Assignment 1

Your solutions should be submitted by the beginning of the lecture on
Thursday, 12 August 2010.

The first two questions use the following definition. The *Borel σ–algebra* of $\mathbb{R}^n$ is the smallest σ–algebra of $\mathbb{R}^n$ containing the open subsets of $\mathbb{R}^n$. It is denoted $\mathcal{B}(\mathbb{R}^n)$. Recall that if for some set S, $\mathcal{E} \subseteq \mathcal{F} \subseteq \mathcal{P}(S)$, then $\Sigma(\mathcal{E}) \subseteq \Sigma(\mathcal{F})$.

Q1 Show that $\mathcal{B}(\mathbb{R})$ is generated by each of the following sets:

(a) the open intervals: $\mathcal{E}_1 = \{(a,b) \mid a < b\}$,

(b) the closed intervals: $\mathcal{E}_2 = \{[a,b] \mid a < b\}$,

(c) the half-open intervals: $\mathcal{E}_3 = \{(a,b] \mid a < b\}$ or $\mathcal{E}_4 = \{[a,b) \mid a < b\}$,

(d) the open rays: $\mathcal{E}_5 = \{(a,\infty) \mid a \in \mathbb{R}\}$ or $\mathcal{E}_6 = \{(-\infty,a) \mid a \in \mathbb{R}\}$,

(e) the closed rays: $\mathcal{E}_7 = \{[a,\infty) \mid a \in \mathbb{R}\}$ or $\mathcal{E}_8 = \{(-\infty,a] \mid a \in \mathbb{R}\}$.

Q2 Let $p_k \colon \mathbb{R}^n \to \mathbb{R}$ be the k^{th} coordinate projection. Then define $\otimes_{k=1}^n \mathcal{B}(\mathbb{R})$ to be the smallest σ–algebra on $\mathbb{R}^n$ containing the set:

$$\{p_k^{-1}(E_k) \mid E_k \in \mathcal{B}(\mathbb{R}), 1 \leq k \leq n\}.$$

Show that

$$\mathcal{B}(\mathbb{R}^n) = \otimes_{k=1}^n \mathcal{B}(\mathbb{R}).$$

Q3 Let $S = \{1,2,3,4\}$, and

$$\mathcal{E} = \{\{1,2\},\{2,4\},\{1,3\},\{3,4\}\}.$$

Find two distinct measures on $\Sigma(\mathcal{E})$, which agree on $\mathcal{E}$.

Q4 Give a complete proof of the "Completion of measure" theorem 1.15.