Assignment 5

"Isometries"

Due Thursday, 13 October, at 16:00 in the assignment box for MATH3405. The box is located on **Level 4** in the Mathematics (Priestley) building (67). It is number 035 in the **brown lot of boxes** (there are two lots, to find ours, turn right as you come from the stairs).

Please use a cover sheet!

Clearly state your assumptions and conclusions, and justify all steps in your work. Marks will be deducted for sloppy or incomplete working.

Q1 Show that a diffeomorphism $F\colon S_1 \to S_2$ of surfaces is an isometry if and only if it preserves speeds of curves (i.e. if $\alpha\colon I \to S_1$ is any regular curve on S_1, then $||\alpha'(t)|| = ||(F \circ \alpha)'(t)||$ for all $t \in I$).

Q2 You know that an isometry preserves Gauß curvature. This exercise shows that a Gauß curvature preserving map need not be an isometry.

Let $U = (0, \infty) \times (0, 2\pi)$, and define $\Phi, \Psi\colon U \to \mathbb{R}^3$ by:

$$\Phi(u, v) = (u \cos v, u \sin v, \log u)$$

and

$$\Psi(u, v) = (u \cos v, u \sin v, v).$$

You may assume that $S_1 = \Phi(U)$ and $S_2 = \Psi(U)$ are regular surfaces, and that the map $\Psi \circ \Phi^{-1}\colon S_1 \to S_2$ is a diffeomorphism.

(a) Show that $\kappa_1(\Phi(u, v)) = \kappa_2(\Psi(u, v))$ for all $(u, v) \in U$, where $\kappa_k\colon S_k \to \mathbb{R}$ is the Gauß curvature.

(b) Show that $\Psi \circ \Phi^{-1}$ is not an isometry.

Q3 Suppose S_1 and S_2 are regular surfaces, and that S_1 has a chart $\Phi_1\colon U_1 \to S_1$ with $S_1 = \Phi_1(U_1)$. Suppose $F\colon S_1 \to S_2$ is a differentiable map that can be written in the form

$$F(\Phi_1(u, v)) = \Phi_2(f(u), g(v))$$

for some chart $\Phi_2\colon U_2 \to S_2$ and functions f, g with $U \ni (u, v) \to (f(u), g(v)) \in U_2$.

(a) Describe the effect of F on the parameter curves of Φ_1, i.e. the images of the curves $u =$constant or $v =$constant.

(b) Show that F is an isometry onto its image, $S_1 \to F(S_1)$, if and only if the following three equations are satisfied:

$$E_1(u, v) = E_2(f(u), g(v))\Big(\frac{df}{du}(u)\Big)^2,$$

$$F_1(u, v) = F_2(f(u), g(v))\Big(\frac{df}{du}(u)\Big)\Big(\frac{dg}{dv}(v)\Big),$$

$$G_1(u, v) = G_2(f(u), g(v))\Big(\frac{dg}{dv}(v)\Big)^2.$$