Problem Set 7

Q59 Which of the following subsets of $C(\mathbb{R}) = C(\mathbb{R}, \mathbb{R})$ are pointwise bounded? Which are equicontinuous?

(a) The set $\{f_n\}$, where $f_n(x) = x + \sin(nx)$.
(b) The set $\{g_n\}$, where $g_n(x) = n + \sin(x)$.
(c) The set $\{h_n\}$, where $h_n(x) = |x|^{1/n}$.
(d) The set $\{k_n\}$, where $k_n(x) = n \sin(x/n)$.

Q60 Let $\mathcal{F}$ be a set of differentiable functions $f\colon [a,b] \to \mathbb{R}$ with the property that there is a constant $M > 0$ and a point $x_0 \in [a,b]$, such that $|f(x_0)| \leq M$ and $|f'(x)| \leq M$ for all $x \in [a,b]$ and all $f \in \mathcal{F}$.
Show that $\overline{\mathcal{F}}$ is compact in the space $C[a,b]$. (Hint: Use the mean value theorem.)

Q61 Let $(X, \mathcal{O})$ be a topological space, (Y, d) be a metric space and $\mathrm{Fun}(X, Y)$ be the set of all continuous functions $X \to Y$. Suppose $\mathcal{F} \subseteq \mathrm{Fun}(X, Y)$ has the property that for all $f, g \in \mathcal{F}$, the set

$$\{d(f(x), g(x)) \mid x \in X\} \subseteq \mathbb{R}$$

is bounded with respect to the Euclidean metric on $\mathbb{R}$. Define $d_\infty\colon \mathcal{F} \times \mathcal{F} \to \mathbb{R}$ by

$$d_\infty(f, g) = \sup\{\rho(f(x), g(x)) \mid x \in X\}.$$

(a) The function d_∞ defines a metric on $\mathcal{F}$.
(b) If X is compact, then the set $\mathcal{F} = C(X, Y)$ has the above property, and hence $(C(X, Y), d_\infty)$ is a metric space.
(c) If X is compact and Y is complete, then $(C(X, Y), d_\infty)$ is a complete metric space.
Hint: This is a little lengthy. Pick an arbitrary Cauchy sequence (f_n) in $C(X, Y)$, and use the completeness of Y to construct a *candidate* limit function. Is this function continuous? Is it the limit of the sequence?