u) - Y(v)\|_{L^2(B_R)} \le C_{Y,R}\|u - v\|_{L^2(B_R)},$
(3) $\|Y(u) - Y(v)\|_{H^1(B_R)} \le C_{Y,R}\|u - v\|_{H^1(B_R)} \left(1 + \|u\|_{H^1(B_R)} + \|v\|_{H^1(B_R)}\right).$

Now we state the main result of this section for the following small noise Cauchy problem

$$\begin{cases} \partial_{tt}u^\varepsilon = \partial_{xx}u^\varepsilon + A_{u^\varepsilon}(\partial_t u^\varepsilon, \partial_t u^\varepsilon) - A_{u^\varepsilon}(\partial_x u^\varepsilon, \partial_x u^\varepsilon) + \sqrt{\varepsilon}Y(u^\varepsilon)\dot{W}, \\ (u^\varepsilon(0), \partial_t u^\varepsilon(0)) = (u_0, v_0), \end{cases} \tag{5.2}$$

with the hypothesis that $(u_0, v_0) \in H^2_{\text{loc}} \times H^1_{\text{loc}}(\mathbb{R}, TM)$ is $\mathfrak{F}_0$-measurable random variable, such that $u_0(x, \omega) \in M$ and $v_0(x, \omega) \in T_{u_0(x,\omega)}M$ hold for every $\omega \in \Omega$ and $x \in \mathbb{R}$. Since the small noise problem (5.2), with initial data $(u_0, v_0) \in \mathscr{H}_{\text{loc}}(\mathbb{R}; M)$, is a particular case of Theorem A.1, for given $\varepsilon > 0$ and $T > 0$, there exists a unique global strong solution to (5.2), which we denote by $z^\varepsilon := (u^\varepsilon, \partial_t u^\varepsilon)$, with values in the Polish space

$$\mathcal{X}_T := \mathcal{C}\left([0, T]; H^2_{\text{loc}}(\mathbb{R}; \mathbb{R}^n)\right) \times \mathcal{C}\left([0, T]; H^1_{\text{loc}}(\mathbb{R}; \mathbb{R}^n)\right),$$

and satisfy the properties mentioned in Appendix A. Then there exists a Borel measurable function, see for e.g. [19] and [43, Theorems 12.1 and 13.2],

$$J^\varepsilon : {}_0\mathcal{C}([0, T], E) \to \mathcal{X}_T, \tag{5.3}$$

where space E can be taken as in Example 3.1, such that $z^\varepsilon(\cdot) = J^\varepsilon(W(\cdot))$, $\mathbb{P}$-almost surely.

Recall from Section 3 that the random perturbation W we consider is a cylindrical Wiener process on H_μ and there exists a separable Hilbert space H such that the embedding of H_μ in H is Hilbert-Schmidt. Hence we can apply the general theory from previous section with the notations defined by taking H_μ instead of K.

Let us define a Borel map

$$J^0 : {}_0\mathcal{C}([0,T],E) \to \mathcal{X}_T.$$

If $h \in {}_0\mathcal{C}([0,T],E) \setminus {}_0H^{1,2}(0,T;H_\mu)$, then we set $J^0(h) = 0$. If $h \in {}_0H^{1,2}(0,T;H_\mu)$ then by Theorem 4.1 there exists a function in $\mathcal{X}_T$, say z_h, that solves

$$\begin{cases} \partial_{tt}u = \partial_{xx}u + A_u(\partial_t u, \partial_t u) - A_u(\partial_x u, \partial_x u) + Y(u)\,\dot{h}, \\ u(0,\cdot) = u_0, \partial_t u(0,\cdot) = v_0, \end{cases} \tag{5.4}$$

uniquely and we set $J^0(h) = z_h$.

Remark 5.5. *At some places in the paper we denote $J^0(h)$ by $J^0\left(\int_0^\cdot \dot{h}(s)\,ds\right)$ to make it clear that in the differential equation we have control $\dot{h}$ not h.*

The main result of this section is as follows:

Theorem 5.6. *The family of laws $\{\mathscr{L}(z^\varepsilon) : \varepsilon \in (0,1]\}$ on $\mathcal{X}_T$, where $z^\varepsilon := (u^\varepsilon, \partial_t u^\varepsilon)$ is the unique solution to* (5.2) *satisfies the large deviation principle with rate function $\mathcal{I}$ defined in* (5.1).

Note that, in the light of Theorem 5.2, in order to prove the Theorem 5.6 it is sufficient to show the following two statements:

Statement 1: For each $M > 0$, the set $K_M := \{J^0(h) : h \in S_M\}$ is a compact subset of $\mathcal{X}_T$, where $S_M \subset {}_0H^{1,2}(0,T;H_\mu)$ is the centred closed ball of radius M endowed with the weak topology.

Statement 2: Assume that $M > 0$, that $\{\varepsilon_n\}_{n\in\mathbb{N}}$ is an $(0,1]$-valued sequence convergent to 0, that $\{h_n\}_{n\in\mathbb{N}} \subset \mathscr{S}_M$ converges in law to $h \in \mathscr{S}_M$ as $\varepsilon \to 0$. Then the processes

$${}_0\mathcal{C}([0,T],E) \ni \omega \mapsto J^{\varepsilon_n}\left(W(\cdot) + \frac{1}{\sqrt{\varepsilon_n}}\int_0^\cdot \dot{h}_n(s)\,ds\right) \in \mathcal{X}_T, \tag{5.5}$$

converges in law on $\mathcal{X}_T$ to $J^0\left(\int_0^\cdot \dot{h}(s)\,ds\right)$.

Remark 5.7. *By combining the proofs of Theorem A.1, which is proven by the first author and M. Ondreját in* [13], *and Theorem 4.1 we infer that the map* (5.5) *is well-defined and $J^{\varepsilon_n}\left(W(\cdot) + \frac{1}{\sqrt{\varepsilon_n}}\int_0^\cdot \dot{h}_n(s)\,ds\right)$ solves the following stochastic control Cauchy problem*

$$\begin{cases} \partial_{tt}u^{\varepsilon_n} = \partial_{xx}u^{\varepsilon_n} + A_{u^{\varepsilon_n}}(\partial_t u^{\varepsilon_n}, \partial_t u^{\varepsilon_n}) - A_{u^{\varepsilon_n}}(\partial_x u^{\varepsilon_n}, \partial_x u^{\varepsilon_n}) + Y(u^{\varepsilon_n})\dot{h}_n \\ \qquad\qquad + \sqrt{\varepsilon_n}Y(u^{\varepsilon_n})\dot{W}, \\ (u^{\varepsilon_n}(0), \partial_t u^{\varepsilon_n}(0)) = (u_0, v_0)\,, \end{cases} \tag{5.6}$$

for the initial data $(u_0, v_0) \in H^2_{loc} \times H^1_{loc}(\mathbb{R}; TM)$.

Remark 5.8. *It is clear by now that verification of the LDP consists in proving two convergence results, see* [11, 10, 18, 23, 55]*. As it was shown first in* [9]*, the second convergence result follows easily from the first one via the Jakubowski version of the Skorokhod representation theorem. Therefore, establishing LDP, de facto, reduces to proving a single convergence result: for the deterministic controlled problem known also as the skeleton equation. This convergence result is specific to the SPDE in question and requires techniques related to the equation considered. Thus, for instance, the proof of* [9, Lemma 6.3] *for the stochastic Landau-Lifshitz-Gilbert equation, is different from the proof of* [23, Proposition 3.5] *for stochastic Navier-Stokes equation. The proof of this convergence result, i.e. Statement 1, is the main contribution of our paper.*

5.2. **Proof of Statement 1.** Let $\{z_n = (u_n, v_n) := J^0(h_n)\}_{n\in\mathbb{N}}$ be a sequence in the set K_M corresponding to the sequence of controls $\{h_n\}_{n\in\mathbb{N}} \subset S_M$. Since S_M is a boudned subset of Hilbert space ${}_0H^{1,2}(0,T;H_\mu)$, S_M is weakly compact. Consequently, see [5], there exists a subsequence of $\{h_n\}_{n\in\mathbb{N}}$, we still denote this by $\{h_n\}_{n\in\mathbb{N}}$, which converges weakly to a limit $h \in {}_0H^{1,2}(0,T;H_\mu)$. But, since S_M is weakly closed, $h \in S_M$. Hence to complete the proof of Statement 1 we need to show that the subsequence of solutions $\{z_n\}_{n\in\mathbb{N}}$ to (5.4), corresponding to the subsequence of controls $\{h_n\}_{n\in\mathbb{N}}$, converges to $z_h = (u_h, v_h)$ which solves the skeleton Cauchy problem (5.4) for the control h. Before delving into the proof of this we will establish the following a priori estimate which is a preliminary step required to prove, Proposition 5.14, the main result of this section.

Lemma 5.9. *Fix any $T > 0$, $x \in \mathbb{R}$. Then there exists a constant $\mathcal{B} > 0$, which depends only on $\|(u_0, v_0)\|_{\mathcal{H}(B(x,T))}, M$ and T, such that*

$$\sup_{h\in S_M} \sup_{t\in[0,T/2]} e(t, z_h(t)) \le \mathcal{B}. \tag{5.7}$$

Here z_h is the unique global strong solution to problem (5.4) *and*

$$\begin{aligned} e(t,z) : &= \frac{1}{2}\|z\|^2_{\mathcal{H}_{B(x,T-t)}} = \frac{1}{2}\Big\{\|u\|^2_{L^2(B(x,T-t))} + \|\partial_x u\|^2_{L^2(B(x,T-t))} + \|v\|^2_{L^2(B(x,T-t))} \\ &\quad + \|\partial_{xx} u\|^2_{L^2(B(x,T-t))} + \|\partial_x v\|^2_{L^2(B(x,T-t))}\Big\}, \quad z = (u,v) \in \mathcal{H}_{loc}. \end{aligned}$$

Moreover, if we restrict x on an interval $[-a,a] \subset \mathbb{R}$, then the positive constant $\mathcal{B} := \mathcal{B}(M,T,a)$, which also depends on 'a' now, can be chosen such that

$$\sup_{x\in[-a,a]} \sup_{h\in S_M} \sup_{t\in[0,T/2]} e(t, z_h(t)) \le \mathcal{B}.$$

Proof of Lemma 5.9. First note that the last part follows from the first one. Indeed, by assumption $(u_0, v_0) \in \mathcal{H}_{loc}$, in particular, $\|(u_0, v_0)\|_{\mathcal{H}(-a-T,a+T)} < \infty$ and therefore,

$$\sup_{x\in[-a,a]} \|(u_0, v_0)\|_{\mathcal{H}(B(x,T))} \le \|(u_0, v_0)\|_{\mathcal{H}(-a-T,a+T)} < \infty.$$

The procedure to prove (5.7) is based on the proof of Proposition 4.10. Let us fix h in S_M and denote the corresponding solution $z_h := (u_h, v_h)$ which exists due to

Theorem 4.1. Since x is fixed we will avoid writing it explicitly the norm. Define

$$l(t) := \frac{1}{2}\|(u_h(t), v_h(t)\|^2_{H^1(B_{T-t})\times L^2(B_{T-t})}, \quad t \in [0,T].$$

Then invoking Proposition B.1, with $k=0$ and $L=I$, implies, for $t \in [0,T]$,

$$\begin{aligned} l(t) \leq l(0) &+ \int_0^t \langle u_h(r), v_h(s)\rangle_{L^2(B_{T-s})}\, ds + \int_0^t \langle v_h(s), f_h(s)\rangle_{L^2(B_{T-s})}\, ds \\ &+ \int_0^t \langle v_h(s), Y(u_h(s))\dot{h}(s)\rangle_{L^2(B_{T-s})}\, ds, \end{aligned} \tag{5.8}$$

where

$$f_h(r) := A_{u_h(r)}(v_h(r), v_h(r) - A_{u_h(r)}(\partial_x u_h(r), \partial_x u_h(r).$$

Since $v_h(r) \in T_{u_h(r)}M$ and by definition $A_{u_h(r)}(\cdot,\cdot) \in N_{u_h(r)}M$, the second integral in (5.8) vanishes. Because $u_h(r) \in M$, invoking the Cauchy-Schwartz inequality, Lemmata 3.2 and 5.4 implies

$$l(t) \leq l(0) + \left(\frac{C_Y^2 C_T^2}{2} + 2\right) \int_0^t (1 + l(s))(1 + \|\dot{h}(s)\|^2_{H_\mu})\, ds.$$

Consequently, by appying the Gronwall Lemma and followed by using $h \in S_M$ we get

$$l(t) \lesssim_{C_Y, C_T} (1 + l(0)) \left[T + \|\dot{h}\|^2_{L^2(0,T;H_\mu)}\right] \leq (T + M)(1 + l(0)). \tag{5.9}$$

Next we define

$$q(t) := \log(1 + \|z_h(t)\|^2_{\mathcal{H}_{T-t}}).$$

Then Proposition B.1, with $k=1$ and $L(x) = \log(1+x)$, gives, for $t \in [0, T/2]$,

$$\begin{aligned} q(t) \leq q(0) &+ \int_0^t \frac{\|z_h(s)\|^2_{\mathcal{H}_{T-s}}}{1 + \|z_h(s)\|^2_{\mathcal{H}_{T-s}}}\, ds \\ &+ \int_0^t \frac{\langle v_h(s), f_h(s)\rangle_{L^2(B_{T-s})}}{1 + \|z_h(s)\|^2_{\mathcal{H}_{T-s}}}\, ds + \int_0^t \frac{\langle \partial_x v_h(s), \partial_x[f_h(s)]\rangle_{L^2(B_{T-s})}}{1 + \|z_h(s)\|^2_{\mathcal{H}_{T-s}}}\, ds \\ &+ \int_0^t \frac{\langle v_h(s), Y(u_h(s))\dot{h}(s)\rangle_{L^2(B_{T-s})}}{1 + \|z_k(s)\|^2_{\mathcal{H}_{T-s}}}\, ds + \int_0^t \frac{\langle \partial_x v_h(s), \partial_x[Y(u_h(s))\dot{h}(s)]\rangle_{L^2(B_{T-s})}}{1 + \|z_h(s)\|^2_{\mathcal{H}_{T-s}}}\, ds. \end{aligned}$$

Since by perpendicularity second integral vanishes, by doing the calculation based on (4.31) and (4.35) we deduce

$$\begin{aligned} q(t) &\lesssim_T 1 + q(0) + \int_0^t \frac{l(s)\|z_h(s)\|^2_{\mathcal{H}_{T-s}}}{1 + \|z_h(s)\|^2_{\mathcal{H}_{T-s}}}\, ds \\ &\quad + \int_0^t \frac{(1 + l(s))\ (1 + \|z_h(s)\|^2_{\mathcal{H}_{T-s}})(1 + \|\dot{h}(s)\|^2_{H_\mu})}{1 + \|z_k(s)\|^2_{\mathcal{H}_{T-s}}}\, ds \\ &\leq 1 + q(0) + \int_0^t (1 + l(s))(1 + \|\dot{h}(s)\|^2_{H_\mu})\, ds, \end{aligned}$$

which further implies, due to (5.9) and $h \in S_M$,

$$q(t) \lesssim 1 + q(0) + (T+M)^2(1+l(0)).$$

In terms of z_h, that is, for each $x \in \mathbb{R}$ and $t \in [0, T/2]$,

$$\|z_h(t)\|^2_{\mathcal{H}_{B(x,T-t)}} \lesssim \exp\left[\|(u_0, v_0)\|^2_{\mathcal{H}_{B(x,T)}}(T+M)^2\right].$$

Since above holds for every $t \in [0, T/2], h \in S_M$, by taking supremum on t and h we get the required (5.7), and hence the proof of Lemma 5.9. $\square$

Remark 5.10. *Since $B(x, T/2) \subseteq B(x, T-t)$ for every $t \in [0, T/2]$, Lemma 5.9 also implies*

$$\sup_{x\in[-a,a]} \sup_{h\in S_M} \sup_{t\in[0,T/2]} \frac{1}{2}\left\{\|u_h(t)\|^2_{H^2(B(x,R))} + \|v_h(t)\|^2_{H^1(B(x,R))}\right\} \le \mathcal{B}(M, T, a),$$

for $R = T/2$.

Now we prove the main result of this subsection which will allow to complete the proof of Statement 1.

Proposition 5.11. *Fix $\mathcal{T} > 0$. The sequence of solutions $\{z_n\}_{n\in\mathbb{N}}$ to the skeleton problem* (5.4) *converges to z_h in the $\mathcal{X}_\mathcal{T}$-norm (strong topology). In particular, for every $\mathcal{T}, M > 0$, the mapping*

$$S_M \ni h \mapsto J^0(h) \in \mathcal{X}_\mathcal{T},$$

is Borel.

Proof of Proposition 5.11. First note that second conclusion follows from first immediately because continuous maps are Borel. Towards proving the first conclusion, let us fix any $n \in \mathbb{N}$. Recall that in our notation, by Theorem 4.1, $z_h = (u_h, v_h)$ and $z_n = (u_n, v_n)$, respectively, are the unique global strong solutions to

$$\begin{cases} \partial_{tt} u_h = \partial_{xx} u_h + A_{u_h}(\partial_t u_h, \partial_t u_h) - A_{u_h}(\partial_x u_h, \partial_x u_h) + Y(u_h)\dot{h}, \\ (u_h(0), v_h(0)) = (u_0, v_0), \quad \text{where } v_n h := \partial_t u_h, \end{cases} \tag{5.10}$$

and

$$\begin{cases} \partial_{tt} u_n = \partial_{xx} u_n + A_{u_n}(\partial_t u_n, \partial_t u_n) - A_{u_n}(\partial_x u_n, \partial_x u_n) + Y(u_n)\dot{h}_n, \\ (u_n(0), v_n(0)) = (u_0, v_0), \quad \text{where } v_n := \partial_t u_n. \end{cases} \tag{5.11}$$

Hence $\mathfrak{z}_n := (\mathfrak{u}_n, \mathfrak{v}_n) = z_h - z_n$ is the unique global strong solution to, with null initial data,

$$\begin{aligned} \partial_{tt}\mathfrak{u}_n = \partial_{xx}\mathfrak{u}_n &- A_{u_h}(\partial_x u_h, \partial_x u_h) + A_{u_n}(\partial_x u_n, \partial_x u_n) + A_{u_h}(\partial_t u_h, \partial_t u_h) \\ &- A_{u_n}(\partial_t u_n, \partial_t u_n) + Y(u_h)\dot{h} - Y(u_n)\dot{h}_n, \end{aligned} \tag{5.12}$$

where $\mathfrak{v}_n := \partial_t \mathfrak{u}_n$. This implies that

$$\mathfrak{z}_n(t) = \int_0^t S_{t-s}\begin{pmatrix} 0 \\ f_n(s) \end{pmatrix} ds + \int_0^t S_{t-s}\begin{pmatrix} 0 \\ g_n(s) \end{pmatrix} ds, \quad t \in [0, T].$$

Here

$$f_n(s) := -A_{u_h(s)}(\partial_x u_h(s), \partial_x u_h(s)) + A_{u_n(s)}(\partial_x u_n(s), \partial_x u_n(s)) + A_{u_h(s)}(\partial_t u_h(s), \partial_t u_h(s)) \\ - A_{u_n(s)}(\partial_t u_n(s), \partial_t u_n(s)),$$

and

$$g_n(s) := Y(u_h(s))\dot{h}(s) - Y(u_n(s))\dot{h}_n(s).$$

We aim to show that

$$\mathfrak{z}_n \xrightarrow[n\to 0]{} 0 \quad \text{in} \quad \mathcal{C}\left([0,\mathcal{T}], H^2_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^n)\right) \times \mathcal{C}\left([0,\mathcal{T}], H^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^n)\right),$$

that is, for every $R > 0$ and $x \in \mathbb{R}$,

$$\sup_{t\in[0,\mathcal{T}]} \left[\|\mathfrak{u}_n(t)\|^2_{H^2(B(x,R))} + \|\mathfrak{v}_n(t)\|^2_{H^1(B(x,R))}\right] \to 0 \text{ as } n \to \infty. \tag{5.13}$$

Without loss of generality we assume $x = 0$. Since a compact set in $\mathbb{R}$ can be convered by a finite number of any given closed interval of non-zero length, it is sufficient to prove above for a fixed $R > 0$ whose value will be set later. Let φ is a bump function which takes value 1 on B_R and vanishes outside $\overline{B_{2R}}$. Define $\bar{u}_n(t,x) := u_n(t,x)\varphi(x)$ and $\bar{u}_h(t,x) := u_h(t,x)\varphi(x)$, so

$$\bar{v}_n(t,x) = \varphi(x)v_n(t,x), \qquad \bar{v}_h(t,x) = \varphi(x)v_h(t,x),$$

and with notation $\bar{\mathfrak{u}}_n := \bar{u}_n - \bar{u}_h$,

$$\begin{aligned}
&\partial_{tt}\bar{\mathfrak{u}}_n - \partial_{xx}\bar{\mathfrak{u}}_n \\
&\quad = \left[A_{u_n}(\partial_t u_n, \partial_t u_n) - A_{u_n}(\partial_x u_n, \partial_x u_n) - A_{u_h}(\partial_t u_h, \partial_t u_h) + A_{u_h}(\partial_x u_h, \partial_x u_h)\right]\varphi \\
&\qquad - (u_n - u_h)\partial_{xx}\varphi - 2(\partial_x u_n - \partial_x u_h)\partial_x\varphi + \left[Y(u_n)\dot{h}_n - Y(u_h)\dot{h}\right]\varphi \\
&\quad =: \bar{f}_n + \bar{g}_n.
\end{aligned}$$

Here

$$\bar{f}_n(s) := \left[A_{u_n(s)}(\partial_t u_n(s), \partial_t u_n(s)) - A_{u_n(s)}(\partial_x u_n(s), \partial_x u_n(s)) - A_{u_h(s)}(\partial_t u_h(s), \partial_t u_h(s)) \right. \\ \left. + A_{u_h(s)}(\partial_x u_h(s), \partial_x u_h(s))\right]\varphi - (u_n(s) - u_h(s))\partial_{xx}\varphi - 2(\partial_x u_n(s) - \partial_x u_h(s))\partial_x\varphi,$$

and

$$\bar{g}_n(s) := \left[Y(u_n(s))\dot{h}_n(s) - Y(u_h(s))\dot{h}(s)\right]\varphi, \quad s \in [0,T].$$

Next, by direct computation we can find constants $C_\varphi, \bar{C}_\varphi > 0$, depend on $\varphi, \varphi', \varphi''$, such that, for $t \in [0,T]$,

$$\begin{aligned}
\|\bar{\mathfrak{u}}_n(t)\|^2_{H^2(-R,R)} + \|\bar{\mathfrak{v}}_n(t)\|^2_{H^1(-R,R)} &\le C_\varphi \|\mathfrak{u}_n(t)\|^2_{H^2(-R,R)} + \|\mathfrak{v}_n(t)\|^2_{H^1(-R,R)} \\
&\le \bar{C}_\varphi \|\bar{\mathfrak{u}}_n(t)\|^2_{H^2(-R,R)} + \|\bar{\mathfrak{v}}_n(t)\|^2_{H^1(-R,R)}.
\end{aligned} \tag{5.14}$$

Hence, instead of (5.13) it is enough to prove the following, for a fixed R,

$$\sup_{t\in[0,\mathcal{T}]} \left[\|\bar{\mathfrak{u}}_n(t)\|^2_{H^2(-R,R)} + \|\bar{\mathfrak{v}}_n(t)\|^2_{H^1(-R,R)}\right] \to 0 \text{ as } n \to \infty. \tag{5.15}$$

Let us set

$$T := 4\mathcal{T} \quad \text{and} \quad R := \frac{T}{4} = \mathcal{T}.$$

The reason of such choice is due to the fact that (5.15) follows from

$$\sup_{t\in[0,R]}\left[\|\bar{\mathfrak{u}}_n(t)\|^2_{H^2(B_{T-t})}+\|\bar{\mathfrak{v}}_n(t)\|^2_{H^1(B_{T-t})}\right]\to 0 \text{ as } n\to\infty. \tag{5.16}$$

Indeed, because for every $t\in[0,R]$, $T-t>2R$, and we have

$$\begin{aligned}\|\bar{\mathfrak{u}}_n(t)\|^2_{H^2(B_R)}+\|\bar{\mathfrak{v}}_n(t)\|^2_{H^1(B_R)} &\le \|\bar{\mathfrak{u}}_n(t)\|^2_{H^2(B_{2R})}+\|\bar{\mathfrak{v}}_n(t)\|^2_{H^1(B_{2R})}\\ &\le \sup_{t\in[0,R]}\left[\|\bar{\mathfrak{u}}_n(t)\|^2_{H^2(B_{T-t})}+\|\bar{\mathfrak{v}}_n(t)\|^2_{H^1(B_{T-t})}\right].\end{aligned}$$

Next, we set $l(t,z):=\frac{1}{2}\|z\|^2_{\mathcal{H}_{T-t}}$, for $z=(u,v)\in\mathcal{H}_{\mathrm{loc}}$ and $t\in[0,R]$. Invoking Proposition B.1, with null diffusion part and $k=1, L=I, x=0$, implies, for every $t\in[0,R]$,

$$l(t,\bar{\mathfrak{z}}_n(t))\le\int_0^t \mathbb{V}(r,\bar{\mathfrak{z}}_n(r))\,dr, \tag{5.17}$$

where $\bar{\mathfrak{z}}_n(t)=(\bar{\mathfrak{u}}_n(t),\bar{\mathfrak{v}}_n(t))$ and

$$\begin{aligned}\mathbb{V}(t,\bar{\mathfrak{z}}_n(t)) &= \langle\bar{\mathfrak{u}}_n(t),\bar{\mathfrak{v}}_n(t)\rangle_{L^2(B_{T-t})}+\langle\bar{\mathfrak{v}}_n(t),\bar{f}_n(t)\rangle_{L^2(B_{T-t})}\\ &\quad+\langle\partial_x\bar{\mathfrak{v}}_n(t),\partial_x\bar{f}_n(t)\rangle_{L^2(B_{T-t})}+\langle\bar{\mathfrak{v}}_n(t),\bar{g}_n(t)\rangle_{L^2(B_{T-t})}\\ &\quad+\langle\partial_x\bar{\mathfrak{v}}_n(t),\partial_x\bar{g}_n(t)\rangle_{L^2(B_{T-t})}\\ &=: \mathbb{V}_f(t,\bar{\mathfrak{z}}_n(t))+\mathbb{V}_g(t,\bar{\mathfrak{z}}_n(t)).\end{aligned}$$

We estimate $\mathbb{V}_f(t,\bar{\mathfrak{z}}_n(t))$ and $\mathbb{V}_g(t,\bar{\mathfrak{z}}_n(t))$ separately as follows. Since $T-t>2R$ for every $t\in[0,R]$ and $\varphi(y),\varphi'(y)=0$ for $y\notin\overline{B_{2R}}$, we have

$$\begin{aligned}\int_0^t\mathbb{V}_f(r,\bar{\mathfrak{z}}(r))\,dr &= \int_0^t\Big[\int_{B_{2R}}\big\{\varphi(y)\mathfrak{u}_n(r,y)\varphi(y)\mathfrak{v}_n(r,y)+\varphi(y)\mathfrak{v}_n(r,y)\bar{f}_n(r,y)\\ &\qquad+\varphi'(y)\mathfrak{v}_n(r,y)\partial_x\bar{f}_n(r,y)+\varphi(y)\partial_x\mathfrak{v}_n(r,y)\partial_x\bar{f}_n(r,y)\big\}\,dy\Big]\,dr\\ &\lesssim_{\varphi,\varphi'}\int_0^t l(r,\bar{\mathfrak{z}}_n(r))\,dr+\int_0^t\|\bar{f}_n(r)\|^2_{H^1(B_{2R})}\,dr,\end{aligned}$$

and

$$\begin{aligned}&\int_0^t\big(\langle\bar{\mathfrak{v}}_n(r),\bar{g}_n(r)\rangle_{L^2(B_{T-r})}+\langle\partial_x\bar{\mathfrak{v}}_n(r),\partial_x\bar{g}_n(r)\rangle_{L^2(B_{T-r})}\big)\,dr\\ &\quad=\int_0^t\big(\langle\bar{\mathfrak{v}}_n(r),\bar{g}_n(r)\rangle_{L^2(B_{2R})}+\langle\partial_x\bar{\mathfrak{v}}_n(r),\partial_x\bar{g}_n(r)\rangle_{L^2(B_{2R})}\big)\,dr.\end{aligned}$$

Let us estimate the terms involving $\bar{f}_n$ first. Since u_n,u_h takes values on manifold M, by using the properties of φ and invoking interpolation inequality (4.5), as pursued in Lemma 4.4, followed by Lemma 5.9 we deduce that

$$\begin{aligned}\|\bar{f}_n(r)\|^2_{L^2(B_{2R})} &\lesssim_{\varphi,\varphi',\varphi''}\|A_{u_n(r)}(v_n(r),v_n(r))-A_{u_h(r)}(v_n(r),v_n(r))\|^2_{L^2(B_{2R})}\\ &\quad+\|A_{u_h(r)}(v_n(r),v_n(r))-A_{u_h(r)}(v_n(r),v_h(r))\|^2_{L^2(B_{2R})}\\ &\quad+\|A_{u_h(r)}(v_n(r),v_h(r))-A_{u_h(r)}(v_h(r),v_h(r))\|^2_{L^2(B_{2R})}\\ &\quad+\|A_{u_n(r)}(\partial_xu_n(r),\partial_xu_n(r))-A_{u_h(r)}(\partial_xu_n(r),\partial_xu_n(r))\|^2_{L^2(B_{2R})}\end{aligned}$$

$$
\begin{aligned}
&+ \|A_{u_h(r)}(\partial_x u_n(r), \partial_x u_n(r)) - A_{u_h(r)}(\partial_x u_n(r), \partial_x u_h(r))\|^2_{L^2(B_{2R})} \\
&+ \|A_{u_h(r)}(\partial_x u_n(r), \partial_x u_h(r)) - A_{u_h(r)}(\partial_x u_h(r), \partial_x u_h(r))\|^2_{L^2(B_{2R})} \\
&+ \|u_n(r) - u_h(r)\|^2_{L^2(B_{2R})} + 2\|\partial_x u_n(r) - \partial_x u_h(r)\|^2_{L^2(B_{2R})} \\
\lesssim_{L_A,B_A,R}& \|u_n(r) - u_h(r)\|_{L^2(B_{2R})}\|v_n(r)\|^2_{L^\infty(B_{2R})} \\
&+ \|v_n(r) - v_h(r)\|^2_{L^2(B_{2R})}\left(\|v_n(r)\|_{L^\infty(B_{2R})} + \|v_h(r)\|_{L^\infty(B_{2R})}\right) \\
&+ \|u_n(r) - u_h(r)\|_{L^2(B_{2R})}\|\partial_x u_n(r)\|^2_{L^\infty(B_{2R})} \\
&+ \|\partial_x u_n(r) - \partial_x u_h(r)\|^2_{L^2(B_{2R})}\left(\|\partial_x u_n(r)\|_{L^\infty(B_{2R})} + \|\partial_x u_h(r)\|_{L^\infty(B_{2R})}\right) \\
&+ \|u_n(r) - u_h(r)\|^2_{L^2(B_{2R})} + 2\|\partial_x u_n(r) - \partial_x u_h(r)\|^2_{L^2(B_{2R})} \\
\lesssim_{L_A,B_A,R,k_e,\mathcal{B}}& \ \|\mathfrak{z}_n(r)\|^2_{\mathcal{H}(B_{2R})} \lesssim l(r, \mathfrak{z}_n(r)).
\end{aligned} \tag{5.18}
$$

Similarly by using the interpolation inequality (4.5) and Lemma 5.9, based on the computation of (4.7), we get

$$
\|\partial_x \bar{f}_n(r)\|^2_{L^2(B_{2R})} \lesssim_{L_A,B_A,R,k_e,\mathcal{B}} \ l(r, \mathfrak{z}_n(r)),
$$

where constant of inequality is independent of n but depends on the properties of φ and its first two derivatives, consequently, we have, for some $C_{\bar{f}} > 0$,

$$
\int_0^t \|\bar{f}_n(r)\|^2_{H^1(B_{2R})}\, dr \le C_{\bar{f}} \int_0^t l(r, \mathfrak{z}_n(r))\, dr. \tag{5.19}
$$

Now we move to the crucial estimate of integral involving $\bar{g}_n$. It is the part where we follow the idea of [23, Proposition 3.4] and [28, Proposition 4.4]. Let m be a natural number, whose value will be set later. Define the following partition of $[0, R]$,

$$
\left\{0, \frac{1\cdot R}{2^m}, \frac{2\cdot R}{2^m}, \cdots, \frac{2^m\cdot R}{2^m}\right\},
$$

and set

$$
r_m := \frac{(k+1)\cdot R}{2^m} \text{ and } t_{k+1} := \frac{(k+1)\cdot R}{2^m} \text{ if } r \in \left[\frac{k\cdot R}{2^m}, \frac{(k+1)\cdot R}{2^m}\right).
$$

Now observe that

$$
\begin{aligned}
&\int_0^t \langle \bar{\mathfrak{v}}_n(r), \bar{g}_n(r)\rangle_{H^1(B_{2R})}\, dr \\
&= \int_0^t \langle \bar{\mathfrak{v}}_n(r), \varphi(Y(u_n(r)) - Y(u_h(r)))\dot{h}_n(r)\rangle_{H^1(B_{2R})}\, dr \\
&\quad + \int_0^t \langle \bar{\mathfrak{v}}_n(r) - \bar{\mathfrak{v}}_n(r_m), \varphi Y(u_h(r))(\dot{h}_n(r) - \dot{h}(r))\rangle_{H^1(B_{2R})}\, dr \\
&\quad + \int_0^t \langle \bar{\mathfrak{v}}_n(r_m), \varphi(Y(u_h(r)) - Y(u_h(r_m)))(\dot{h}_n(r) - \dot{h}(r))\rangle_{H^1(B_{2R})}\, dr \\
&\quad + \int_0^t \langle \bar{\mathfrak{v}}_n(r_m), \varphi Y(u_h(r_m))(\dot{h}_n(r) - \dot{h}(r))\rangle_{H^1(B_{2R})}\, dr \\
&=: G_1(t) + G_2(t) + G_3(t) + G_4(t).
\end{aligned} \tag{5.20}
$$

For G_1, since $T-r>2R$, Lemmata 3.2, 5.4 and 5.9 followed by (5.14) implies

$$\begin{aligned}
|G_1(t)| &\lesssim_\varphi \int_0^t \|\bar{\mathfrak{v}}_n(r)\|^2_{H^1(B_{2R})}\,dr + \int_0^t \|Y(u_n(r)) - Y(u_h(r))\|^2_{H^1(B_{2R})}\|\dot{h}_n(r)\|^2_{H_\mu}\,dr \\
&\lesssim_R \int_0^t \|\bar{\mathfrak{v}}_n(r)\|^2_{H^1(B_{2R})}\,dr \\
&\quad + \int_0^t \|u_n(r)-u_h(r)\|^2_{H^1(B_{2R})}\left(1+\|u_n(r)\|^2_{H^1(B_{2R})} + \|u_h(r)\|^2_{H^1(B_{2R})}\right)\|\dot{h}_n(r)\|^2_{H_\mu}\,dr \\
&\lesssim_{\mathcal{B}} \int_0^t (1+l(r,\mathfrak{z}_n(r)))\left(1+\|\dot{h}_n(r)\|^2_{H_\mu}\right)\,dr. \qquad (5.21)
\end{aligned}$$

To estimate $G_2(t)$ we invoke $\langle h,k\rangle_{H^1(B_{2R})} \leq \|h\|_{L^2(B_{2R})}\|k\|_{H^2(2R))}$ followed by the Hölder inequality and Lemmata 3.2, 5.4, 5.9 and 5.13 to get

$$\begin{aligned}
|G_2(t)| &\lesssim_{R,\varphi} \int_0^t \|\mathfrak{v}_n(r)-\mathfrak{v}_n(r_m)\|_{L^2(B_{2R})}\|Y(u_h(r))\|_{H^2(B_{2R})}\|\dot{h}_n(r)-\dot{h}(r)\|_{H_\mu}\,dr \\
&\lesssim_R \left(\int_0^t \|\mathfrak{v}_n(r)-\mathfrak{v}_n(r_m)\|^2_{L^2(B_{2R})}\,dr\right)^{\frac{1}{2}} \\
&\quad\times\left(\int_0^t \|u_h(r)\|^2_{H^2(B_{2R})}\left[1+\|u_h(r)\|^2_{H^2(B_{2R})}\right]\|\dot{h}_n(r)-\dot{h}(r)\|^2_{H_\mu}\,dr\right)^{\frac{1}{2}} \\
&\lesssim \sqrt{M_\mu}\left(\int_0^t |r-r_m|\,dr\right)^{\frac{1}{2}} \sup_{r\in[0,T/2]} \|u_h(r)\|^2_{H^2(B_{T-r})}\left[1+\|u_h(r)\|^2_{H^2(B_{T-r})}\right] \\
&\lesssim \frac{R\sqrt{M_\mu}}{2^{m/2}} \sup_{r\in[0,T/2]} l(r,z_h(r))\left[1+l(r,z_h(r))\right] \leq \frac{R\sqrt{M_\mu}}{2^{m/2}}\,\mathcal{B}(1+\mathcal{B}),
\end{aligned}$$

where in the second last step we have used

$$\left(\int_0^t |r-r_m|\,dr\right)^{\frac{1}{2}} \leq \left(\int_0^R |r-r_m|\,dr\right)^{\frac{1}{2}} = \left(\sum_{k=1}^{2^m}\int_{t_{k-1}}^{t_k}\left|r-\frac{kR}{2^m}\right|\,dr\right)^{\frac{1}{2}} \leq \frac{R}{2^{m/2}}.$$

Moreover, in the third last step we have also applied the following: since $\dot{h}_n \to \dot{h}$ weakly in $L^2(0,T;H_\mu)$, the sequence $\dot{h}_n - \dot{h}$ is bounded in $L^2(0,T;H_\mu)$ i.e. $\exists M_\mu > 0$ such that

$$\int_0^t \|\dot{h}_n(r)-\dot{h}(r)\|^2_{H_\mu}\,dr \leq M_\mu, \quad \forall n. \qquad (5.22)$$

Before moving to $G_3(t)$ note that, since $2R = T/2$, due to Remark 5.10, for every $s,t\in[0,T/2]$,

$$\|u_h(t)-u_h(s)\|_{H^1(B_{2R})} \leq \int_s^t \|v_h(r)\|_{H^1(B_{2R})}\,dr \lesssim \sqrt{\mathcal{B}}|t-s|.$$

Consequently, by the Hölder inequality followed by Lemmata 3.2, 5.13, and 5.4 we obtain

$$\begin{aligned}|G_3(t)| &\lesssim_\varphi \left(\int_0^t \left[\|v_n(r_m)\|^2_{H^1(B_{2R})} + \|v_h(r_m)\|^2_{H^1(B_{2R})}\right] dr\right)^{\frac{1}{2}} \\ &\qquad\times \left(\int_0^t \|Y(u_h(r)) - Y(u_h(r_m))\|^2_{H^1(B_{2R})} \|\dot{h}_n(r) - \dot{h}(r)\|^2_{H_\mu} dr\right)^{\frac{1}{2}} \\ &\lesssim_{T,\mathcal{B}} \left(\int_0^t \|u_h(r) - u_h(r_m)\|^2_{H^1(B_{2R})} \left[1 + \|u_h(r)\|^2_{H^1(B_{2R})} + \|u_h(r_m)\|^2_{H^1(B_{2R})}\right]\right. \\ &\qquad \left.\times \|\dot{h}_n(r) - \dot{h}(r)\|^2_{H_\mu} dr\right)^{\frac{1}{2}} \\ &\lesssim_{T,\mathcal{B}} \left(\int_0^t |r - r_m| \, \|\dot{h}_n(r) - \dot{h}(r)\|^2_{H_\mu} dr\right)^{\frac{1}{2}} \\ &\le \left(\sum_{k=1}^{2^m} \int_{t_{k-1}}^{t_k} \left|r - \frac{kR}{2^m}\right| \|\dot{h}_n(r) - \dot{h}(r)\|^2_{H_\mu} dr\right)^{\frac{1}{2}} \\ &\le \sqrt{\frac{R}{2^m}} \left(\int_0^t \|\dot{h}_n(r) - \dot{h}(r)\|^2_{H_\mu} dr\right)^{\frac{1}{2}} \le \sqrt{T\frac{M_\mu}{2^m}}, \quad t \in [0, T].\end{aligned}$$

Finally we start estimating $G_4(t)$ by noting that for every $t \in [0, R]$,

$$\text{there exists} \quad K_t \le 2^m \quad \text{such that} \quad t \in \left[\frac{(k_t - 1)\cdot R}{2^m}, \frac{k_t \cdot R}{2^m}\right).$$

Note that on such interval $r_m = \frac{k_t \cdot R}{2^m}$. Then by Lemma 5.9 we have

$$\begin{aligned}|G_4(t)| &\le \left|\sum_{k=1}^{k_t - 1} \int_{t_{k-1}}^{t_k} \left\langle \mathfrak{v}_n\left(\frac{k\cdot R}{2^m}\right), \varphi Y\left(u_h\left(\frac{k\cdot R}{2^m}\right)\right)(\dot{h}_n(r) - \dot{h}(r))\right\rangle_{H^1(B_{2R})} dr\right. \\ &\quad \left.+ \int_{t_{k_t - 1}}^t \left\langle \mathfrak{v}_n\left(\frac{(k_t - 1)\cdot R}{2^m}\right), \varphi Y\left(u_h\left(\frac{(k_t-1)\cdot R}{2^m}\right)\right)(\dot{h}_n(r) - \dot{h}(r))\right\rangle_{H^1(B_{2R})} dr\right| \\ &\le \sum_{k=1}^{2^m}\left|\left\langle \mathfrak{v}_n\left(\frac{k\cdot R}{2^m}\right), \varphi Y\left(u_h\left(\frac{k\cdot R}{2^m}\right)\right)\int_{t_{k-1}}^{t_k} (\dot{h}_n(r) - \dot{h}(r))\, dr\right\rangle_{H^1(B_{2R})}\right| \\ &\quad + \sup_{1\le k\le 2^m} \sup_{t_k \le t \le t_{k-1}} \left|\left\langle \mathfrak{v}_n\left(\frac{(k-1)\cdot R}{2^m}\right), \varphi Y\left(u_h\left(\frac{(k-1)\cdot R}{2^m}\right)\right)\int_{t_{k-1}}^{t} (\dot{h}_n(r) - \dot{h}(r))\, dr\right\rangle_{H^1(B_{2R})}\right| \\ &\le \sum_{k=1}^{2^m}\left\|\mathfrak{v}_n\left(\frac{k\cdot R}{2^m}\right)\right\|_{H^1(B_{2R})}\left\|\varphi Y\left(u_h\left(\frac{k\cdot R}{2^m}\right)\right)\int_{t_{k-1}}^{t_k} (\dot{h}_n(r) - \dot{h}(r))\, dr\right\|_{H^1(B_{2R})} \\ &\quad + \sup_{1\le k\le 2^m} \sup_{t_k \le t \le t_{k-1}} \left\|\mathfrak{v}_n\left(\frac{(k-1)\cdot R}{2^m}\right)\right\|_{H^1(B_{2R})}\left\|\varphi Y\left(u_h\left(\frac{(k-1)\cdot R}{2^m}\right)\right)\int_{t_{k-1}}^{t} (\dot{h}_n(r) - \dot{h}(r))\, dr\right\|_{H^1(B_{2R})} \\ &\lesssim_{\varphi,\mathcal{B}} \sum_{k=1}^{2^m}\left\|Y\left(u_h\left(\frac{k\cdot R}{2^m}\right)\right)\int_{t_{k-1}}^{t_k} (\dot{h}_n(r) - \dot{h}(r))\, dr\right\|_{H^1(B_{2R})}\end{aligned}$$

$$+\sup_{1\le k\le 2^m}\sup_{t_k\le t\le t_{k-1}}\left\|Y\left(u_h\left(\frac{(k-1)\cdot R}{2^m}\right)\right)\int_{t_{k-1}}^{t}(\dot h_n(r)-\dot h(r))\,dr\right\|_{H^1(B_{2R})}$$
$$=: G_4^1+G_4^2, \tag{5.23}$$

where the right hand side does not depend on t. By invoking Lemmata 3.2, 5.4, the Hölder inequality, and Lemma 5.9 we estimate G_4^1 as

$$\begin{aligned}
G_4^1 &\lesssim_{R,T} \sup_{1\le k\le 2^m}\sup_{t_k\le t\le t_{k-1}}\left\|Y\left(u_h\left(\frac{(k-1)\cdot R}{2^m}\right)\right)\right\|_{H^1(B_{2R})}\left(\int_{t_{k-1}}^{t}\|\dot h_n(r)-\dot h(r)\|_{H_\mu}^2\,dr\right)^{\frac12}\\
&\lesssim_{R,T} \sup_{1\le k\le 2^m}\sup_{t_k\le t\le t_{k-1}}\left[1+\left\|u_h\left(\frac{(k-1)\cdot R}{2^m}\right)\right\|_{H^1(B_{2R})}\right]\left(\int_{t_{k-1}}^{t}\|\dot h_n(r)-\dot h(r)\|_{H_\mu}^2\,dr\right)^{\frac12}\\
&\lesssim_{R,T,\mathcal B} \sup_{1\le k\le 2^m}\sup_{t_k\le t\le t_{k-1}}\left(\int_{t_{k-1}}^{t}\|\dot h_n(r)-\dot h(r)\|_{H_\mu}^2\,dr\right)^{\frac12}\\
&\le \sup_{1\le k\le 2^m}\left(\int_{t_{k-1}}^{t_k}\|\dot h_n(r)-\dot h(r)\|_{H_\mu}^2\,dr\right)^{\frac12}.
\end{aligned}$$

For G_4^2 recall that, by Lemma 3.2, for every $\phi\in H^1(B(x,r))$ the multiplication operator

$$Y(\phi)\cdot : K\ni k\mapsto Y(\phi)\cdot k\in H^1(B(x,r)),$$

is γ-radonifying and hence compact. So Lemma 5.12 implies the following, for every k,

$$\left\|Y\left(u_h\left(\frac{k\cdot R}{2^m}\right)\right)\int_{t_{k-1}}^{t_k}(\dot h_n(r)-\dot h(r))\,dr\right\|_{H^1(B_{2R})}\to 0 \text{ as } n\to 0. \tag{5.24}$$

Hence, for fix m, each term of the sum in G_4^2 goes to 0 as $n\to\infty$. Consequently, by substituting the computation between (5.21) and (5.23) into (5.20) we obtain

$$\begin{aligned}
\int_0^t\langle\bar{\mathfrak v}_n(r),\bar g_n(r)\rangle_{H^1(B_{2R})}\,dr &\lesssim_{R,L_A,B_A,\varphi,\mathcal B}\int_0^t(1+l(r,\mathfrak z_n(r)))\left(1+\|\dot h_n(r)\|_{H_\mu}^2\right)dr\\
&\quad+\sqrt{T\frac{M_\mu}{2^m}}+\sup_{1\le k\le 2^m}\left(\int_{t_{k-1}}^{t_k}\|\dot h_n(r)-\dot h(r)\|_{H_\mu}^2\,dr\right)^{\frac12}\\
&\quad+\sum_{k=1}^{2^m}\left\|Y\left(u_h\left(\frac{k\cdot R}{2^m}\right)\right)\int_{t_{k-1}}^{t_k}(\dot h_n(r)-\dot h(r))\,dr\right\|_{H^1(B_{2R})}.
\end{aligned}$$

Therefore, with (5.19) and (5.14), from (5.17) we have

$$\begin{aligned}
l(t,\mathfrak z_n(t)) &\lesssim \int_0^t(1+l(r,\mathfrak z_n(r)))\left(1+\|\dot h_n(r)\|_{H_\mu}^2\right)dr\\
&\quad+\sqrt{T\frac{M_\mu}{2^m}}+\sup_{1\le k\le 2^m}\left(\int_{t_{k-1}}^{t_k}\|\dot h_n(r)-\dot h(r)\|_{H_\mu}^2\,dr\right)^{\frac12}
\end{aligned}$$

$$+\sum_{k=1}^{2^m}\left\|Y\left(u_h\left(\frac{k\cdot R}{2^m}\right)\right)\int_{t_{k-1}}^{t_k}(\dot h_n(r)-\dot h(r))\,dr\right\|_{H^1(B_{2R})},\quad t\in[0,T],$$

and by the Gronwall Lemma, with the observation that all the terms in right hand side except the first are independent of t, and $h_n\in S_M$ further we get

$$\sup_{t\in[0,R]} l(t,\mathfrak{z}_n(t))\lesssim e^{T+M}\left\{\sqrt{T\frac{M_\mu}{2^m}}+\sup_{1\le k\le 2^m}\left(\int_{t_{k-1}}^{t_k}\|\dot h_n(r)-\dot h(r)\|_K^2\,dr\right)^{\frac12}\right.$$
$$\left.+\sum_{k=1}^{2^m}\left\|Y\left(u_h\left(\frac{k\cdot R}{2^m}\right)\right)\int_{t_{k-1}}^{t_k}(\dot h_n(r)-\dot h(r))\,dr\right\|_{H^1(B_{2R})}\right\}. \quad (5.25)$$

Now by [52, Theorem 6.11], for every $\alpha>0$ we can choose m such that

$$\sup_{1\le k\le 2^m}\left(\int_{t_{k-1}}^{t_k}\|\dot h_n(r)-\dot h(r)\|_{H_\mu}^2\,dr\right)^{\frac12}+\sqrt{T\frac{M_\mu}{2^m}}<\alpha,$$

and for such chosen m, due to (5.24) by taking $n\to\infty$ in (5.25) we conclude that, for every $\alpha>0$,

$$0<\lim_{n\to\infty}\sup_{t\in[0,R]} l(t,\mathfrak{z}_n(t))<\alpha.$$

Therefore, due to (5.14) we get (5.16) and hence the Proposition 5.11. □

Now we come back to the proof of Statement 1. Previous Proposition shows, for fix $T>0$, that every sequence in K_M has a convergent subsequence. Hence K_M is sequentially relatively compact subset of $\mathcal{X}_T$. Let $\{z_n\}_{n\in\mathbb{N}}\subset K_M$ which converges to $z\in\mathcal{X}_T$. But Proposition 5.11 shows that there exists a subsequence $\{u_{n_k}\}_{k\in\mathbb{N}}$ which converges to some element z_h of K_M in the same strong topology of $\mathcal{X}_T$. Hence $z=z_h$ and K_M is a closed subset of $\mathcal{X}_T$. This completes the proof of Statement 1.

Below is a basic result that we have used in the proof of Proposition 5.11.

Lemma 5.12. *Let X,Y be separable Hilbert spaces such that the embedding $i:X\to Y$ is compact. If $g_n\to g$ weakly in $L^2(0,T;X)$, then*

$$i\int_0^{\cdot} g_n(s)\,ds-i\int_0^{\cdot} g(s)\,ds\to 0 \text{ as } n\to\infty \quad \text{in} \quad \mathcal{C}([0,T];Y).$$

Proof of Lemma 5.12. Define $G_n:[0,T]\ni t\mapsto\int_0^t g_n(s)\,ds\in X$. Then the sequence of functions $\{G_n\}_{n\in\mathbb{N}}\subset\mathcal{C}([0,T];X)$. Next, since weakly convergence sequence is bounded, the Hölder inequality gives

$$\|G_n(t_2)-G_n(t_1)\|_X\le\int_{t_1}^{t_2}\|g_n(s)\|_X\,ds\le|t_2-t_1|^{\frac12}\left(\int_0^T\|g_n(s)\|_X^2\,ds\right)\le C_g|t_2-t_1|^{\frac12},$$

for some $C_g>0$. So the sequence $\{G_n\}_{n\in\mathbb{N}}$ is equicontinuous and uniformly bounded on $[0,T]$. Hence, $\{G_n\}_{n\in\mathbb{N}}$ is a bounded subset of $L^2(0,T;X)$ because $\mathcal{C}([0,T];X)\subset L^2(0,T;X)$. Consequently, since the embedding $X\overset{i}{\hookrightarrow}Y$ is compact, due to Dubinsky Theorem [56, Theorem 4.1, p. 132], $\{iG_n\}_{n\in\mathbb{N}}$ is relatively compact in $\mathcal{C}([0,T];Y)$, where $iG_n:[0,T]\ni t\mapsto i(G_n(t))\in Y$. Therefore, there exist a subsequence, which we again indexed by $n \in \mathbb{N}$, $\{iG_n\}_{n\in\mathbb{N}}$ and $F \in \mathcal{C}([0,T];Y)$ such that $iG_n \to F$, as $n \to \infty$, in $\mathcal{C}([0,T];Y)$. This implies, for each $t \in [0,T]$, $G_n(t) \to F(t)$ in Y.

On the other hand, by weak convergence of g_n to g, we have, for every $x \in X$ and $t \in [0,T]$,

$$\begin{aligned}\langle G_n(t), x\rangle_X = \int_0^T \langle g_n(s), x\mathbb{1}_{[0,t]}(s)\rangle_X \, ds &= \langle g_n, x\mathbb{1}_{[0,t]}\rangle_{L^2(0,T;X)} \\ &\xrightarrow[n\to\infty]{} \langle g, x\mathbb{1}_{[0,t]}\rangle_{L^2(0,T;X)} = \langle G(t), x\rangle_X.\end{aligned}$$

Hence, for each $t \in [0,T]$, $\{G_n(t)\}_{n\in\mathbb{N}}$ is weakly convergent to $G(t)$ in X. Since $X \overset{i}{\hookrightarrow} Y$ is compact, $\{i(G_n(t))\}_{n\in\mathbb{N}}$ strongly converges to $i(G(t))$ in Y. So by the uniqueness of limit in Y, $i(G(t)) = F(t)$ for $t \in [0,T]$ and we have proved that every weakly convergent sequence $\{g_n\}_{n\in\mathbb{N}}$ has a subsequence, indexed again by $n \in \mathbb{N}$, such that $i\int_0^\cdot g_n(s)\, ds$ converges to $i\int_0^\cdot g(s)\, ds$ in $\mathcal{C}([0,T];Y)$.

Since the same argument proves that from every weakly convergent subsequence in $L^2(0,T;X)$ we can extract a subsubsequence such that the last statement convergence holds, we have proved the Lemma 5.12. $\square$

The following Lemma is about the Lipschitz property of the difference of solutions that we have used in proving Proposition 5.11.

Lemma 5.13. *Let $h_n, h \in S_M$ and $I = [-a,a]$. There exists a positive constant $C := C(R,\mathcal{B},M,a)$ such that for $t,s \in [0,T/2]$ the following holds*

$$\sup_{x\in I} \|\mathfrak{v}_n(t) - \mathfrak{v}_n(s)\|_{L^2(B(x,R))} \lesssim C\, |t-s|^{\frac{1}{2}}, \tag{5.26}$$

for $R = T/2$, where $\mathfrak{v}_n$ is defined just after (5.11)*.*

Proof of Lemma 5.13. Due to triangle inequality it is sufficient to show

$$\sup_{x\in I} \|v_h(t) - v_h(s)\|_{L^2(B(x,R))} \lesssim C|t-s|^{\frac{1}{2}}, \quad t,s \in [0,T/2].$$

From the proof of existence part in Theorem 4.1 we have, for $t,s \in [0,T/2]$,

$$\begin{aligned}\|v_h(t) - v_h(s)\|_{L^2(B(x,R))} \le & \int_s^t \|\partial_{xx} u_h(r)\|_{L^2(B(x,R))}\, dr \\ & + \int_s^t \left[\|f_h(r)\|_{L^2 B(x,R))} + \|g_h(r)\|_{L^2(B(x,R))}\right] dr,\end{aligned} \tag{5.27}$$

where

$$f_h(r) := A_{u_h(r)}(v_h(r), v_h(r)) - A_{u_h(r)}(\partial_x u_h(r), \partial_x u_h(r)), \text{ and } g_h(r) := Y(u_h(r))\dot{h}(r).$$

But, since $h \in S_M$, the Hölder inequality followed by Lemmata 3.2, 5.4 and 5.9 yields

$$\begin{aligned}\sup_{x\in I}\int_s^t \|g_h(r)\|_{L^2(B(x,R))}\, dr &\le |t-s|^{\frac{1}{2}} \left(\int_s^t \sup_{x\in I} \|Y(u_h(r))\|^2_{L^2(B(x,R))} \|\dot{h}(s)\|^2_{H_\mu}\, ds\right)^{\frac{1}{2}} \\ &\lesssim_{R,\mathcal{B},M} |t-s|^{\frac{1}{2}}, \qquad \text{for} \quad t,s \in [0,T/2],\end{aligned}$$

and, based on (5.18), we also have

$$
\begin{aligned}
&\sup_{x\in I}\int_s^t \|f_h(r)\|_{L^2(B(x,R))}\,dr \\
&\le |t-s|^{\frac12}\left(\int_s^t \sup_{x\in I}\|A_{u_h(r)}(v_h(r),v_h(r))\|^2_{L^2(B(x,R))}\,dr\right)^{\frac12} \\
&\quad + |t-s|^{\frac12}\left(\int_s^t \sup_{x\in I}\|A_{u_h(s)}(\partial_x u_h(r),\partial_x u_h(r))\|^2_{L^2(B(x,R))}\,dr\right)^{\frac12} \\
&\lesssim |t-s|^{\frac12}\left(\int_s^t \sup_{x\in I}\|u_h(r)\|^2_{L^2(B(x,R))}\{\|v_h(s)\|^4_{L^2(B(x,R))}+\|\partial_x u_h(s)\|^4_{L^2(B(x,R))}\}\,ds\right)^{\frac12} \\
&\lesssim |t-s|\ \mathcal{B}^{\frac32} \qquad \text{for} \quad t,s\in[0,T/2].
\end{aligned}
$$

Finally, by the Hölder inequality and Lemma 5.9, we obtain, for $t,s\in[0,T/2]$,

$$
\begin{aligned}
\sup_{x\in I}\int_s^t \|\partial_{xx}u_h(s)\|_{L^2(B(x,R))}\,dr &\le \left(\int_s^t 1\,dr\right)^{\frac12}\left(\int_s^t \sup_{x\in I}\|u_h(r)\|^2_{H^2(B(x,R))}\,dr\right)^{\frac12} \\
&\lesssim \sqrt{\mathcal{B}}|t-s|.
\end{aligned}
$$

Therefore, by collecting the estimates in (5.27) we get the required inequality (5.26) and we are done with the proof of Lemma 5.13. □

5.3. **Proof of Statement 2.** It will be useful to introduce the following notation for the processes

$$
Z_n := (U_n,V_n) = J^{\varepsilon_n}\left(W+\frac{1}{\sqrt{\varepsilon_n}}h_n\right), \quad z_n := (u_n,v_n) = J^0(h_n).
$$

Let us fix any $x\in\mathbb{R}$ and $T>0$. Then set N a natural number such that $N > \|(u_0,v_0)\|_{\mathcal{H}(B(x,T))}$. To simplify the notation we write $\mathcal{H}_T$ instead $\mathcal{H}(B(x,T))$. For each $n\in\mathbb{N}$ we define an $\mathscr{F}_t$-stopping time

$$
\tau_n(\omega) := \inf\{t>0 : \|Z_n(t,\omega)\|_{\mathcal{H}_{T-t}} \ge N\}\wedge T, \quad \omega\in\Omega. \tag{5.28}
$$

Define, for $z=(u,v)\in\mathcal{H}_{loc}$,

$$
\mathbf{e}(t,z) := \frac12\left\{\|u\|^2_{H^2(B_{T-t})}+\|v\|^2_{H^1(B_{T-t})}\right\} = \frac12\|z\|^2_{\mathcal{H}_{T-t}}, \quad t\in[0,T]. \tag{5.29}
$$

In this framework we prove the following key result.

Proposition 5.14. *Let us define $\mathcal{Z}_n := Z_n - z_n$. For τ_n defined in* (5.28) *we have*

$$
\lim_{n\to\infty}\mathbb{E}\left[\sup_{t\in[0,T]}\mathbf{e}(t\wedge\tau_n,\mathcal{Z}_n(t\wedge\tau_n))\right] = 0.
$$

Proof of Proposition 5.14. Let us fix any $n\in\mathbb{N}$. First note that under our notation $Z_n=(U_n,V_n)$ and $z_n=(u_n,v_n)$, respectively, are the unique global strong solutions to the Cauchy problem

$$\begin{cases} \partial_{tt} U_n = \partial_{xx} U_n + A_{U_n}(\partial_t U_n, \partial_t U_n) - A_{U_n}(\partial_x U_n, \partial_x U_n) + Y(U_n)\dot{h}_n, \\ \qquad\qquad + \sqrt{\varepsilon_n} Y(U_n)\dot{W}, \\ (U_n(0), \partial_t U_n(0)) = (u_0, v_0)\,, \quad \text{ where } V_n := \partial_t U_n, \end{cases}$$

and

$$\begin{cases} \partial_{tt} u_n = \partial_{xx} u_n + A_{u_n}(\partial_t u_n, \partial_t u_n) - A_{u_n}(\partial_x u_n, \partial_x u_n) + Y(u_n)\dot{h}_n, \\ (u_n(0), \partial_t u_n(0)) = (u_0, v_0)\,, \quad \text{ where } v_n := \partial_t u_n. \end{cases}$$

Hence $\mathcal{Z}_n$ solves uniquely the Cauchy problem, with null initial data,

$$\begin{aligned} \partial_{tt}\mathcal{U}_n = \partial_{xx}\mathcal{U}_n - A_{U_n}(\partial_x U_n, \partial_x U_n) + A_{u_n}(\partial_x u_n, \partial_x u_n) + A_{U_n}(\partial_t U_n, \partial_t U_n) \\ - A_{u_n}(\partial_t u_n, \partial_t u_n) + Y(U_n)\dot{h}_n - Y(u_n)\dot{h}_n + \sqrt{\varepsilon_n} Y(U_n)\dot{W}, \end{aligned}$$

where $\mathcal{V}_n := \partial_t \mathcal{U}_n$. This is equivalent to say, for all $t \in [0,T]$,

$$\mathcal{Z}_n(t) = \int_0^t S_{t-s} \begin{pmatrix} 0 \\ f_n(s) \end{pmatrix} ds + \int_0^t S_{t-s} \begin{pmatrix} 0 \\ g_n(s) \end{pmatrix} dW(s). \tag{5.30}$$

Here

$$\begin{aligned} f_n(s) := -A_{U_n(s)}(\partial_x U_n(s), \partial_x U_n(s)) + A_{u_n(s)}(\partial_x u_n(s), \partial_x u_n(s)) + A_{U_n(s)}(V_n(s), V_n(s)) \\ - A_{u_n(s)}(v_n(s), v_n(s)) + Y(U_n(s))\dot{h}_n(s) - Y(u_n(s))\dot{h}_n(s), \end{aligned}$$

and

$$g_n(s) := \sqrt{\varepsilon_n} Y(U_n(s)).$$

Invoking Proposition B.1, with that by taking $k = 1, L = I$, implies for every $t \in [0,T]$,

$$\begin{aligned} \mathbf{e}(t, \mathcal{Z}_n(t)) \leq \int_0^t \mathbb{V}(r, \mathcal{Z}_n(r))\, dr + \int_0^t \langle \mathcal{V}_n(r), g_n(r) dW(r) \rangle_{L^2(B_{T-r})} \\ + \int_0^t \langle \partial_x \mathcal{V}_n(r), \partial_x [g_n(r) dW(r)] \rangle_{L^2(B_{T-r})}, \end{aligned} \tag{5.31}$$

with

$$\begin{aligned} \mathbb{V}(t, \mathcal{Z}_n(t)) &= \langle \mathcal{U}_n(t), \mathcal{V}_n(t) \rangle_{L^2(B_{T-t})} + \langle \mathcal{V}_n(t), f_n(t) \rangle_{L^2(B_{T-t})} \\ &\quad + \langle \partial_x \mathcal{V}_n(t), \partial_x f_n(t) \rangle_{L^2(B_{T-t})} + \frac{1}{2} \sum_{j=1}^{\infty} \|g_n(t) e_j\|^2_{L^2(B_{T-t})} \\ &\quad + \frac{1}{2} \sum_{j=1}^{\infty} \|\partial_x [g_n(t) e_j]\|^2_{L^2(B_{T-t})}, \end{aligned}$$

for a given sequence $\{e_j\}_{j \in \mathbb{N}}$ of orthonormal basis of H_μ. Let us define

$$\Psi_n(t) := \mathbb{E}\left[\sup_{0 \leq s \leq t} \mathbf{e}(s \wedge \tau_n, \mathcal{Z}_n(s \wedge \tau_n)) \right] = \mathbb{E}\left[\sup_{0 \leq s \leq t \wedge \tau_n} \mathbf{e}(s, \mathcal{Z}_n(s)) \right].$$

Observe that, for any $\tau \in [0,T]$, by the Cauchy-Schwartz inequality

$$\sup_{0\le t\le \tau} \int_0^{t\wedge\tau_n} \mathbb{V}(r,\mathcal{Z}_n(r))\,dr \le 2\int_0^{\tau\wedge\tau_n} \mathbf{e}(r,\mathcal{Z}_n(r))\,dr + \frac{1}{2}\int_0^{\tau\wedge\tau_n} \Big(\|f_n(r)\|^2_{H^1(B_{T-r})} + \|g_n(r)\cdot\|^2_{\mathscr{L}_2(H_\mu,H^1(B_{T-r}))}\Big)\,dr, \tag{5.32}$$

where $g_n(r)\cdot$ denotes the multiplication operator in the space $\mathscr{L}_2(H_\mu, H^1(B(x,R)))$, see Lemma 3.2.

Next, we define the process

$$\mathcal{Y}(t) := \int_0^t \langle \mathcal{V}_n(r), g_n(r)dW(r)\rangle_{H^1(B_{T-r})}. \tag{5.33}$$

By taking $\int_0^t \xi(r)\,dW(r)$ with

$$\xi(r) : H_\mu \ni k \mapsto \langle \mathcal{V}_n(r), g_n(r)(k)\rangle_{H^1(B_{T-r})} \in \mathbb{R},$$

a Hilbert-Schmidt operator, note that

$$\mathcal{Q}(t) := \int_0^t \xi(r)\circ\xi(r)^\star\,dr,$$

is quadratic variation of $\mathbb{R}$-valued martingale $\mathcal{Y}$. Then

$$\begin{aligned}\mathcal{Q}(t) &\le \int_0^t \|\xi(r)\|_{\mathscr{L}_2(H_\mu,\mathbb{R})}\|\xi(r)^\star\|_{\mathscr{L}_2(\mathbb{R},H_\mu)}\,dr = \int_0^t \|\xi(r)\|^2_{\mathscr{L}_2(H_\mu,\mathbb{R})}\,dr \\ &= \int_0^t \sum_{j=1}^\infty |\xi(r)(e_j)|^2\,dr = \int_0^t \sum_{j=1}^\infty |\langle \mathcal{V}_n(r), g_n(r)(e_j)\rangle_{H^1(B_{T-r})}|^2\,dr, \quad t\in[0,T].\end{aligned} \tag{5.34}$$

On the other hand by the Cauchy-Schwartz inequality

$$\sum_{j=1}^\infty |\langle \mathcal{V}_n(r), g_n(r)(e_j)\rangle_{H^1(B_{T-r})}|^2 \le \|\mathcal{V}_n(r)\|^2_{H^1(B_{T-r})}\|g_n(r)\cdot\|^2_{\mathscr{L}_2(H_\mu,H^1(B_{T-r}))}.$$

Therefore,

$$\mathcal{Q}(t) \le \int_0^t \|\mathcal{V}_n(r)\|^2_{H^1(B_{T-r})}\|g_n(r)\cdot\|^2_{\mathscr{L}_2(H_\mu,H^1(B_{T-r}))}\,dr. \tag{5.35}$$

Invoking the Davis inequality with (5.35) followed by the Young inequality gives

$$\begin{aligned}\mathbb{E}\left[\sup_{0\le t\le\tau}|\mathcal{Y}(t\wedge\tau_n)|\right] &\le 3\mathbb{E}\left[\sqrt{\mathcal{Q}(\tau\wedge\tau_n)}\right] \\ &\le 3\mathbb{E}\left[\sup_{0\le t\le\tau\wedge\tau_n}\|\mathcal{V}_n(t\wedge\tau_n)\|_{H^1(T-t)}\left\{\int_0^{\tau\wedge\tau_n}\|g_n(r)\cdot\|^2_{\mathscr{L}_2(H_\mu,H^1(B_{T-r}))}\,dr\right\}^{\frac{1}{2}}\right] \\ &\le 3\mathbb{E}\left[\varepsilon\sup_{0\le t\le\tau\wedge\tau_n}\|\mathcal{V}_n(t)\|^2_{H^1(T-t)} + \frac{1}{4\varepsilon}\int_0^{\tau\wedge\tau_n}\|g_n(r)\cdot\|^2_{\mathscr{L}_2(H_\mu,H^1(B_{T-r}))}\,dr\right]\end{aligned}$$

$$\leq 6\varepsilon\, \mathbb{E}\left[\sup_{0\leq t\leq \tau\wedge\tau_n} \mathbf{e}(t, \mathcal{Z}_n(t))\right] + \frac{3}{4\varepsilon}\mathbb{E}\left[\int_0^{\tau\wedge\tau_n} \|g_n(r)\cdot\|^2_{\mathscr{L}_2(H_\mu, H^1(B_{T-r}))}\, dr\right]. \tag{5.36}$$

By choosing ε such that $6\varepsilon = \frac{1}{2}$ and taking $\sup_{0\leq s\leq t}$ followed by expectation $\mathbb{E}$ on the both sides of (5.31) after evaluating it at $\tau\wedge\tau_n$ we obtain

$$\mathbb{E}\left[\sup_{0\leq s\leq t\wedge\tau_n} \mathbf{e}(s, \mathcal{Z}_n(s))\right] \leq \mathbb{E}\left[\sup_{0\leq s\leq t}\int_0^{s\wedge\tau_n} \mathbb{V}(r, \mathcal{Z}_n(r))\, dr\right] + \mathbb{E}\left[\sup_{0\leq s\leq t} \mathcal{Y}(s\wedge\tau_n)\right].$$

Consequently, using (5.32) and (5.36) we infer that

$$\mathbb{E}\left[\sup_{0\leq s\leq t\wedge\tau_n} \mathbf{e}(s, \mathcal{Z}_n(s))\right] \leq 4\mathbb{E}\left[\int_0^{t\wedge\tau_n} \mathbf{e}(r, \mathcal{Z}_n(r))\, dr\right] + \mathbb{E}\left[\int_0^{t\wedge\tau_n} \|f_n(r)\|^2_{H^1(B_{T-r})}\, dr\right]$$
$$+ 19\mathbb{E}\left[\int_0^{t\wedge\tau_n} \|g_n(r)\cdot\|^2_{\mathscr{L}_2(H_\mu, H^1(B_{T-r}))}\, dr\right]. \tag{5.37}$$

Now since the Hilbert-Schmidt operator $g_n(r)\cdot$ is defined as

$$H_\mu \ni k \mapsto g_n(r)\cdot k \in H^1(B_{T-r}),$$

Lemmata 3.2 and 5.4 gives,

$$\begin{aligned}
\mathbb{E}\left[\int_0^{t\wedge\tau_n} \|g_n(r)\cdot\|^2_{\mathscr{L}_2(H_\mu, H^1(B_{T-r}))}\, dr\right] &\lesssim_T\ \mathbb{E}\left[\int_0^{t\wedge\tau_n} \|\sqrt{\varepsilon_n} Y(U_n(r))\|^2_{H^1(B_{T-r})}\, dr\right] \\
&\lesssim_T \varepsilon_n\ \mathbb{E}\left[\int_0^{t\wedge\tau_n} \left(1 + \|U_n(r)\|^2_{H^1(B_{T-r})}\right)\, dr\right] \\
&\leq \varepsilon_n\ \mathbb{E}\left[\int_0^{t\wedge\tau_n} \left(1 + \|Z_n(r)\|^2_{\mathcal{H}_{T-r}}\right)\, dr\right] \\
&\lesssim_T \varepsilon_n\ (1+N^2).
\end{aligned} \tag{5.38}$$

To estimate the terms involving f_n we have

$$\begin{aligned}
\|f_n(r)\|^2_{H^1(B_{T-r})} &\lesssim \|A_{U_n(r)}(\partial_x U_n(r), \partial_x U_n(r)) - A_{u_n(r)}(\partial_x u_n(r), \partial_x u_n(r))\|^2_{H^1(B_{T-r})} \\
&\quad + \|A_{U_n(r)}(V_n(r), V_n(r)) - A_{u_n(r)}(v_n(r), v_n(r))\|^2_{H^1(B_{T-r})} \\
&\quad + \|Y(U_n(r))\dot{h}_n(r) - Y(u_n(r))\dot{h}_n(r)\|^2_{H^1(B_{T-r})} \\
&=: f_n^1 + f_n^2 + f_n^3.
\end{aligned} \tag{5.39}$$

By doing the computation based on Lemmata 4.4 and 5.4 we obtain

$$\begin{aligned}
f_n^1 &\lesssim \|A_{U_n(r)}(\partial_x U_n(r), \partial_x U_n(r)) - A_{u_n(r)}(\partial_x U_n(r), \partial_x U_n(r))\|^2_{H^1(B_{T-r})} \\
&\quad + \|A_{u_n(r)}(\partial_x U_n(r), \partial_x U_n(r)) - A_{u_n(r)}(\partial_x u_n(r), \partial_x U_n(r))\|^2_{H^1(B_{T-r})} \\
&\quad + \|A_{u_n(r)}(\partial_x u_n(r), \partial_x U_n(r)) - A_{u_n(r)}(\partial_x u_n(r), \partial_x u_n(r))\|^2_{H^1(B_{T-r})} \\
&\lesssim_T \|U_n(r) - u_n(r)\|^2_{H^2(B_{T-r})}\left(1 + \|\partial_x U_n(r)\|^2_{H^1(B_{T-r})} + \|\partial_x U_n(r)\|^2_{H^1(B_{T-r})}\right)\times \\
&\quad \times\left(1 + \|u_n(r)\|^2_{H^2(B_{T-r})}\right) \\
&\quad + \|u_n(r)\|^2_{H^2(B_{T-r})}\|\partial_x[U_n(r) - u_n(r)]\|^2_{H^1(B_{T-r})}\|\partial_x[u_n(r)]\|^2_{H^1(B_{T-r})}
\end{aligned}$$

$$\lesssim \|\mathcal{Z}_n(r)\|^2_{\mathcal{H}_{T-r}} \left[\left(1+\|Z_n(r)\|^2_{\mathcal{H}_{T-r}}\right)\left(1+\|z_n(r)\|^2_{\mathcal{H}_{T-r}}\right)+\|z_n(r)\|^4_{\mathcal{H}_{T-r}}\right], \tag{5.40}$$

and, by similar calculations,

$$f_n^2 \lesssim_T \|\mathcal{Z}_n(r)\|^2_{\mathcal{H}_{T-r}} \left[\left(1+\|Z_n(r)\|^2_{\mathcal{H}_{T-r}}\right)\left(1+\|z_n(r)\|^2_{\mathcal{H}_{T-r}}\right)+\|z_n(r)\|^4_{\mathcal{H}_{T-r}}\right]. \tag{5.41}$$

Furthermore, Lemmata 5.4 and 3.2 implies

$$\begin{aligned} f_n^3 &\lesssim_T \|U_n(r)-u_n(r)\|^2_{H^1(B_{T-r})}\left[1+\|U_n(r)|^2_{H^1(B_{T-r})}+\|u_n(r)|^2_{H^1(B_{T-r})}\right]\|\dot{h}_n(r)\|^2_{H_\mu} \\ &\lesssim_T \|\mathcal{Z}_n(r)\|^2_{\mathcal{H}_{T-r}}\left(1+\|Z_n(r)\|^2_{\mathcal{H}_{T-r}}+\|z_n(r)\|^2_{\mathcal{H}_{T-r}}\right)\|\dot{h}_n(r)\|^2_{H_\mu}. \end{aligned} \tag{5.42}$$

Hence by substituting (5.40)-(5.42) in (5.39) we get

$$\begin{aligned} \|f_n(r)\|^2_{H^1(B_{T-r})} \lesssim_T \|\mathcal{Z}_n(r)\|^2_{\mathcal{H}_{T-r}} &\left[\left(1+\|Z_n(r)\|^2_{\mathcal{H}_{T-r}}\right)\left(1+\|z_n(r)\|^2_{\mathcal{H}_{T-r}}\right)+\|z_n(r)\|^4_{\mathcal{H}_{T-r}}\right] \\ &+\|\mathcal{Z}_n(r)\|^2_{\mathcal{H}_{T-r}}\left(1+\|Z_n(r)\|^2_{\mathcal{H}_{T-r}}+\|z_n(r)\|^2_{\mathcal{H}_{T-r}}\right)\|\dot{h}_n(r)\|^2_{H_\mu}, \end{aligned}$$

consequently, the definition of τ_n and Lemma 5.9 suggest

$$\begin{aligned} \mathbb{E}\left[\int_0^{t\wedge\tau_n}\|f_n(r)\|^2_{H^1(B_{T-r})}\,dr\right] &\lesssim \mathbb{E}\left[\int_0^{t\wedge\tau_n}\left\{\|\mathcal{Z}_n(r)\|^2_{\mathcal{H}_{T-r}}\left[\left(1+N^2\right)\left(1+\mathcal{B}^2\right)+\mathcal{B}^4\right]\right.\right. \\ &\qquad \left.\left.+\|\mathcal{Z}_n(r)\|^2_{\mathcal{H}_{T-r}}\left(1+N^2+\mathcal{B}^2\right)\left(1+\mathcal{B}^2\right)\|\dot{h}_n(r)\|^2_{H_\mu}\right\}dr\right] \\ &\lesssim \mathbb{E}\left[\int_0^{t\wedge\tau_n}\mathbf{e}(r,\mathcal{Z}_n(r))\,C_{N,\mathcal{B}}\left(1+\|\dot{h}_n(r)\|^2_{H_\mu}\right)dr\right], \end{aligned} \tag{5.43}$$

for some constant $C_{N,\mathcal{B}}>0$ depends on $N,\mathcal{B}$, Then substitution of (5.38) and (5.43) in (5.37) implies

$$\begin{aligned} \mathbb{E}\left[\sup_{0\le s\le t\wedge\tau_n}\mathbf{e}(s,\mathcal{Z}_n(s))\right] &\lesssim_T \varepsilon_n\,(1+N^2) \\ &\quad + C_{N,\mathcal{B}}\mathbb{E}\left[\int_0^{t\wedge\tau_n}[\sup_{0\le s\le r\wedge\tau_n}\mathbf{e}(s,\mathcal{Z}_n(s))]\left(1+\|\dot{h}_n(r)\|^2_{H_\mu}\right)dr\right]. \end{aligned}$$

Therefore, invoking the stochastic Gronwall Lemma, see [28, Lemma 3.9], gives,

$$\mathbb{E}\left[\sup_{0\le s\le t\wedge\tau_n}\mathbf{e}(s,\mathcal{Z}_n(s))\right] \lesssim_T \varepsilon_n\,(1+N^2)\exp\left[C_{N,\mathcal{B}}(T+M)\right]. \tag{5.44}$$

Since $\varepsilon_n\to 0$ as $n\to\infty$ and

$$\mathbb{E}\left[\sup_{0\le s\le t\wedge\tau_n}\mathbf{e}(s,\mathcal{Z}_n(s))\right]=\mathbb{E}\left[\sup_{0\le s\le t}\mathbf{e}(s\wedge\tau_n,\mathcal{Z}_n(s\wedge\tau_n))\right],$$

inequality (5.44) gives $\lim_{n\to\infty}\mathbb{E}\left[\sup_{0\le t\le T}\mathbf{e}(t\wedge\tau_n,\mathcal{Z}_n(t\wedge\tau_n))\right]=0$. Hence we are done with the proof of Proposition 5.14. □

To proceed further we also need the following stochastic analogue of Lemma 5.9.

Lemma 5.15. *There exists a constant* $\mathscr{B} := \mathscr{B}(N,T,M) > 0$ *such that*

$$\limsup_{n\to\infty} \mathbb{E}\left[\sup_{t\in[0,T]} \mathbf{e}(t\wedge\tau_n, Z_n(t\wedge\tau_n))\right] \leq \mathscr{B}.$$

Proof of Lemma 5.15. Let us fix sequence $\{e_j\}_{j\in\mathbb{N}}$ of orthonormal basis of H_μ. Let us also fix any $n\in\mathbb{N}$. With the notation of this subsection, Proposition B.1, with $k=1, L=I$, implies for every $t\in[0,T]$,

$$\mathbf{e}(t, Z_n(t)) \leq \int_0^t \mathbb{V}(r, Z_n(r))\,dr + \int_0^t \langle V_n(r), g_n(r)dW(r)\rangle_{H^1(B_{T-r})},$$

with

$$\mathbb{V}(t, Z_n(t)) = \langle U_n(t), V_n(t)\rangle_{L^2(B_{T-t})} + \langle V_n(t), f_n(t)\rangle_{H^1(B_{T-t})} + \frac{1}{2}\sum_{j=1}^\infty \|g_n(t)e_j\|^2_{H^1(B_{T-t})},$$

and

$$\begin{aligned} f_n(s) &:= A_{U_n(s)}(V_n(s), V_n(s)) - A_{U_n(s)}(\partial_x U_n(s), \partial_x U_n(s)) + Y(U_n(s))\dot{h}_n(s),\\ g_n(s) &:= \sqrt{\varepsilon_n}Y(U_n(s)). \end{aligned}$$

Next, we set

$$\psi_n(t) := \mathbb{E}\left[\sup_{0\leq s\leq t} \mathbf{e}(s\wedge\tau_n, Z_n(s\wedge\tau_n))\right], \quad t\in[0,T].$$

We intent to follow the procedure of Proposition 5.14. By the Cauchy-Schwartz inequality, for $\tau\in[0,T]$, we have

$$\begin{aligned} \sup_{0\leq t\leq\tau} \int_0^{t\wedge\tau_n} \mathbb{V}(r, Z_n(r))\,dr &\leq 2\int_0^{\tau\wedge\tau_n} \mathbf{e}(r, Z_n(r))\,dr\\ &\quad + \frac{1}{2}\int_0^{\tau\wedge\tau_n} \left(\|f_n(r)\|^2_{H^1(B_{T-r})} + \|g_n(r)\cdot\|^2_{\mathscr{L}_2(H_\mu, H^1(B_{T-r}))}\right)\,dr. \end{aligned}$$

Since the g_n here is same as in Proposition 5.14, the computation of (5.33)-(5.38) fits here too and we have

$$\begin{aligned} \mathbb{E}\left[\sup_{0\leq s\leq t\wedge\tau_n} \mathbf{e}(s, Z_n(s))\right] &\lesssim_T \mathbb{E}\left[\int_0^{t\wedge\tau_n} \mathbf{e}(r, Z_n(r))\,dr\right] + \mathbb{E}\left[\int_0^{t\wedge\tau_n} \|f_n(r)\|^2_{H^1(B_{T-r})}\,dr\right]\\ &\quad + \varepsilon_n(1+N^2). \end{aligned} \tag{5.45}$$

Invoking Lemmata 3.2 and 5.4 implies

$$\begin{aligned} \|f_n(r)\|^2_{H^1(B_{T-r})} &\lesssim \|A_{U_n(r)}(\partial_x U_n(r), \partial_x U_n(r))\|^2_{H^1(B_{T-r})} + \|A_{U_n(r)}(V_n(r), V_n(r))\|^2_{H^1(B_{T-r})}\\ &\quad + \|Y(U_n(r))\dot{h}_n(r)\|^2_{H^1(B_{T-r})}\\ &\lesssim_T \left(1+\|U_n(r)\|^2_{H^1(B_{T-r})}\right)\Big[1+\|\partial_x U_n(r)\|^2_{H^1(B_{T-r})}\\ &\quad +\|V_n(r)\|^2_{H^1(B_{T-r})} + \|\dot{h}_n(r)\|^2_{H_\mu}\Big]\\ &\lesssim \left(1+\|Z_n(r)\|^2_{\mathcal{H}_{T-r}}\right)\left[1+\|Z_n(r)\|^2_{\mathcal{H}_{T-r}} + \|\dot{h}_n(r)\|^2_{H_\mu}\right]. \end{aligned}$$

So from (5.45) and the definition (5.28) we get

$$\begin{aligned}\mathbb{E}\left[\sup_{0\le s\le t\wedge\tau_n}\mathbf{e}(s,Z_n(s))\right] &\lesssim_T N^2\mathbb{E}\left[t\wedge\tau_n\right]+\varepsilon_n(1+N^2)\\ &\quad+(1+N^2)\mathbb{E}\left[\int_0^{t\wedge\tau_n}\left(1+N^2+\dot{h}_n(r)\|^2_{H_\mu}\right)\,dr\right]\\ &\lesssim_T N^2T+(1+N^2)T+M+\varepsilon_n(1+N^2).\end{aligned}$$

Since $\lim_{n\to\infty}\varepsilon_n=0$, taking $\limsup_{n\to\infty}$ on both the sides we get the required bound, and hence the Lemma 5.15. □

Lemma 5.16. *Given $T>0$, the sequence of $\mathcal{X}_T$-valued process $\{\mathcal{Z}_n\}_{n\in\mathbb{N}}$ converges in probability to* 0.

Proof of Lemma 5.16. It is sufficient to show that, for every $R,\delta,\alpha>0$ there exists n_0 such that

$$\mathbb{P}\left[\omega\in\Omega:\sup_{t\in[0,T]}\|\mathcal{Z}_n(t,\omega)\|_{\mathcal{H}_R}>\delta\right]<\alpha\quad\text{ for all }\quad n\ge n_0.$$

As before it is sufficient to prove above for $R=T/2$ which will be a particular case if we show the existence of n_0 such that

$$\mathbb{P}\left[\omega\in\Omega:\sup_{t\in[0,T/2]}\mathbf{e}(t,\mathcal{Z}_n(t))>\delta\right]<\alpha\quad\text{ for all }\quad n\ge n_0.$$

Let us fix $\delta,\alpha>0$. Choose $N>\|(u_0,v_0)\|_{\mathcal{H}_T}$ such that, based on Lemma 5.15,

$$\frac{1}{N}\sup_{n\in\mathbb{N}}\mathbb{E}\left[\sup_{t\in[0,T/2]}\mathbf{e}(t\wedge\tau_n,Z_n(t\wedge\tau_n))\right]<\frac{\alpha}{2},\tag{5.46}$$

and $n_0\in\mathbb{N}$, due to Proposition 5.14,

$$\mathbb{E}\left[\sup_{t\in[0,T/2]}\mathbf{e}(t\wedge\tau_n,\mathcal{Z}_n(t\wedge\tau_n))\right]<\frac{\delta\alpha}{2}\quad\text{ for all }\quad n\ge n_0.\tag{5.47}$$

Then the Markov inequality followed by using of (5.46) ad (5.47), for $n\ge n_0$, gives

$$\begin{aligned}&\mathbb{P}\left[\sup_{t\in[0,T/2]}\mathbf{e}(t,\mathcal{Z}_n(t))\ge\delta\right]\\ &\le\mathbb{P}\left[\sup_{t\in[0,T/2]}\mathbf{e}(t,\mathcal{Z}_n(t))\ge\delta\text{ and }\tau_n=T\right]\mathbb{P}\left[\sup_{t\in[0,T/2]}\mathbf{e}(t,Z_n(t))\ge N\right]\\ &\le\frac{1}{\delta}\mathbb{E}\left[\sup_{t\in[0,T/2]}\mathbf{e}(t,\mathcal{Z}_n(t,\omega))\right]+\frac{1}{N}\mathbb{E}\left[\sup_{t\in[0,T/2]}\mathbf{e}(t,Z_n(t,\omega))\right]\\ &<\alpha.\end{aligned}$$

Hence the Lemma 5.16. □

Now we come back to the proof of Statement 2. Recall that S_M is a separable metric space. Since the sequence $\{\mathscr{L}(h_n)\}_{n\in\mathbb{N}}$ of laws on S_M converges weakly to the law $\mathscr{L}(h)$ by assumption, the Skorokhod representation theorem, see for example [38, Theorem 3.30], yields the existsence of a probability space $(\tilde{\Omega}, \tilde{\mathfrak{F}}, \tilde{\mathbb{P}})$, and processes $(\tilde{h}_n, \tilde{h}, \tilde{W})$ defined on this space, such that the joint distribution of $(\tilde{h}_n, \tilde{W})$ is same as that of (h_n, W), the distribution of $\tilde{h}$ coincide with that of h, and $\tilde{h}_n \xrightarrow[n\to\infty]{} \tilde{h}$, $\tilde{\mathbb{P}}$-a.s. pointwise on $\tilde{\Omega}$, in the weak topology of S_M. By Lemma 5.11 this implies that

$$J^0 \circ \tilde{h}_n \to J^0 \circ \tilde{h} \quad \text{in} \quad \mathcal{X}_T \ \ \mathbb{P}\text{-a.s. pointwise on } \tilde{\Omega}.$$

Next, we claim that

$$\mathscr{L}(z_n) = \mathscr{L}(\tilde{z}_n), \qquad \text{for all } n$$

where

$$z_n := J^0 \circ h : \Omega \to \mathcal{X}_T \qquad \text{and} \qquad \tilde{z}_n := J^0 \circ \tilde{h}_n : \tilde{\Omega} \to \mathcal{X}_T.$$

To avoid complexity, we will write $J^0(h)$ for $J^0 \circ h$. Let B be an arbitrary Borel subset of $\mathcal{X}_T$. Thus, since from Lemma 5.11 $J^0 : S_M \to \mathcal{X}_T$ is Borel, $(J^0)^{-1}(B)$ is Borel in S_M. So we have

$$\mathscr{L}(z_n)(B) = \mathbb{P}\left[J^0(h_n)(\omega) \in B\right] = \mathbb{P}\left[h_n^{-1}\left((J^0)^{-1}(B)\right)\right] = \mathscr{L}(h_n)\left((J^0)^{-1}(B)\right).$$

But, since $\mathscr{L}(h_n) = \mathscr{L}(\tilde{h}_n)$ on $\mathcal{X}_T$, this implies $\mathscr{L}(z_n)(B) = \mathscr{L}(\tilde{z}_n)(B)$. Hence the claim and by a similar argument we also have $\mathscr{L}(z_h) = \mathscr{L}(z_{\tilde{h}})$.

Before moving forward note that from Proposition 5.14, the sequence of $\mathcal{X}_T$-valued random variables, defined from Ω, $J^{\varepsilon_n}(h_n) - J^0(h_n)$ converges in probability to 0. Consequently, because $\mathscr{L}(h_n) = \mathscr{L}(\tilde{h}_n)$ and $J^{\varepsilon_n} - J^0$ is measurable, we infer that $J^{\varepsilon_n}(\tilde{h}_n) - J^0(\tilde{h}_n) \xrightarrow{\tilde{\mathbb{P}}} 0$ as $n \to \infty$. Therefore, we can choose a subsequence $\{J^{\varepsilon_n}(\tilde{h}_n) - J^0(\tilde{h}_n)\}_{n\in\mathbb{N}}$, indexed again by n, of $\mathcal{X}_T$-valued random variables converges to 0, $\tilde{\mathbb{P}}$-almost surely.

Now we can conclude the proof of Statement 2. Indeed, for any globally Lipschitz and bounded function $\psi : \mathcal{X}_T \to \mathbb{R}$, see [29, Theorem 11.3.3], we have

$$\begin{aligned}
&\left|\int_{\mathcal{X}_T} \psi(x)\, d\mathscr{L}(J^{\varepsilon_n}(h_n)) - \int_{\mathcal{X}_T} \psi(x)\, d\mathscr{L}(J^0(h))\right| \\
&= \left|\int_{\mathcal{X}_T} \psi(x)\, d\mathscr{L}(J^{\varepsilon_n}(\tilde{h}_n)) - \int_{\mathcal{X}_T} \psi(x)\, d\mathscr{L}(J^0(\tilde{h}))\right| \\
&= \left|\int_{\tilde{\Omega}} \psi\left(J^{\varepsilon_n}(\tilde{h}_n)\right) d\tilde{\mathbb{P}} - \int_{\tilde{\Omega}} \psi\left(J^0(\tilde{h})\right) d\tilde{\mathbb{P}}\right| \\
&\le \left|\int_{\tilde{\Omega}} \left\{\psi\left(J^{\varepsilon_n}(\tilde{h}_n)\right) - \psi\left(J^0(\tilde{h}_n)\right)\right\} d\tilde{\mathbb{P}}\right| \\
&\quad + \left|\int_{\tilde{\Omega}} \psi\left(J^0(\tilde{h}_n)\right) d\tilde{\mathbb{P}} - \int_{\tilde{\Omega}} \psi\left(J^0(\tilde{h})\right) d\tilde{\mathbb{P}}\right|.
\end{aligned}$$

Since $J^0(\tilde{h}_n) \xrightarrow[n\to\infty]{} J^0(\tilde{h})$, $\mathbb{P}$-a.s. and ψ is bounded and continuous, we deduce that the 2nd term in right hand side above converges to 0 as $n \to \infty$. Moreover we claim that the 1st term also goes to 0. Indeed, it follows from the dominated convergence theorem because the term is bounded by

$$L_\psi \int_{\tilde{\Omega}} |J^{\varepsilon_n}(\tilde{h}_n) - J^0(\tilde{h}_n)| \, d\tilde{\mathbb{P}},$$

where L_ψ is Lipschitz constant of ψ, and the sequence $\{J^{\varepsilon_n}(\tilde{h}_n) - J^0(\tilde{h}_n)\}_{n\in\mathbb{N}}$ converges to 0, $\tilde{\mathbb{P}}$-a.s.

Therefore, Statement 2 holds true and we complete the proof of Theorem 5.6.

Appendix A. Existence and uniqueness result

In this part we recall the existence and uniqueness result for global solution, in strong sense, to problem

$$\begin{cases} \partial_{tt} u = \partial_{xx} u + A_u(\partial_t u, \partial_t u) - A_u(\partial_x u, \partial_x u) + Y_u\left(\partial_t u, \partial_x u\right) \dot{W}, \\ u(0,\cdot) = u_0, \partial_t u(t,\cdot)_{|t=0} = v_0. \end{cases} \tag{A.1}$$

In this framework, [13, Theorem 11.1] gives the following.

Theorem A.1. *Fix $T > 0$ and $R > T$. For every $\mathfrak{F}_0$-measurable random variable u_0, v_0 with values in $H^2_{loc}(\mathbb{R}, M)$ and $H^1_{loc}(\mathbb{R}, TM)$, there exists a process $u : [0,T) \times \mathbb{R} \times \Omega \to M$, which will be denoted by $u = \{u(t), t < T\}$, such that the following hold:*

1. *$u(t,x,\cdot) : \Omega \to M$ is $\mathscr{F}_t$-measurable for every $t < T$ and $x \in \mathbb{R}$,*
2. *$[0,T) \ni t \mapsto u(t,\cdot,\omega) \in H^2((-R,R);\mathbb{R}^n)$ is continuous for almost every $\omega \in \Omega$,*
3. *$[0,T) \ni t \mapsto u(t,\cdot,\omega) \in H^1((-R,R);\mathbb{R}^n)$ is continuously differentiable for almost every $\omega \in \Omega$,*
4. *$u(t,x,\omega) \in M$, for every $t < T, x \in \mathbb{R}$, $\mathbb{P}$-almost surely,*
5. *$u(0,x,\omega) = u_0(x,\omega)$ and $\partial_t u(0,x,\omega) = v_0(x,\omega)$ holds, for every $x \in \mathbb{R}$, $\mathbb{P}$-almost surely,*
6. *for every $t \geq 0$ and $R > 0$,*

$$\begin{aligned} \partial_t u(t) = v_0 &+ \int_0^t \left[\partial_{xx} u(s) - A_{u(s)}(\partial_x u(s), \partial_x u(s)) + A_{u(s)}(\partial_t u(s), \partial_t u(s))\right] ds \\ &+ \int_0^t Y_{u(s)}(\partial_t u(s), \partial_x u(s)) \, dW(s), \end{aligned}$$

holds in $L^2((-R,R);\mathbb{R}^n)$, $\mathbb{P}$-almost surely.

Moreover, if there exists another process $U = \{U(t); t \geq 0\}$ satisfy the above properties, then $U(t,x,\omega) = u(t,x,\omega)$ for every $|x| < R - t$ and $t \in [0,T)$, $\mathbb{P}$-almost surely.

Appendix B. Energy inequality for stochastic wave equation

Recall the following slightly modified version of [13, Proposition 6.1] for a one (spatial) dimensional linear inhomogeneous stochastic wave equation. For $l \in \mathbb{N}$, we use the symbol $D^l h$ to denote the $\mathbb{R}^{n\times 1}$-vector $\left(\frac{d^l h^1}{dx^l}, \frac{d^l h^2}{dx^l}, \cdots, \frac{d^l h^n}{dx^l}\right)$.

Proposition B.1. *Assume that $T > 0$ and $k \in \mathbb{N}$. Let W be a cylindrical Wiener process on a Hilbert space K. Let f and g be progressively measurable processes with values in $H^k_{loc}(\mathbb{R};\mathbb{R}^n)$ and $\mathscr{L}_2(K, H^k_{loc}(\mathbb{R};\mathbb{R}^n))$ respectively such that, for every $R > 0$,*

$$\int_0^T \left\{ \|f(s)\|_{H^k((-R,R);\mathbb{R}^n)} + \|g(s)\|^2_{\mathscr{L}_2(K,H^k((-R,R);\mathbb{R}^n))} \right\} ds < \infty,$$

$\mathbb{P}$-almost surely. Let z_0 be an $\mathcal{F}_0$-measurable random variable with values in

$$\mathcal{H}^k_{loc} := H^{k+1}_{loc}(\mathbb{R};\mathbb{R}^n) \times H^k_{loc}(\mathbb{R};\mathbb{R}^n).$$

Assume that an $\mathcal{H}^k_{loc}$-valued process $z = z(t)$, $t \in [0,T]$, satisfies

$$z(t) = S_t z_0 + \int_0^t S_{t-s} \begin{pmatrix} 0 \\ f(s) \end{pmatrix} ds + \int_0^t S_{t-s} \begin{pmatrix} 0 \\ g(s) \end{pmatrix} dW(s), \qquad 0 \le t \le T.$$

Given $x \in \mathbb{R}$, we define the energy function $\mathbf{e} : [0,T] \times \mathcal{H}^k_{loc} \to \mathbb{R}^+$ by, for $z = (u,v) \in \mathcal{H}^k_{loc}$,

$$\mathbf{e}(t,z) = \frac{1}{2} \left\{ \|u\|^2_{L^2(B(x,T-t))} + \sum_{l=0}^k \left[\|D^{l+1}u\|^2_{L^2(B(x,T-t))} + \|D^l v\|^2_{L^2(B(x,T-t))} \right] \right\}.$$

Assume that $L : [0,\infty) \to \mathbb{R}$ is a non-decreasing $\mathcal{C}^2$-smooth function and define the second energy function $E : [0,T] \times \mathcal{H}^k_{loc} \to \mathbb{R}$, by

$$\mathbf{E}(t,z) = L(\mathbf{e}(t,z)), \ z = (u,v) \in \mathcal{H}^k_{loc}.$$

Let $\{e_j\}$ be an orthonormal basis of K. We define a function $V : [0,T] \times \mathcal{H}^k_{loc} \to \mathbb{R}$, by

$$\begin{aligned} V(t,z) &= L'(\mathbf{e}(t,z)) \left[\langle u, v \rangle_{L^2(B(x,T-t))} + \sum_{l=0}^k \langle D^l v, D^l f(t) \rangle_{L^2(B(x,T-t))} \right] + \\ &+ \frac{1}{2} L'(\mathbf{e}(t,z)) \sum_j \sum_{l=0}^k |D^l[g(t)e_j]|^2_{L^2(B(x,T-t))} + \\ &+ \frac{1}{2} L''(\mathbf{e}(t,z)) \sum_j \left[\sum_{l=0}^k \langle D^l v, D^l[g(t)e_j] \rangle_{L^2(B(x,T-t))} \right]^2, \ (t,z) \in [0,T] \times \mathcal{H}^k_{loc}. \end{aligned}$$

Then $\mathbf{E}$ is continuous on $[0,T] \times \mathcal{H}^k_{loc}$, and for every $0 \le t \le T$,

$$\begin{aligned} \mathbf{E}(t, z(t)) &\le \mathbf{E}(0, z_0) + \int_0^t V(r, z(r)\, dr \\ &+ \sum_{l=0}^k \int_0^t L'(\mathbf{e}(r, z(r))) \langle D^l v(r), D^l[g(r)\, dW(r)] \rangle_{L^2(B(x,T-r))}, \quad \mathbb{P}\text{-a.s.}. \end{aligned}$$

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