# Assignment 2

Your solutions should be submitted by the beginning of the lecture on Monday, 6 September 2010.

Q1 Consider $(\mathbb{R}, \mathcal{B}(\mathbb{R}), m)$, where $m$ is Lebesgue measure, and denote $(\mathbb{R}, \mathcal{L}, m)$ the completion. Show that for each $A \in \mathcal{L}$, we have

$$\begin{aligned} m(A) &= \inf\{m(U) \mid A \subseteq U \text{ and } U \text{ is open}\} \\ &= \sup\{m(K) \mid K \subseteq A \text{ and } K \text{ is compact}\} \end{aligned}$$

Q2 Let $A \subset \mathbb{R}$ be a Lebesgue measurable set of positive measure. Prove the following:

(a) $A$ contains a non-measurable set.

(b) For every $r < 1$, there is an open interval $I$ such that $m(A \cap I) > rm(I)$.

(c) The set $\{x - y \mid x, y \in A\}$ contains an interval centred at 0.

Q3 Let $(X, \Sigma)$ be a measurable space and consider $\mathbb{R}^*$ with the Borel $\sigma$–algebra.

(a) Show that $f\colon X \to \mathbb{R}^*$ is measurable if and only if the restriction of $f$ to $f^{-1}(\mathbb{R})$ is measurable and both $f^{-1}(\{\infty\})$ and $f^{-1}(\{-\infty\})$ are in $\Sigma$.

(b) Show that if $f, g\colon X \to \mathbb{R}^*$ are measurable, then $fg$ is also measurable, where we use the convention $0 \cdot (\pm\infty) = 0$.

(c) Show that if $f, g\colon X \to \mathbb{R}^*$ are measurable, then the function $h$ defined as follows is also measurable. Fix $a \in \mathbb{R}^*$ and let

$$h(x) = \begin{cases} a & \text{if } (f(x), g(x)) = \pm(\infty, -\infty) \\ f(x) + g(x) & \text{otherwise} \end{cases}$$

Q4 Use the Monotone Convergence Theorem from the lecture on Monday, 30 August, to prove the following version of Fatou's Lemma:

If $(f_n)_n$ is a sequence of measurable, non-negative functions $(X, \Sigma) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, then

$$\int (\liminf f_n) \leq \liminf \int f_n.$$