Problem Set 2

Ask for hints if you'd like some!

Q1 Show that a linear operator between two normed spaces is continuous if and only if it is continuous at some point.

Q2 Show that $\mathfrak{B}(X, Y)$ is a subspace of $\mathfrak{L}(X, Y)$.

Q3 Verify that the operator norm is indeed a norm and check that the five (or six?) given descriptions are all equivalent.

Q4 Let $X = (\ C[a,b],\ ||\ \cdot\ ||\ )$, where $C[a,b]$ is the real vector space of all continuous functions defined on the interval $[a,b]$, and

$$||\ f\ || = \max_{t \in [a,b]} |f(t)|$$

is the supremum norm. Show that

$$I(f) = \int_a^b f(t)dt$$

is a bounded linear functional and determine $||\ I\ ||$.

Q5 Given a linear operator $T\colon\ X \to Y$, recall that the adjoint $T^*\colon\ Y^* \to X^*$ satisfies:

$$\langle x, T^*F \rangle = \langle Tx, F \rangle$$

for all $x \in X$ and all $F \in Y^*$.

(a) Show that T^* is the unique linear operator $Y^* \to X^*$ satisfying this equality.
(b) Show that $||T^*|| \leq ||T||$.
(c) Show that $||T^*|| = ||T||$.

Q6 For $T \in \mathfrak{B}(X,Y)$ define

$$\gamma(T) = \inf\{\frac{||Tx||}{||x||} \mid x \neq 0\}.$$

Show that T is invertible with $T^{-1} \in \mathfrak{B}(Y,X)$ if and only if T is surjective and $\gamma(T) > 0$.

Q7 Let $p \geq 1$ and let R and L be the right and left shift operators on l_p:

$$R(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, x_3, \ldots),$$
$$L(x_1, x_2, x_3, \ldots) = (x_2, x_3, \ldots).$$

Show that

(a) R and L are linear and bounded and find $||R||$ and $||L||$;

(b) $LR = I$ but $RL \neq I$;

(c) $||L^n x|| \to 0$ for each $x \in l_p$, but $||L^n||$ does not converge to zero.

Q8 Let $X = C^1[0,1]$ be the vector space of all continuous differentiable functions and $Y = C[0,1]$. Let $|| \cdot ||$ be the supremum norm on both X and Y, and define on X the norm

$$||f||_1 = ||f|| + ||f'||,$$

where $f'(t)$ is the derivative with respect to t. Let D be the differential operator $D(f) = f'$. Show that

(a) $D\colon (X, || \cdot ||_1) \to (Y, || \cdot ||)$ is a bounded linear operator with $||D|| = 1$, and

(b) $D\colon (X, || \cdot ||) \to (Y, || \cdot ||)$ is an unbounded linear operator.