MATH 3963
NONLINEAR ODES WITH APPLICATIONS

R MARANGELL

Contents

1. More Examples of Bifurcations 1
1.1. A sub-critical Hopf bifurcation 1
1.2. A simplified Lotka-Volerra model 1
1.3. A subcritical pitchfork 3
1.4. An easy lemniscate example 4

1. More Examples of Bifurcations

1.1. **A sub-critical Hopf bifurcation.** In this example we are going to describe a system which exhibits a subcritical Hopf bifurcation. A bifurcation diagram is supplied in the accompanying pictures to the notes this week for completeness.

$$\begin{aligned} \dot{x} &= \mu x - y + xy^2 \\ \dot{y} &= x + \mu y + y^3. \end{aligned} \tag{1.1}$$

You can see for yourself that $(0,0)$ is a critical point and that at this equilibrium the linearisation is $\begin{pmatrix} \mu & -1 \\ 1 & -\mu \end{pmatrix}$ which has eigenvalues $\mu \pm i$. So as μ passes through 0 we have that the origin goes from a stable to an unstable focus. However, as you can see for yourself in figure 1, when $\mu < 0$ (the left) the far field behaviour is such that things are shooting off towards infinity. This suggests the existence of an unstable (i.e. repelling) limit cycle. On the right $\mu > 0$ and it can be seen that the origin is now an unstable focus, so the unstable limit cycle has evidently disappeared, and we therefore have a subcritical Hopf bifurcation. Again, as with the case of the van der Pol oscillator, we have not actually proven the existence of a limit cycle, but just inferred it quantitatively from the behaviour of the phase plane.

1.2. **A simplified Lotka-Volerra model.** In this next example (which might feel a bit ad hoc, but that's bifurcation analysis) we need to be kind of careful with what is going, on and bookkeeping is half the battle. We'll start with a simplified Lotka-Volterra model

$$\begin{aligned} \dot{x} &= ax - xy \\ \dot{y} &= -by + xy. \end{aligned} \tag{1.2}$$

Here a and b are the parameters, but this time, we're going to consider what happens as a and b change sign. Also, we are going to want to allow our dependent variables (x, y) to range all over the phase plane, and will not restrict ourselves to only the first quadrant. The

Date: September 6, 2017.

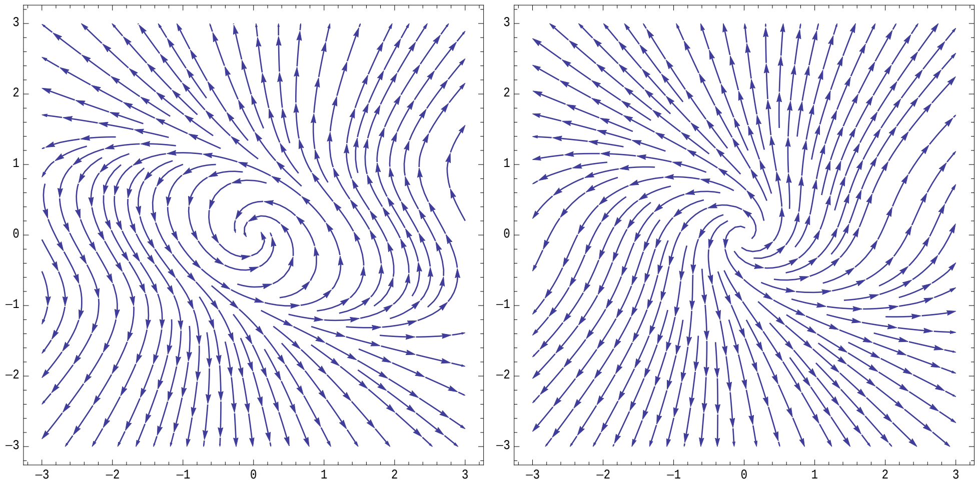


FIGURE 1. On the left $\mu < 0$ and you can (maybe) make out that there is an unstable limit cycle (kind of an oval around the origin where the arrows switch from pointing out to pointing in), while in the picture on the right $\mu > 0$ and the origin is an unstable focus and (evidently) the limit cycle has disappeared.

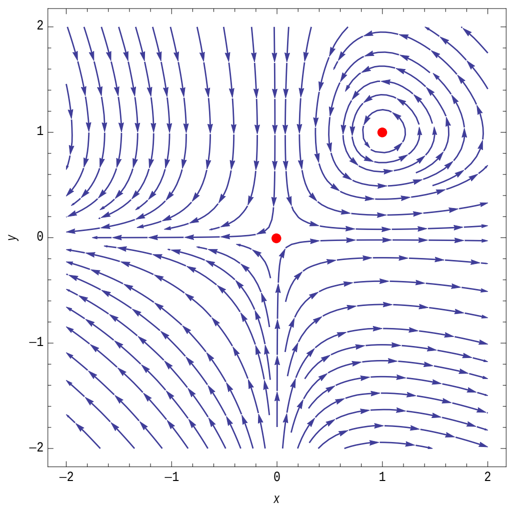
first step (as always) is to identify the critical points. In this case we have that $(0,0)$ and (b,a) are the critical points. Further we have that the linearisations are given by

$$DF(0,0) = \begin{pmatrix} a & 0 \\ 0 & -b \end{pmatrix} \quad \text{and,} \quad DF(b,a) = \begin{pmatrix} 0 & -b \\ a & 0 \end{pmatrix}.$$

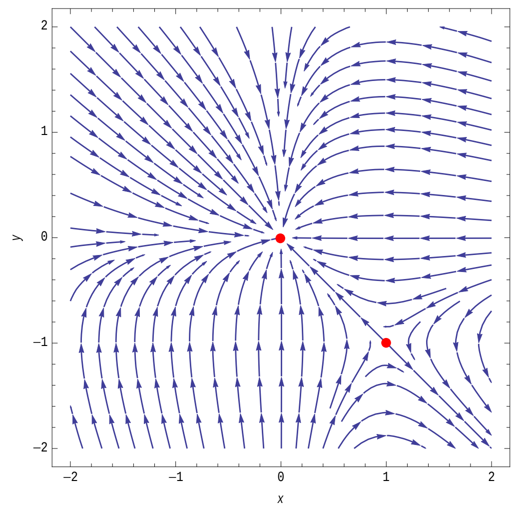
Now we are going to study what happens to these matrices as we move around in our parameter space.

(1) $a, b > 0$. This is the case we began with, the case that we originally studied when we were investigating the Lotka-Volterra. We have a saddle at $(0,0)$ and a centre at (b,a). See fig. 2(I).
(2) $a < 0$ and $b > 0$. Here we consider b to be fixed and look at what happens as we move a. As we decrease a, we can see that $(0,0)$ goes from a saddle to a stable node, and (b,a) goes from a centre to a saddle. See fig. 2(II)
(3) $a > 0, b < 0$. Here we consider a to be fixed as in (1) and look at what happens when we decrease b. You can see that the origin is an unstable node, while (a,b) becomes a saddle. See fig. 2(III)
(4) $a < 0, b < 0$. Finally we look at what happens if we fix a as in (2) and then decrease b. Here we can see that $(0,0)$ goes from a stable node to a saddle, while (b,a) goes from a saddle to back to a centre. See fig. 2(IV)

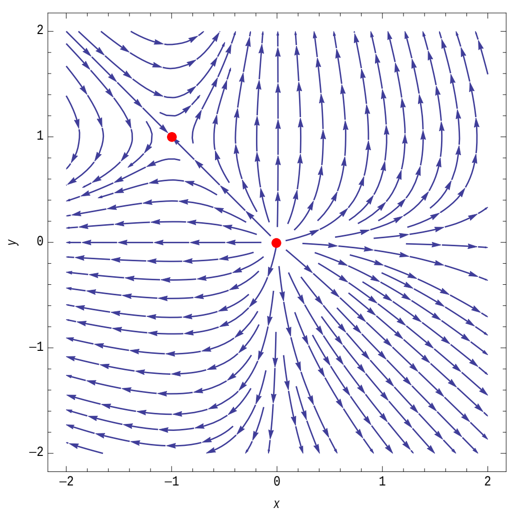
The bifurcation diagrams for what is going on are in fig. 3. In this instance, we just plot the x or y coordinate of the critical points, considered as a function of the 'moving' parameter b, or a. The bifurcation behaviour is a bit complicated and the methods used for dealing with it might seem a bit ad hoc, (as well as seemingly organisational in flavour), but that is what needs to be done in order to analyse more complicated bifurcations as they arise. While there are some systematic ways of organising parts of bifurcation theory, there is no one way universally organses such diverse behaviour.



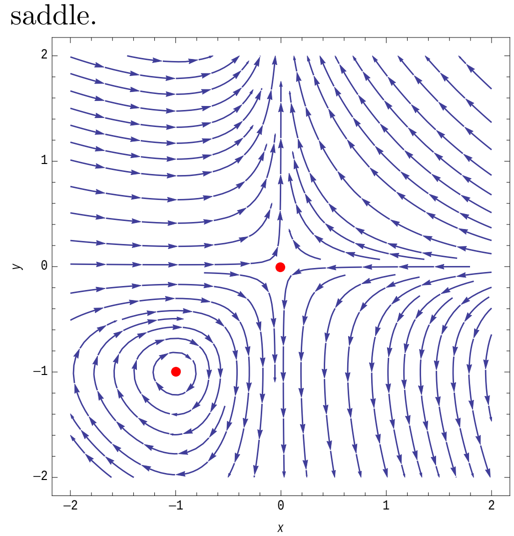
(I) $a = 1, b = 1$. We see that the origin is a saddle while (b, a) is a centre.



(II) $a = -1, b = 1$. We see that the origin is a stable node while (b, a) is a saddle.



(III) $a = 1, b = -1$. We see that the origin is an unstable node while (b, a) is a saddle.



(IV) $a = -1, b = -1$. We see that the origin is a saddle while (b, a) is a centre.

FIGURE 2. Plots of the phase plane for the salient values of the parameters a and b in the modified Lotka-Volterra equations.

1.3. **A subcritical pitchfork.** This example done entirely via looking at the graph of $(x, f(x; r))$ while varying the bifurcation parameter. Consider the ODE

$$\dot{x} = x + \frac{rx}{1 + x^2}. \tag{1.3}$$

You can set $f(x)$ equal to zero and see that equilibria occur when $x_* = 0$ or $x_* = \pm\sqrt{-(r+1)}$ so that when $r < -1$ there are three equilibria, and when $r > -1$ there is only 1. We look at the graph of $f(x; r)$ as we vary r and determine the stability and tease out the bifurcation diagrams. We see that we have for $r < -1$ a stable equilibrium at $x_* = 0$, and two unstable equilibria at $\pm\sqrt{-(r+1)}$, which then disappear, and the equilibrium at $x_* = 0$ switches

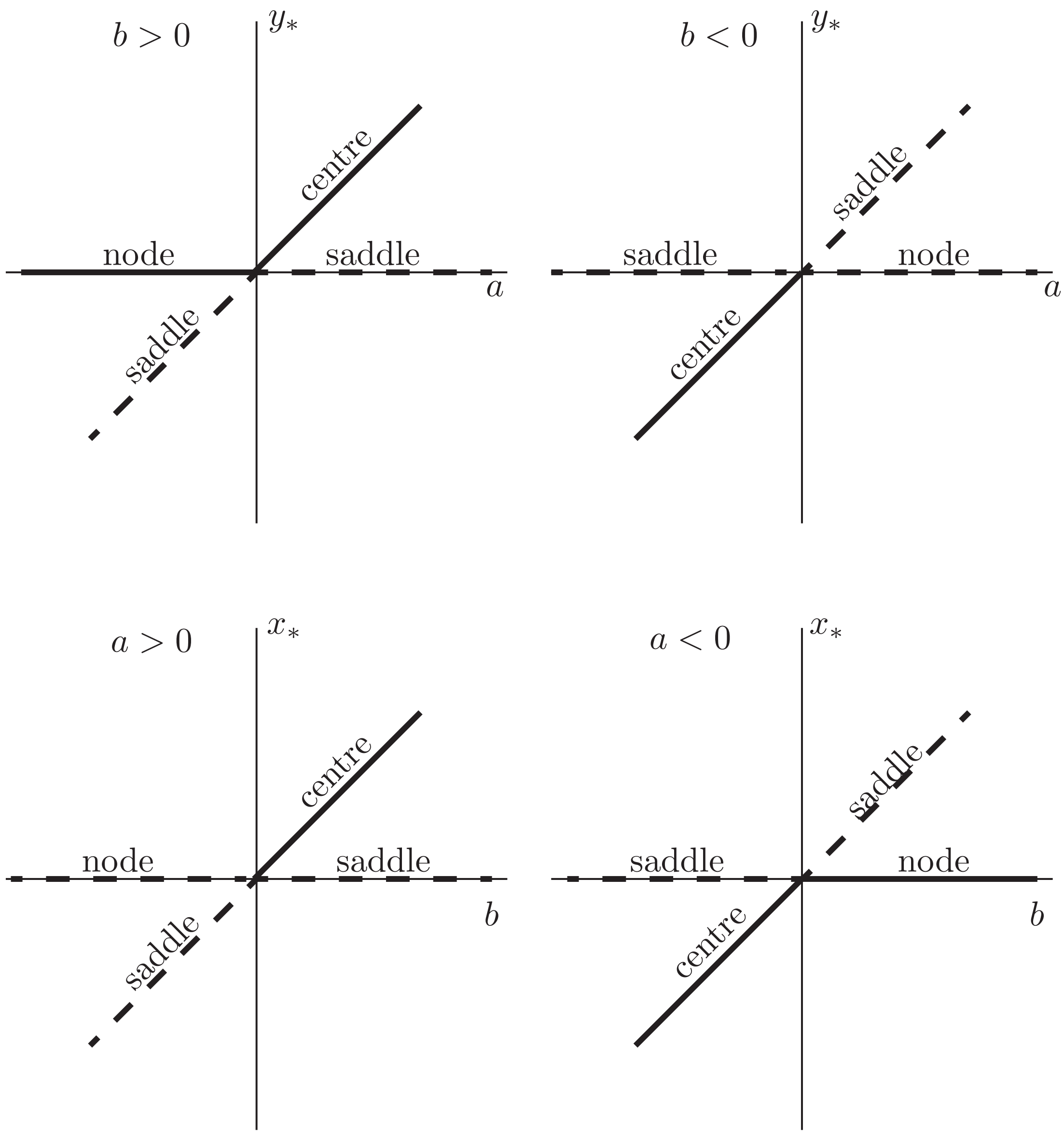


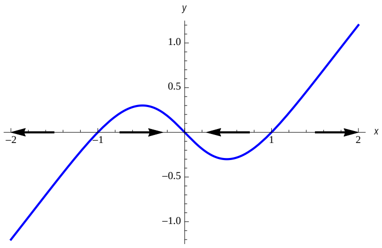
FIGURE 3. The bifurcation diagrams for the various parameter regimes of eq. (1.2)

stability. Thus we conclude that we have a *subcritical pitchfork bifurcation*. See figs. 4 and 5.

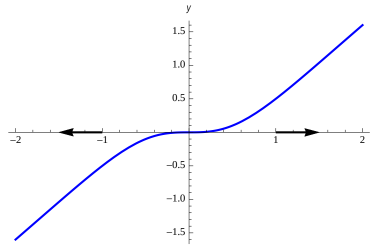
1.4. **An easy lemniscate example.** Name and classify all the types of bifurcations and sketch the bifurcation diagram for the following ODE:

$$\dot{x} = x^2 - \mu^2 + \frac{\mu^4}{2} \tag{1.4}$$

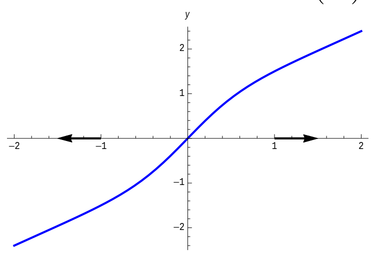
Setting the right hand side equal to 0 we get $x_*^2 - \mu^2 + \dfrac{\mu^4}{2} = 0$ which is the equation for a lemniscate in x_* and μ. The derivative of the right hand side is just $f'(x_*) = 2x_*$ which it is easy to see has the same sign as x_*. The critical points above the $\mu = 0$ axis are therefore unstable, while those below it are stable. There are three changes in stability. One



(I) $r = -2$



(II) $r = -1$



(III) $r = 1$

FIGURE 4. Plots of the right hand side of eq. (1.3) for various, distinguishing values of the bifurcation parameter.

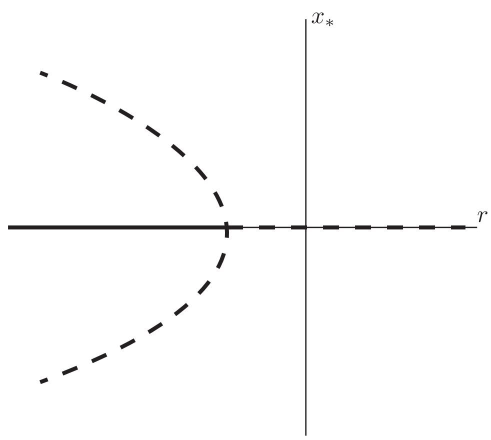


FIGURE 5. The bifurcation diagram for eq. (1.3)

at $(\mu, x_*) = (\pm\sqrt{2}, 0)$ and one at $(\mu, x_*) = (0, 0)$. The first two are saddle-node bifurcations while the third is a transcritical. The bifurcation diagram is in fig. 6

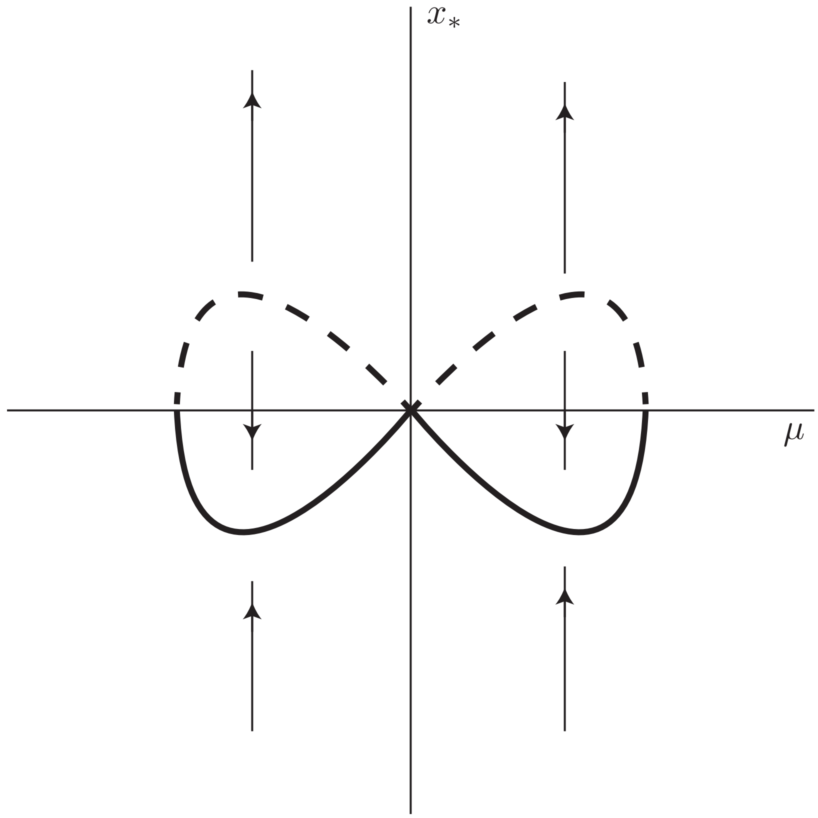


FIGURE 6. The bifurcation diagram for eq. (1.4)