Assignment 2

"More hyperbolic geometry in dimension 2"

Due Tuesday, 24 January, at the start of the 9:00 lecture. Clearly state your assumptions and conclusions, and justify all steps in your work. Marks will be deducted for sloppy or incomplete working.

Q1 (Once-punctured torus)

Construct a complete hyperbolic structure on the once-punctured torus that is not isometric to the one given in class. (Note that the one given in class has a lot of symmetry!) I expect to see a fundamental domain, the Möbius transformations realising the edge pairings, the first few tiles in a tesselation of the hyperbolic plane, as well as a proof of completeness and an argument why the structures are not isometric.

Q2 (Busemann functions)

For this question, I write $\mathbb{H}^2$ for the upper half-space model and $\partial\mathbb{H}^2 = \mathbb{R} \cup \{\infty\}$, both considered as subsets of the Riemann sphere, $\mathbb{H}^2 \cup \partial\mathbb{H}^2 \subset \widehat{\mathbb{C}}$.

(a) Fix a point $P_0 \in \mathbb{H}^2$ and a point $\xi \in \partial\mathbb{H}^2$. Show that as $Q \in \mathbb{H}^2$ tends to ξ on the Riemann sphere, the limit

$$h(P) = \lim_{Q\to\xi} \Big(d(Q,P) - d(Q,P_0) \Big)$$

exists for every $P \in \mathbb{H}^2$.

(b) Suppose $\xi = \infty$, $P_0 = (x_0, y_0)$ and $P = (x, y)$. Show that $h(P) = \log \frac{y_0}{y}$.

(c) Show that the set of all $P \in \mathbb{H}^2$ with $h(P) = 0$ is exactly the horocircle centred at ξ and passing through P_0.

(d) Show that $h\colon \mathbb{H}^2 \to \mathbb{R}$ is surjective and that each of its level sets is a horocircle.

(Bonahon, Exercises 6.10, 6.11 and 6.12)

Q3 (Discontinuous actions)

(a) Suppose $G \leq \operatorname{Isom}\mathbb{H}^2$, and there are $P_0 \in \mathbb{H}^2$ and $\varepsilon > 0$ such that $g \cdot P_0 \in B(P_0, \varepsilon)$ for at most finitely many $g \in G$. Show that G acts discontinuously on $\mathbb{H}^2$.

(b) Suppose that the group $G \leq \operatorname{Isom}(X, d)$ acts by isometries on the complete metric space (X, d). Show that the orbit space with the quotient metric is complete.

(Bonahon, Exercises 7.5 and 7.6)