Problem Set 4

Q31 Let (X, d) be a metric space. Show that if $A \subseteq X$ is totally bounded, then $\overline{A}$ is also totally bounded.

Q32 Let $(X, \mathcal{O})$ be a second countable space. Show that every open cover of X has a countable subcover.

Q33 Suppose $(X, \mathcal{O}_X)$ is a separable space and $(Y, \mathcal{O}_Y)$ is a topological space. If there is a surjective, continuous function $f\colon X \to Y$, then $(Y, \mathcal{O}_Y)$ is also separable.

Q34 The separation properties $T_0, \ldots, T_4$ are topological properties.

Q35 Show that the open unit ball $B_1((0,0)) \subset \mathbb{R}^2$ is homeomorphic to the open square $(-1,1) \times (-1,1)$ with respect to the Euclidean topology.

Q36 Show that the open unit ball $B_1((0,0)) \subset \mathbb{R}^2$ is homeomorphic to the open upper half plane $(0,\infty) \times \mathbb{R}$ with respect to the Euclidean topology.

Q37 Suppose $(X, \mathcal{O}_X)$ is a topological space and $(Y, \mathcal{O}_Y)$ is a Hausdorff space. Let $f, g\colon X \to Y$ be continuous functions. Show that $\{x \in X \mid f(x) = g(x)\}$ is a closed subset of X.

Q38 Let $(X, \mathcal{O}_X)$ be a topological space and suppose that the subset A of X is connected (i.e. $(A, \mathcal{O}_A)$ is a connected topological space, where $\mathcal{O}_A$ is the subspace topology).
If $B \subseteq X$ satisfies $A \subseteq B \subseteq \overline{A}$, then B is connected.

Q39 Let $(X, \mathcal{O}_X)$ be a topological space and suppose that the subsets A and B of X are connected.
If $\overline{A} \cap B \neq \emptyset$, then $A \cup B$ is connected.

Q40 Which of the following sets $X \subset \mathbb{R}^2$ are connected with respect to the subspace topology from $(\mathbb{R}^2, \mathcal{O}_{\mathbb{E}})$?

(a) $X = \{(x,y) \mid xy = 1 \text{ and } x, y > 0\} \cup \{(x, 0) \mid x \in \mathbb{R}\}$
(b) Let $C_n = \{(x,y) \mid (x - \frac{1}{n})^2 + y^2 = \frac{1}{n^2}\}$ for each $n \in \mathbb{Z}$, and $X = \bigcup_{n \in \mathbb{Z}} C_n$
(c) $X = \big((\mathbb{Q} \times \mathbb{R}) \cup (\mathbb{R} \times \mathbb{Q})\big) \setminus (\mathbb{Q} \times \mathbb{Q})$

Q41 Is the set $\{(x,y) \mid x = 0, -1 \leq y \leq 1\} \cup \{(x, \sin\frac{\pi}{x}) \mid 0 < x \leq 1\}$ connected with respect to the subspace topology from $(\mathbb{R}^2, \mathcal{O}_{\mathbb{E}})$? Is it path connected?

Q42 Let $(X, \mathcal{O}_X)$ be a topological space. The point $p \in X$ is a *cut point* of X if $X \setminus \{p\}$ is disconnected. Show that the property of having a cut point is a topological property.

Q43 Let $a, b \in \mathbb{R}$ with $a < b$. Show that no two of the intervals (a,b), $(a,b]$ and $[a,b]$ are homeomorphic with respect to the subspace topology from $(\mathbb{R}, \mathcal{O}_{\mathbb{E}})$.

Q44 Let $S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ be the unit circle in $\mathbb{R}^2$, and suppose $f\colon S^1 \to \mathbb{R}$ is a continuous function (with respect to the Euclidean topologies). Show that there exists a pair of antipodal points with the same image, i.e. there exists $z \in S^1$ such that $f(z) = f(-z)$. (Hint: adapt a trick from the lectures.)