Assignment 6

"Geodesics"

Due Thursday, 27 October, at 16:00 in the assignment box for MATH3405. The box is located on **Level 4** in the Mathematics (Priestley) building (67). It is number 035 in the **brown lot of boxes** (there are two lots, to find ours, turn right as you come from the stairs).

Please use a cover sheet!

Clearly state your assumptions and conclusions, and justify all steps in your work. Marks will be deducted for sloppy or incomplete working.

Fix $a, b \in \mathbb{R}$ satisfying $0 < b < a$, and let

$$\Psi(u, v) = \Big((a + b\cos u)\cos v, (a + b\cos u)\sin v, b\sin u\Big).$$

Let $T = \mathrm{Im}(\Psi)$. This is the torus obtained by revolving the circe of radius b centered at distance a from the origin around the z axis. You know a lot about it; for instance, $E = b^2$ and $F = 0$ are constant, and $G = (a + b\cos u)^2$ only depends on u.

Q1 (Some examples)

Using the differential equations for geodesics (Proposition 4.6), verify that the *meridians* $\Psi(\{v = v_0\})$, *inner equator* $\Psi(\{u = \pi\})$ and *outer equator* $\Psi(\{u = 0\})$ are geodesics. Similarly, determine whether any other *parallel* $\Psi(\{u = u_0\})$ is a geodesic.

Q2 (A useful constant)

Let S be a regular surface. Suppose $\Phi\colon U \to S$ is a chart with the properties that $F = 0$ everywhere and E and G only depend on u. Examples of this are surfaces of revolution, such as the torus.

Let $\alpha\colon \mathbb{R} \to \Phi(U) \subseteq S$ be a unit speed geodesic given by $\alpha(t) = \Phi(u(t), v(t))$. Show that:

(a) The function $c(t) = G(u(t))\, v'(t)$ is constant along α. Write $c = c(t)$.
(b) We have $c = \sqrt{G(u(t))}\sin\varphi(t)$, where $\varphi(t)$ the angle between Φ_u and α' at $\alpha(t)$.
(c) The image of α is contained in the region of S, where $G \geq c^2$.

Q3 (The geodesics on the torus)

(a) Show that if a geodesic on T is tangent to the top circle $\Psi(\{u = \frac{\pi}{2}\})$ at some point, then it remains on the "half facing outside" $(-\frac{\pi}{2} \leq u \leq \frac{\pi}{2}))$ and travels around T oscillating between the top circle and the bottom circle.
(b) Every geodesic that crosses the inner equator also crosses the outer equator, and (unless it is a meridian) spirals around the torus, crossing both equators infinitely often.
(c) Every geodesic on T except for the inner and outer equators crosses the outer equator.

Q3 (Geodesic curvature)

Let Δ be the triangle in $\mathbb{R}^2$ with vertices $(\pi, 0)$, (π, π) and $(\frac{\pi}{2}, \frac{\pi}{2})$ and edges the straight line segments connecting these vertices. Compute the integral of Gaussian curvature over $\Psi(\Delta)$, determine the cosine and sign of each turning angle, and show how to set up the integral for the boundary curvature of $\partial\Psi(\Delta)$ (where the side towards Δ is chosen).