Assignment 4

Due Thursday, 21 April, at 17:00 in the assignment box for MATH3402. The box is located on **Level 3** in the Mathematics (Priestley) building (67). It is in the **white lot of boxes** (there are two lots, to find ours, turn left as you come from the stairs).

Please use a cover sheet!

Clearly state your assumptions and conclusions, and justify all steps in your work.

Marks will be deducted for sloppy working.

You may assume that $(C(X,Y), d_\infty)$ is a complete metric space, where $(X, \mathcal{O})$ is a *compact* topological space, (Y, ρ) is a *complete* metric space, and

$$d_\infty(f,g) = \sup\{\rho(f(x), g(x)) \mid x \in X\}.$$

Q1 Let (X,d) and (Y,ρ) be metric spaces. A surjective function $f\colon X \to Y$ is an *isometry* if $\rho(f(x), f(y)) = d(x,y)$ for all $x, y \in X$.

(a) Show that an isometry is injective.

(b) Show that an isometry is a homeomorphism.

(c) Show that the set of all isometries, $\mathrm{Isom}(X,Y) \subseteq C(X,Y)$, is equicontinuous.

Q2 Let (X,d) be a compact metric space. Suppose $(g_n)_{n\in\mathbb{N}}$ and $(h_n)_{n\in\mathbb{N}}$ are sequences of isometries, $g_n, h_n\colon X \to X$, which converge in $(C(X,X), d_\infty)$ to $g\colon X \to X$ and $h\colon X \to X$ respectively. Show that the sequence $(g_n \circ h_n)_{n\in\mathbb{N}}$ converges to $g \circ h$.

Q3 Let (X,d) be a compact metric space.

(a) Let $(f_n)_{n\in\mathbb{N}}$ be any sequence of isometries $f_n\colon X \to X$. Show that a subsequence of $(f_n)_{n\in\mathbb{N}}$ converges in $C(X,X)$, and that its limit is an isometry.

(b) Let $(f_{n_k})_{k\in\mathbb{N}}$ be a convergent subsequence from part (a), and denote the limiting isometry f. Does the sequence $(f_{n_k}^{-1})_{n\in\mathbb{N}}$ converge to f^{-1}?