MATH 3963
NONLINEAR ODES WITH APPLICATIONS

R MARANGELL

Contents

Date: September 6, 2017.

1. 2-D nonlinear autonomous systems

For this section (and for a large part of the remainder of the course) we are going to consider nonlinear autonomous 2×2 systems of equations. We begin with some examples.

Example 1.1. We start with the second order ODE $\ddot{x} + \sin(x) = 0$, the normalised pendulum equation. We'll write this as a system in the usual fashion

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\sin(x). \end{aligned}$$

Example 1.2 (Modified Logistic Model)**.** Another example is one that arises in cellular transport models (i.e. how cells move through a given material/substrate). We start with the 'well-known' logistic model

$$\dot{u} + u(1-u) = 0$$

and we modify it slightly

$$\ddot{u} + u(1-u) = 0.$$

The effect is that now we want to consider a diffusive process balanced by the logistic reaction term (the $u(1-u)$ bit) rather than a growth process balanced by it. We again write this as a 2-D system in the usual fashion

$$\begin{aligned} \dot{u} &= v \\ \dot{v} &= -u(1-u). \end{aligned}$$

Example 1.3 (Lotka-Volterra)**.** This next example (which is quite rich/informative) comes from mathematical biology. It is used in more complicated population models - so-called predator-prey systems. Sometimes it is called the Lotka-Volterra equation.

$$\begin{aligned} \dot{x} &= ax - cxy \\ \dot{y} &= -by + dxy \end{aligned}$$

Here x and y represent the population of two species the 'prey' (x) and the 'predators' (y). In this system a, b, c, d are all real positive parameters which will change according to what you are trying to model (though we shall see that they don't in fact affect the qualitative behaviour of the system that much).

Example 1.4. This next example comes up directly in my own research - it isn't the whole story, but it does play a role.

$$\begin{aligned} \dot{u} &= \sigma u^2 - \frac{2u^4w^2}{\sigma} \\ \dot{w} &= -\sigma w(1-w) + \frac{2u^3w^3}{\sigma}. \end{aligned}$$

Again in the above $\sigma > 0$ is a parameter. In fact this is a relatively safe rule of thumb - if there is an unknown in the right hand side of the dynamical system (ODE) which does not appear on the left, it can usually be thought of as a parameter of the system. This equation (in case you were curious) arises in what's called 'haptotactic cell migration'.

So we now return to a general case:

$$\dot{x} = f(x,y) \quad \text{or} \quad \begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix} f(x,y)\\ g(x,y)\end{pmatrix} =: F(x,y), \tag{1.1}$$
$$\dot{y} = g(x,y)$$

where f and g are 'nice' (i.e. Lipschitz, C^1 C^∞, etc - depending on what we need) functions.

The idea now is to construct a *phase plane* or *phase portrait* of the nonlinear ODE. The phase plane is the (x, y) plane together with *phase curves* of solutions. These are parametrized curves $(x(t), y(t))$ which are solutions to the differential equation (1.1), and parametrized by the independent variable t. This is exactly what we did for linear systems, only this time we won't necessarily be able to solve for $x(t)$ and $y(t)$ explicitly.

The simplest type of phase curves are single points. These are when $f(x, y) = g(x, y) = 0$. In this case we will have a constant solution to the ODE. These points are called *critical points* or *equilibrium points*.

Example 1.5. Returning to the modified logistic growth equation

$$\begin{aligned}\dot{u} &= v\\ \dot{v} &= -u(1-u).\end{aligned} \tag{1.2}$$

We can see that the right hand side is equal to 0 precisely when $(u, v) = (1, 0)$ or $(u, v) = (0, 0)$. Thus we have two critical points, or equilibrium points or equilibrium solutions in the (u, v) plane. Can we find more solutions? In this case, we can do the following 'trick'. Consider the following:

$$\frac{\dot{v}}{\dot{u}} = \frac{\frac{\mathrm{d}v}{\mathrm{d}t}}{\frac{\mathrm{d}u}{\mathrm{d}t}} = \frac{\mathrm{d}v}{\mathrm{d}u} = \frac{-u(1-u)}{v}.$$

The last two expressions give us an ODE in terms of u and v that our phase curves must satisfy. In this case

$$\frac{\mathrm{d}v}{\mathrm{d}u} = \frac{-u(1-u)}{v}$$

is a separable equation and we can solve it. Doing so we see that our phase curves must satisfy the following equation in the (u, v) plane

$$v^2 - \frac{2u^3}{3} + u^2 = C,$$

where C is the constant of integration. In other words our phase curves (the parametrized solutions to our ODE in the (u, v) plane) must be level sets of the function

$$H(u,v) = v^2 - \frac{2u^3}{3} + u^2.$$

We can use Mathematica to plot a few of these for a few values of C. See Figure 1. Once we have the phase curves, we know that solutions will 'travel' along these curves, but there is more. Solutions will be *parametrized* by the independent variable t. In order to account for this, we add arrows to the phase curves showing the direction of motion of the solution as t increases. How do we find the direction of the arrows? Well, we have continuity of our solutions in initial conditions, so it suffices to find a single value of the vector field $\begin{pmatrix}\dot{u}\\ \dot{v}\end{pmatrix} = F(u, v)$ and then we know that a solution curve must be tangent at such a point to the vector. From here we can extend by continuity. In this instance, choose the point $(0, 1)$ in the phase plane. We have that the tangent vector at that point is the vector $(1, 0)$, so we know our solution must be moving to the right at this point. We can then extend by continuity (see Figure 2).

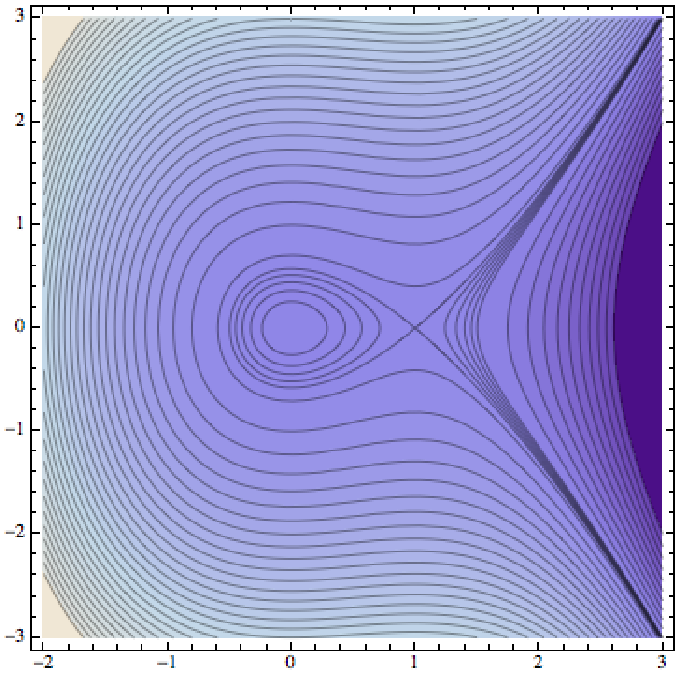


FIGURE 1. Phase curves of eq. (1.2). The value of the constant C ranges from -5 to 10 or so.

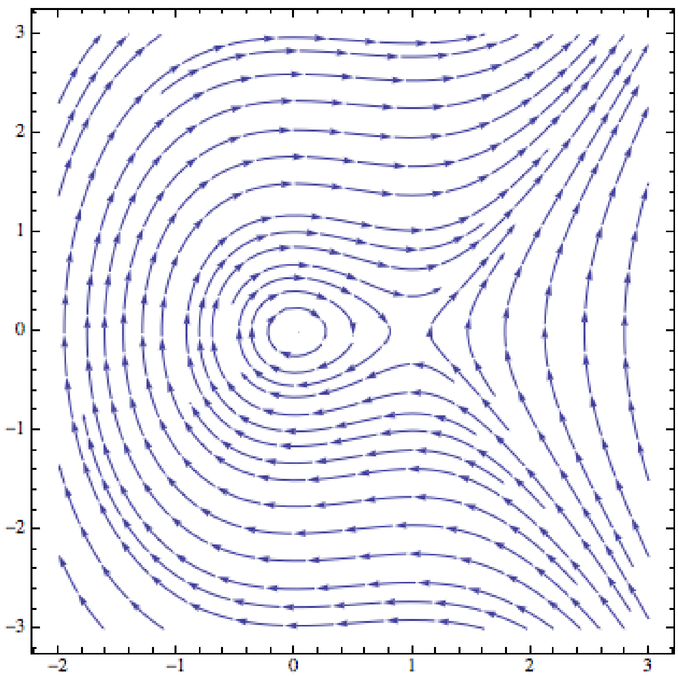


FIGURE 2. The phase portrait of eq. (1.2).

More generally, for eq. (1.1) we can consider the following:

$$\frac{\dot{y}}{\dot{x}} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{g(x,y)}{f(x,y)}$$

and we have an ODE in terms of y and x

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{g(x,y)}{f(x,y)}. \tag{1.3}$$

In many interesting cases it is possible to solve (1.3) directly, and in doing so, we will then know the paths that solutions must follow in the phase plane (we won't know the direction along the paths that solutions will travel though).

Remark. Now I want to point out and emphasise a philosophical shift in how we think about ODEs. We can directly think of these phase curves as *solutions to the ODE.* Typically, we think of a solution as an expression $(x(t), y(t))$ where we write out some explicit formula of x and y in terms of t. This is often impossible to do for the general nonlinear ODE case, and even when it is possible, it isn't very enlightening. Instead you should begin to think about the phase curves (the solutions to $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{g(x,y)}{f(x,y)}$ together with the arrows indicating the direction of movement in t) as *the solutions themselves* to the ODE.

Example 1.6. As an example of what I mean when I say that it is much more fruitful to think of solutions to ODE's as phase curves in the phase plane, let's return to the pendulum equation. The pendulum equation is given by

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\sin(x). \end{aligned} \tag{1.4}$$

It is easy to see that critical points of eq. (1.4) are $(n\pi, 0)$ where $n \in \mathbb{Z}$, and you can check yourself that phase curves must satisfy $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-\sin(x)}{y}$ which is a separable equation that we can explicitly solve. So, we have that phase curves are solutions in the (x, y) plane that satisfy $y^2 - 2\cos(x) = C$, a constant. Using Mathematica, we can plot these for several of values of C and see what happens (see Figure 3). We can also plot the direction

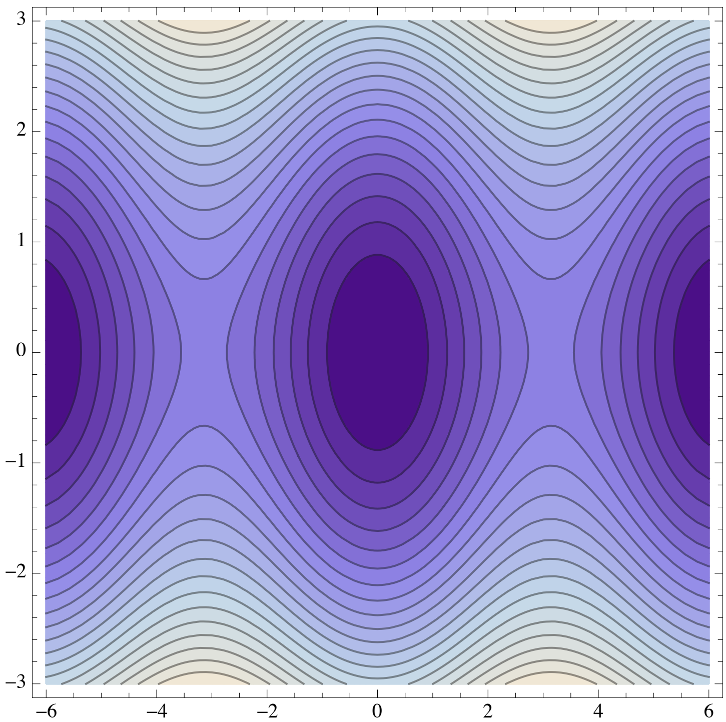


FIGURE 3. Phase curves of the nonlinear pendulum equation. The value of the constant C ranges from -2 to 10.5 or so.

of motion along the phase curves by evaluating the right hand side of eq. (1.4) at a non critical point and then extending by continuity. See Figure 4.

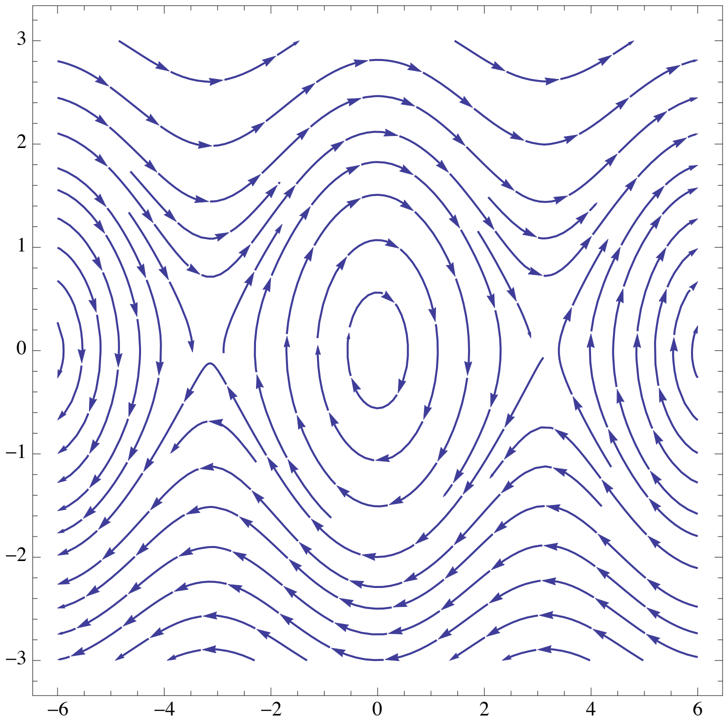


FIGURE 4. The phase portrait of the pendulum equation.

You can look up on the internet that the solution of the nonlinear pendulum equation is given by

$$x(t) = 2\arcsin\left(k\ \mathrm{sn}\,(t;k)\right)$$

where $k = \sin\left(\dfrac{x(0)}{2}\right)$, and sn is a *Jacobi elliptic function*, defined as the inverse of an integral. We let $u(y;k)$ be defined as

$$u(y;k) = \int_0^y \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-k^2t^2)}}.$$

The Jacobi elliptic function $\mathrm{sn}(u;k)$ is then defined as the inverse of u:

$$y = \mathrm{sn}(u;k).$$

This is a particularly nice example and in general you will not be able to find a closed form solution (even in terms of special functions) of your ODE. Effectively the way that this is derived is the following. The expression for $\dot{x}$ together with the formula for the phase curves gives you the following:

$$t = \int_{x_0}^{x(t)} \frac{\mathrm{d}\zeta}{\sqrt{2\cos\zeta + C}}$$

Then if we suppose that the integrand has an antiderivative $\Phi(x)$, we can write

$$t = \Phi(x(t)) - \Phi(x_0),$$

and so if we then assume that the antiderivative has an inverse, we have (technically anyway) an expression for $x(t)$ in terms of t

$$x(t) = \Phi^{-1}(t + \Phi(x_0)).$$

Essentially, what is done in this case is to express the initial conditions and the constant C in such a way so that a Jacobi elliptic integral pops out. This relies on some trigonometric identities and some fairly straightforward substitutions, but is by no means easily extended to very many ODEs.

Remark (Translation in the independent variable)**.** We pause briefly to mention that for an autonomous ODE, then if we have a solution $x(t)$, then $x(t+s)$ is also a solution, provided $(t+s)$ is in the maximal interval of existence of solutions. What this means is that phase curves are really representing *families* of solutions, not just a single one. So if we want to choose a specific one, we need to explicitly say what t_0 and $x(t_0)$ are for some t_0 in the maximal interval of existence.

Now we move on to introduce some more terminology for 2-D autonomous systems of ODEs.

Definition 1.1. The curves where $f(x,y)=0$ or $g(x,y)=0$ are called *nullclines.* Slightly more generically, the curves where $\dfrac{g(x,y)}{f(x,y)} = \dfrac{\mathrm{d}y}{\mathrm{d}x} = C$ a constant are called *isoclines.*

Nullclines and isoclines, while not in general solutions themselves, give you some information as to what the slope of tangent line to the phase curve is along the null/isocline. For example if we're looking at the nullclines then either the vector field is horizontal $\dot{y} = g(x,y) = 0$ or vertical $\dot{x} = f(x,y) = 0$, and so the phase curves will have either a vertical or horizontal tangent. Also you can find critical points by looking for the intersection of the nullcline curves.

2. Linearisation: Hyperbolic, Good. Non-Hyperbolic, Bad

Example 2.1. Consider the modified logistic equation:

$$\begin{aligned} \dot{u} &= v \\ \dot{v} &= -u(1-u) \end{aligned} =: F(u,v). \tag{2.1}$$

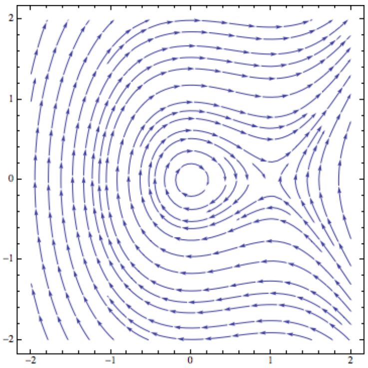
We have that $(1,0)$ is a critical point in the phase plane (i.e $F(1,0)=0$). We want to examine solutions 'near' the critical point. To that end, let's consider solutions of the form $\begin{pmatrix} x(t)+1 \\ y(t) \end{pmatrix}$. Plugging them into eq. (2.1) we have

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = F(x+1, y+0) = F(1,0) + DF(1,0)\begin{pmatrix} x \\ y \end{pmatrix} + h.o.t. \tag{2.2}$$

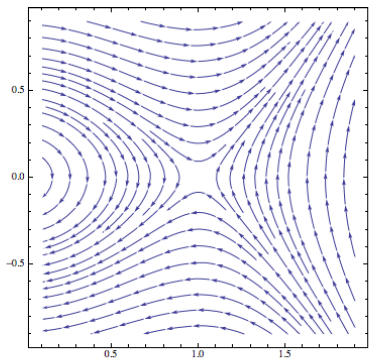
where we have used the Taylor Series expansion of the function F at the point $(1,0)$ and $DF(1,0)$ is the *Jacobian* of the function $F(u,v)$ evaluated at $(1,0)$. This can be computed as $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. If we ignore the higher order terms, we are left with a 2-D *linear* system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \tag{2.3}$$

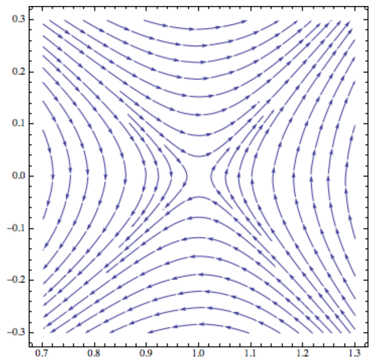
which you can readily check for yourself is a saddle. Further, if we 'zoom' in on our critical point $(1,0)$, we see that the solution to the nonlinear equations seem to be very similar to the solutions to the linearisation at $(1,0)$.



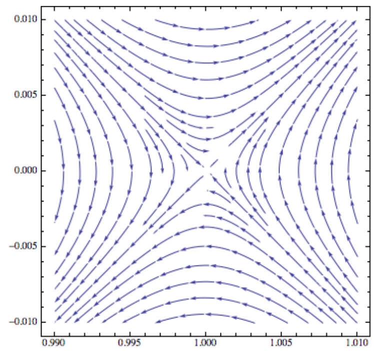
(A) The phase plane of the nonlinear ODE eq. (2.1) in the window $[-2, 2] \times [-2, 2]$.



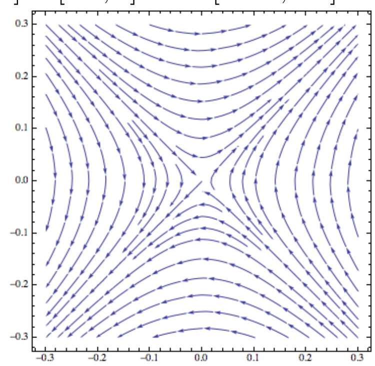
(B) The phase plane of the nonlinear ODE eq. (2.1) in the window $[0, 2] \times [-1, 1]$.



(C) The phase plane of the nonlinear ODE eq. (2.1) in the window $[0.7, 1.3] \times [-0.3, 0.3]$.



(D) The phase plane of the nonlinear ODE eq. (2.1) in the window $[0.99, 1.01] \times [-0.01, 0.01]$.



(E) The phase plane of the *linear* ODE eq. (2.3) in the window $[-0.3, 0.3] \times [-0.3, 0.3]$.

FIGURE 5. Zooming in on the critical point of eq. (2.1) at $(1, 0)$. It is reasonably clear that the last two pictures 'look alike'.

This process of constructing a linear equation for each critical point is totally generic. Associated with every critical point (x_0, y_0) of a 2-D autonomous system:

$$\begin{aligned} \dot{x} &= f(x, y) \\ \dot{y} &= g(x, y) \end{aligned} \quad \text{or} \quad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} f(x, y) \\ g(x, y) \end{pmatrix} =: F(x, y) \tag{2.4}$$

We have a *linear* ODE

$$\begin{pmatrix} \dot{p} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} f_x(x_0, y_0) & f_y(x_0, y_0) \\ g_x(x_0, y_0) & g_y(x_0, y_0) \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = DF(x_0, y_0) \begin{pmatrix} p \\ q \end{pmatrix}. \tag{2.5}$$

The p's and q's are really just there to remind you that this is not a linear system in the original variables that you started with. This linear ODE is called *the linearisation of the system at the critical point.* This is the $2D$ linear constant coefficient ODE found by evaluating the Jacobian of F at the critical point.

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = DF(x_0, y_0) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f_x(x_0, y_0) & f_y(x_0, y_0) \\ g_x(x_0, y_0) & g_y(x_0, y_0) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

We can solve eq. (2.5), classify it, write down solutions, eigenvectors, etc. The question is do the solutions to the nonlinear eq. (2.4) behave 'like' $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ + solutions to eq. (2.5)? The answer evidently is not always.

Example 2.2. Here is an example where the linearisation is deliberately misleading. Consider the following nonlinear autonomous planar system of ODEs

$$\begin{aligned} \dot{x} &= xy \\ \dot{y} &= -y + x^2. \end{aligned} \tag{2.6}$$

We'll set $F(x, y) := \begin{pmatrix} xy \\ -y + x^2 \end{pmatrix}$, the right hand side of eq. (2.6). We have a single critical point $(0, 0)$ and the linearisation at it is given as

$$\begin{pmatrix} \dot{p} \\ \dot{q} \end{pmatrix} = DF(0,0) \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} y & x \\ 2x & -1 \end{pmatrix}\Bigg|_{(x,y)=(0,0)} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}. \tag{2.7}$$

So the linearisation predicts that we will have a degenerate stable equilibrium, and in particular the solutions to the linearised problem are all bounded.

Now, let's use the techniques from earlier to sketch the phase portrait. It is straightforward to see that we have nullclines at $y = x^2$, as well as on the x and y axes. You can also verify for yourself that there are isoclines on the curve $y = \frac{x^2}{1+|x|}$ and on this curve $\frac{dy}{dx} = \pm 1$ depending on the sign of x. Putting all of these together with Mathematica's StreamPlot function gives a fairly reasonable picture as to what is going on, see Figure 6.

This time however, no matter how much we 'zoom' in (shrink the viewing window) we are never going to get the picture to 'look like' the picture of the linearisation at the critical point. In particular, what is happening is that near the critical point, there is actually motion *away* from the origin. So if you were to think about the phase portrait as a bunch of sand, and then you turn on time, and you see that the sand flows according to the arrows along the solutions, what you would see is that if you started near the origin (away from the vertical axis), you would eventually find yourself well far away from it. And so, with the exception of the solutions with initial conditions on the vertical axis you have that that solutions to the nonlinear equation are *unbounded*, while the linearisation predicted bounded behaviour. Effectively what this is saying is that we have a 'saddle-like' behaviour at the origin, but the linearisation predicts that we should have a degenerate stable equilibrium. This is just one example of how non-hyperbolicity is 'bad'. See Figure 7.

In order to formalise what is going on (as much as we're going to) we need the following.

Definition 2.1. A critical point x_* of a nonlinear autonomous ODE $\dot{x} = f(x)$ is called *hyperbolic*, if the linearisation of the full system at the critical point is a hyperbolic linear ODE. That is if the Jacobian of f, evaluated at x_*, $Df(x_*)$, has no eigenvalues with real part equal to zero.

Theorem 2.1 (Hartman-Grobman). *Suppose that x_* is a hyperbolic critical point of an autonomous nonlinear system of ODEs $\dot{x} = f(x)$, and suppose further that $f(x)$ is C^1 in a neighbourhood of x_*. Then the linearisation at x_*, that is the linear equation $\dot{p} = Df(x_*)p$, (where $Df(x_*)$ is the Jacobian matrix of f, evaluated at the critical point) is a 'good approximation' in some neighbourhood of x_*.*

Remark. For those of you interested, when I say 'in a neighbourhood' I mean, in an open set around x_*, and when I say 'good approximation' what I mean is that the solutions

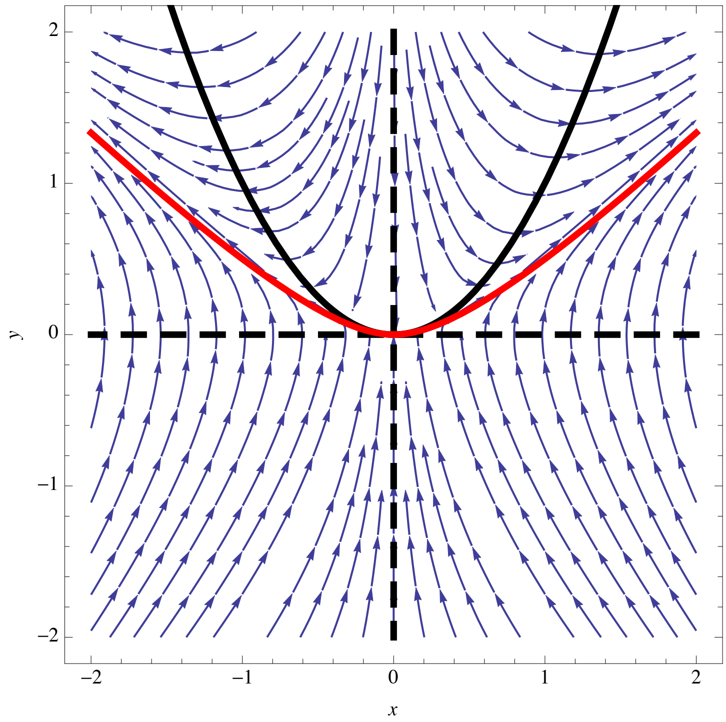


FIGURE 6. Using Mathematica's STREAMPLOT function to get a feel for the phase portrait of eq. (2.6). The black lines are the nullclines (dashed are where $\dot{x} = 0$ and so the solutions have a vertical tangent line, solid are where the tangent line is horizontal). The red line is the isocline where $\frac{dy}{dx} = \text{sgn}x$. That is the curve $y = \dfrac{x^2}{1+|x|}$.

to the linear and nonlinear ODEs are, in that neighbourhood, what is called *topologically conjugate* (look that up on google/wikipedia or come talk to me if you want more information).

Linearisation is an extremely useful and powerful tool for gaining information about a nonlinear planar system of ODEs *but it only works if the critical point under investigation is hyperbolic*. If the critical point is not hyperbolic, then all bets are off - sometimes the linearisation looks like what is going on, and sometimes... not so much.

Example 2.3. Here is another example detailing how non-hyperbolicity is 'bad'. Consider the system

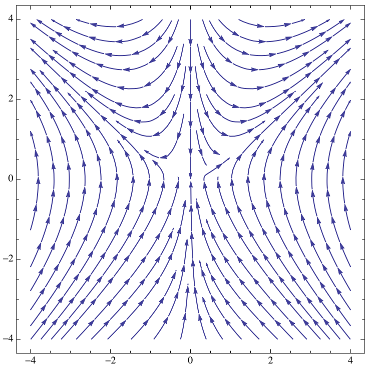
$$\begin{aligned} \dot{x} &= xy \\ \dot{y} &= -x^2. \end{aligned} \tag{2.8}$$

We can readily identify the critical points. We have that there is a line of critical points on the y-axis, which we'll denote by $(0, *)$, and we have that the point $(0, 0)$ is a critical point. So we have a line of equilibria (the y-axis). We can also draw the phase curves. We have

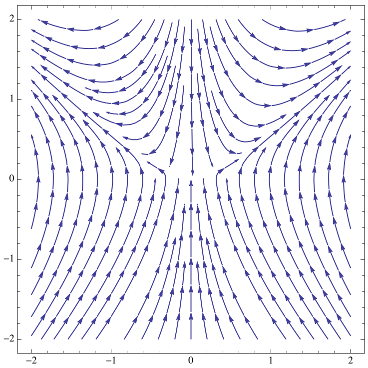
$$\frac{dy}{dx} = \frac{-x^2}{xy} = -\frac{x}{y} \quad \Rightarrow \quad x^2 + y^2 = C$$

so we can see that our phase curves are circles. We now compute the Jacobian and evaluate it at the critical points. We have

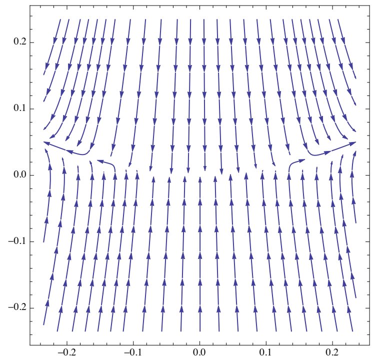
$$DF(0, *) = \begin{pmatrix} y & x \\ -2x & 0 \end{pmatrix}\Bigg|_{(x,y)=(0,*)} = \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix}$$



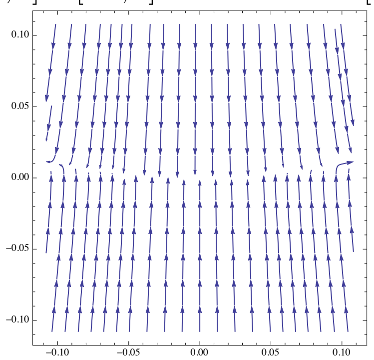
(A) The phase plane of the nonlinear ODE eq. (2.6) in the window $[-4,4]\times[-4,4]$.



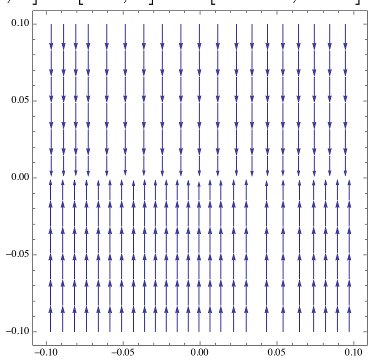
(B) The phase plane of the nonlinear ODE eq. (2.6) in the window $[-2,2]\times[-2,2]$.



(C) The phase plane of the nonlinear ODE eq. (2.6) in the window $[0.25, 0.25]\times[-0.25, 0.25]$.



(D) The phase plane of the nonlinear ODE eq. (2.6) in the window $[-0.1, 0.1]\times[-0.1, 0.1]$.



(E) The phase plane of the *linear* ODE eq. (2.7) in the window $[-0.1, 0.1]\times[-0.1, 0.1]$.

FIGURE 7. Zooming in on the critical point of eq. (2.6) at $(0,0)$. The last two pictures don't really look alike. In fact, you will never be able to 'straighten out' the curvy part of (D), to get it to look like the phase plane of the linear system (E). The reason for this (effectively) is that the critical point of the nonlinear system is not hyperbolic, and so the hypothesis of the Hartman-Grobman Theorem (theorem 2.1) fails.

which for positive y predicts one unstable direction, and one 'null' direction, while for negative y this predicts one stable direction and one 'null' direction, and for $y = 0$ we have $DF = 0$, which predicts no motion whatsoever. This is not hyperbolic for any critical point, and so we can't apply the Hartman-Grobman theorem. What is bad about this example is that the linearisation *correctly* predicts the degenerate stable and unstable equilibria away from the origin, but *incorrectly* predicts the behaviour at $(0,0)$. So non-hyperbolicty can mean that the linearisation is a 'good' approximation and a 'bad' approximation *in the same phase portrait*. What is (perhaps) even worse, is that numerically, this system isn't resolvable near the origin. So we can plug it into Mathematica, and we see that it trajectories are moving away from the critical points on the positive y-axis, and towards the critical points on the negative y-axis, but as we zoom in to the origin, Mathematica breaks down, and is unable to resolve the phase portrait - we can't zoom in enough to see what is happening near the origin... (see Figure 8).

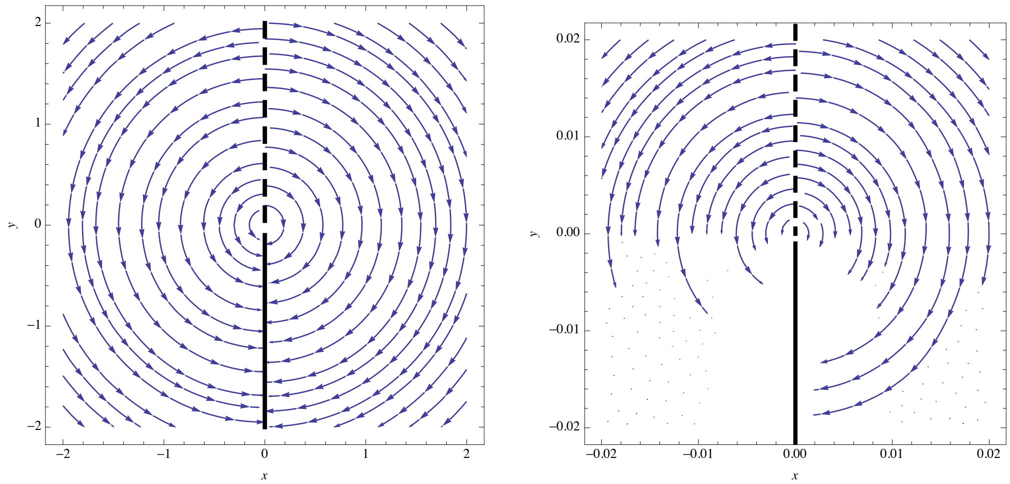


Figure 8. **Left:** a picture of the phase portrait of eq. (2.8). The y axis is a line of equilibria, and for positive y values you can see that the phase curves are flowing away from the y-axis along a circle towards the corresponding negative y values. **Right:** A zoomed in picture of the origin in Mathematica. Essentially, because we are at a non-hyperbolic critical point, we can't even *numerically* resolve what is going in with the dynamics.

3. Some more qualitative results

Next I want to return to some observations about the examples we've described so far. The first thing to point out is that the phase curves do not appear to cross each other. This is not an accident.

Theorem 3.1. *Let $\dot{x} = F(x)$ be an autonomous ODE. There is one and only one phase curve through any point x_0 in the phase plane.*

Sketch of Proof. Without going into details here, the proof is essentially due to the fact that we have uniqueness of solutions and time translation invariance of autonomous systems. Two phase curves passing through the same point (x_0 say) must have the same tangent vector ($F(x_0)$), but by uniqueness of solutions, this means that they must be at most time translations of each other. However, we just saw that phase curves are invariant under time translation. □

Another thing to notice is that in some of these examples we have these closed phase curves. So if we choose a phase curve that is closed, but doesn't contain an equilibrium point, then we see that for some $T > 0$ we must have that $(x(T), y(T)) = (x(0), y(0))$. That is, if we progress long enough in t we will eventually come back to our initial condition. Because phase curves are invariant under translation by the independent variable, this can easily be extended to see that $(x(t+T), y(t+T)) = (x(t), y(t))$ for all $t \in \mathbb{R}$. What this means is that we have just proved the following:

Proposition 3.1. *Closed paths in the phase plane represent periodic solutions.*

As an immediate consequence, we can see that periodic solutions must have a maximal domain of existence of $\mathbb{R}$. Going a little further, if you can write your phase curve as $y = y(x)$ (maybe with a couple of equations $y_1(x)$ and $y_2(x)$) then you can find the period T of a closed path $\mathcal{P}$ (solution) in the phase plane by computing the line integral

$$T = \oint_{\mathcal{P}} \frac{\mathrm{d}x}{f(x, y(x))}$$

where $\dot{x} = f(x, y)$ is the first entry of your vector field. (Likewise you could separate and integrate $\dot{y} = g(x, y)$ if you can write $x = x(y)$ for your phase curve. This is typically just done for numerics however, as these things are usually impossible to do analytically. Also a worthwhile question to ask yourself at this point is: Why isn't this line integral equal to 0? The answer has to do with the following proposition which I am going to state without proof:

Proposition 3.2. *The phase path of a periodic solution in the plane must enclose at least one critical point.*

Example 3.1 (The undamped, unforced Duffing oscillator)**.** Let's look at another example; one coming from biomechanics, neurobiology, nonlinear optics, and quantum mechanics (among other things). We start with the second order nonlinear ODE

$$\ddot{x} + \delta x + \alpha x + \beta x^3 = \gamma \cos(\omega t).$$

This equation represents a damped, forced oscillator. In order to make sense of the phase portraits we are going to set $\alpha = -1$, $\beta = \frac{1}{6}$ and $\delta = \gamma = 0$. Substituting in these parameter values gives

$$\ddot{x} - x + \frac{x^3}{6} = 0. \tag{3.1}$$

We re-write it as a first order system in the usual way, writing

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= x - \frac{x^3}{6}. \end{aligned} \tag{3.2}$$

We can plug this into Mathematica right away and get a picture of the phase portrait (Figure 9), and if you are at a computer, I highly recommend that you do so. But it is also useful to perform some analysis, in order to be able to further study what is going on at key points in the phase plane. We have that the critical points are $(0, 0)$ and $(\pm\sqrt{6}, 0)$.

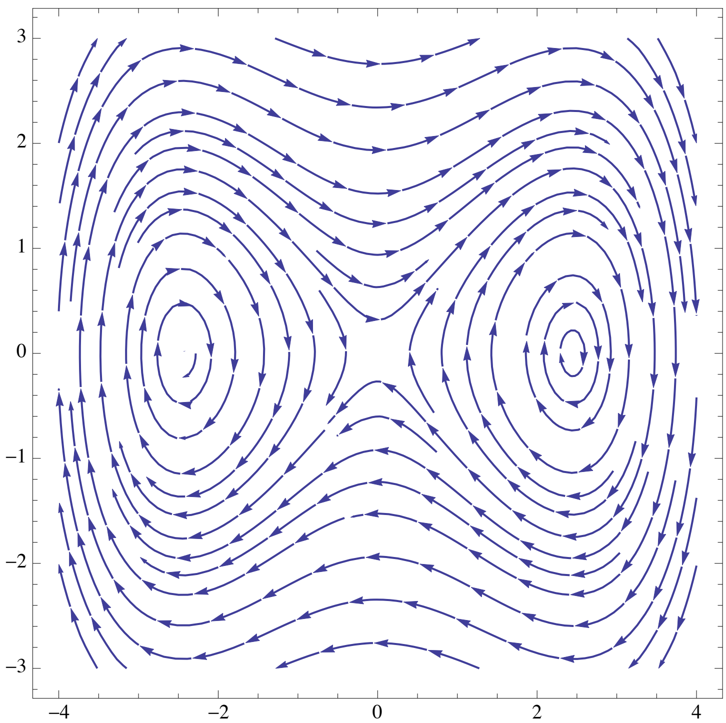
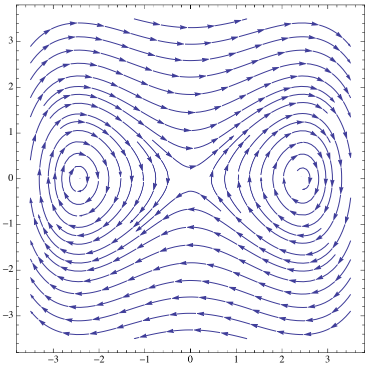


FIGURE 9. Phase curves of the system corresponding to eq. (3.1).

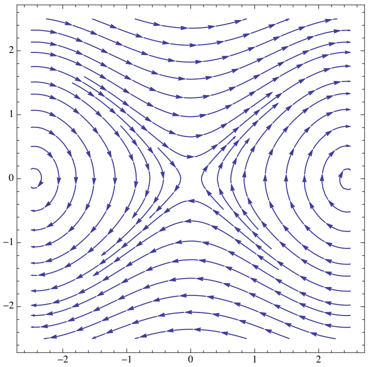
The linearisation at the origin is

$$\begin{pmatrix}\dot{p}\\ \dot{q}\end{pmatrix} = DF(0,0)\begin{pmatrix}p\\ q\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1-\frac{x^2}{2} & 0\end{pmatrix}\Bigg|_{(x,y)=(0,0)}\begin{pmatrix}p\\ q\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}p\\ q\end{pmatrix}. \tag{3.3}$$

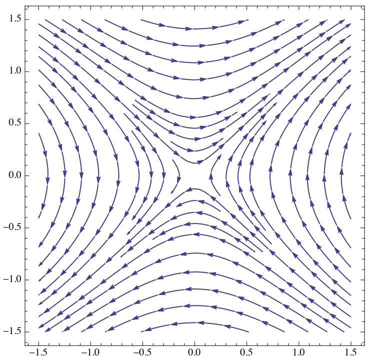
This is easily seen to be a saddle as the determinant of the matrix is negative. The critical point is therefore hyperbolic and so the Hartman-Grobman theorem tells us that this is a reasonable approximation to what is going on. This fact is verified in Figure 10.



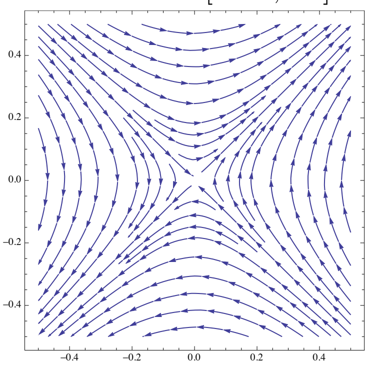
(A) The phase plane of the nonlinear ODE in eq. (3.1) in the window $[-3.5, 3.5] \times [-3.5, 3.5]$.



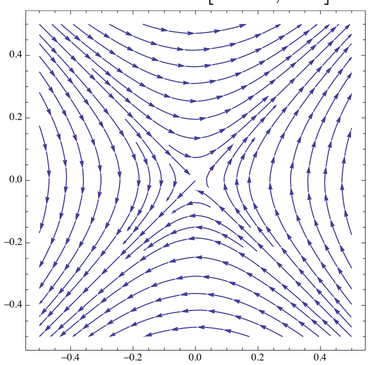
(B) The phase plane of the nonlinear ODE in eq. (3.1) in the window $[-2.5, 2.5] \times [-2.5, 2.5]$.



(C) The phase plane of the nonlinear ODE in eq. (3.1) in the window $[-1.5, 1.5] \times [-1.5, 1.5]$.



(D) The phase plane of the nonlinear ODE in eq. (3.1) in the window $[-0.5, 0.5] \times [-0.5, 0.5]$.



(E) The phase plane of the *linear* ODE in eq. (3.3) in the window $[-0.5, 0.5] \times [-0.5, 0.5]$.

FIGURE 10. Zooming in on the critical point of eq. (3.1) at the origin. It is reasonably clear that the last two pictures 'look alike'.

At the other two critical points $(\pm\sqrt{6}, 0)$, the linearisation is given by

$$\begin{pmatrix}\dot{p}\\ \dot{q}\end{pmatrix} = \begin{pmatrix}0 & 1\\ -2 & 0\end{pmatrix}\begin{pmatrix}p\\ q\end{pmatrix}. \tag{3.4}$$

In these cases the linearisation is not hyperbolic, and so the Hartman-Grobman theorem doesn't tell us anything.

The phase curves are found by solving the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x - \frac{x^3}{6}}{y} \quad \text{and so} \quad y^2 = x^2 - \frac{x^4}{12} + C$$

where C is a constant. That is to say, that phase curves lie on level sets of the function

$$G(x,y) = y^2 - x^2 + \frac{x^4}{12}.$$

The vertical nullcline of this system is when $\dot{x} = 0$ on the line $y = 0$, and you can verify for yourself that $\dot{y} < 0$ when $x \in (-\sqrt{6}, 0) \cup (\sqrt{6}, \infty)$, and so the arrows of the vector field are pointing down in that part of the nullcline. You can also check that $\dot{y} > 0$ when $x \in (-\infty, -\sqrt{6}) \cup (0, \sqrt{6})$ and so the arrows are pointing up there. The horizontal nullclines are the lines $x = 0, \pm\sqrt{6}$. Along these lines you can see that the vector field is pointing to the right when $y > 0$ and to the left when $y < 0$. Lastly, in order to qualitatively get a picture of what is going on, we sketch one more level set of the function $G(x,y)$, the one containing the origin. That is we sketch the curve $y^2 - x^2 + \frac{x^4}{12} = 0$ in the (x,y) plane. This is the 'bowtie' shape in left picture in fig. 11. Putting this all together, along with the fact that phase curves don't cross, and it is reasonably straightforward to sketch out the phase portrait. This can then be verified with the aid of a computer.

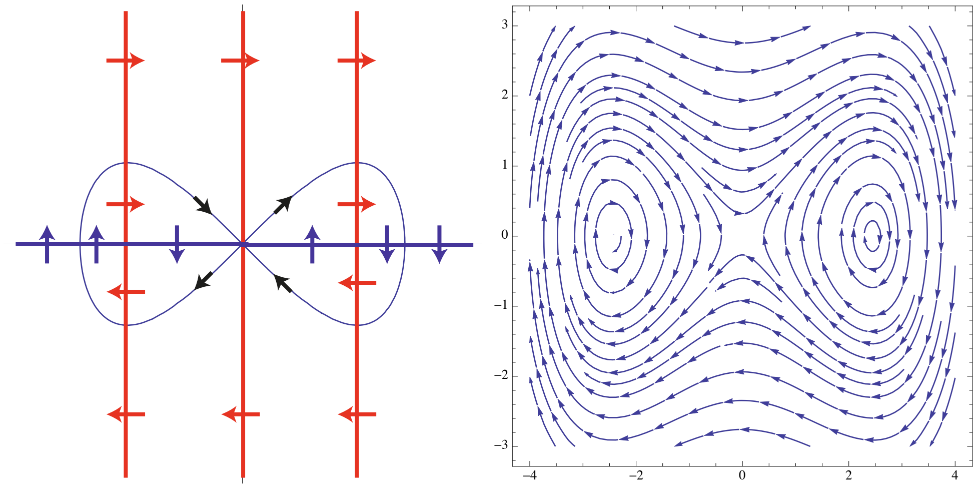


FIGURE 11. The left portrait is a sketch of the nullclines and the curve $G(x,y) = 0$ in the phase portrait.The right is the phase portrait of the full nonlinear ODE eq. (3.1).

Before moving on, it is worth noting that the level set $G(x,y) = 0$ (the 'bow-tie curve in Figure 12) in this example contains the origin, which is a critical point. So this is actually *three* solutions. The left and right lobes and the point $(0,0)$. It is also a decent idea to try and sketch what the plots of x vs t and y vs t for solutions would look like in some examples. In Figure 12 I have done this for you for the case of the right lobe of the 'bow-tie' curve. You should try it for some of the other phase curves.

Example 3.2. Let's return to Example 3.1, and take a look at the other two critical points at $(\pm\sqrt{6}, 0)$. We saw that the linearisation was given as

$$\begin{pmatrix} \dot{p} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} \tag{3.5}$$

which we saw to be a centre. We were therefore unable to use the Hartman-Grobman theorem to analyse what is happening to the nonlinear equation near the critical points. However, we can exploit another feature of this system, namely, that there is an equation $G(x,y)$ such that the level curves of $G(x,y)$ are the solutions in the phase plane. We notice that the critical points of our vector field are also critical points of G. That is we

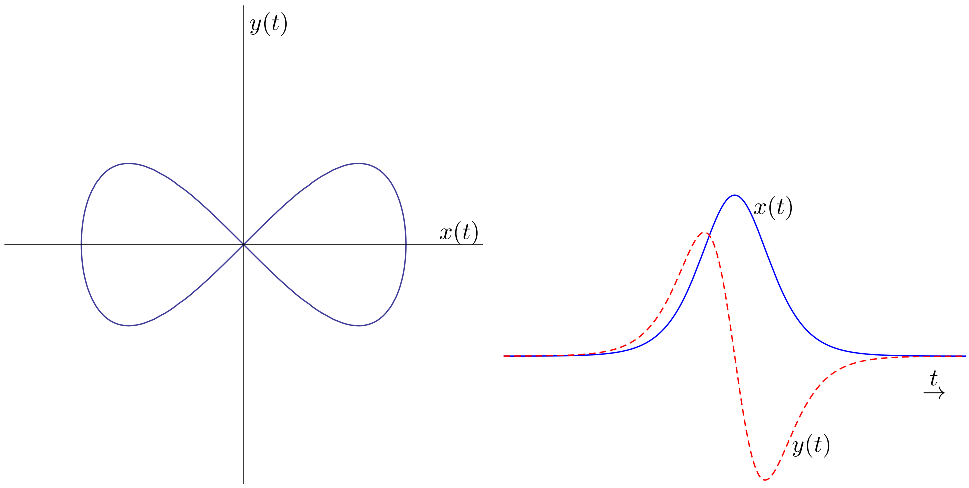


FIGURE 12. **Left**: A plot of the 'bow-tie' phase curve. Note that the origin is a critical point, so that this is actually 3 solutions (the left, and right lobes and the point $(0,0)$)). **Right**: Plots of the profiles $x(t)$ (blue-solid) and $y(t)$ (red-dashed) of the phase curve corresponding to the right lobe of bow-tie curve in Example 3.1. Note that $y(t) = \dot{x}(t)$.

have

$$\nabla G(\pm\sqrt{6}, 0) = (-2x + \frac{x^3}{3}, 2y)\Big|_{(x,y)=(\pm\sqrt{6},0)} = (0,0).$$

We can then use the second derivative test to determine what type of critical point we have. The Hessian of G at $(\pm\sqrt{6}, 0)$ is given by

$$D^2G(\pm\sqrt{6}, 0) = \begin{pmatrix} -2+x^2 & 0 \\ 0 & 2 \end{pmatrix}\Bigg|_{(x,y)=(\pm\sqrt{6},0)} = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix},$$

which has a positive determinant. Therefore you know from the second derivative test that $G(x,y)$ has either a maximum or a minimum at $(\pm\sqrt{6}, 0)$ (in this case it will turn out to be a minimum). Thus we know that near the critical point of G *which is also a critical point of eq.* (3.2) solutions to eq. (3.2) will be closed (i.e. periodic) level sets of the function G. Thus we can conclude from the geometry, that around the critical points $(\pm\sqrt{6}, 0)$ we have a sequence of periodic phase paths (i.e. closed paths in the phase plane) which are going around the critical points. We haven't described a nonlinear centre yet, but this is it. It is important to note that while the linearisation fails to tell us what we have, the geometry of the problem exactly tells what is going on. For completeness, in Figure 13 I have shown that as we 'zoom' in on the critical point $(\sqrt{6}, 0)$ we do indeed see behaviour that looks like a centre.

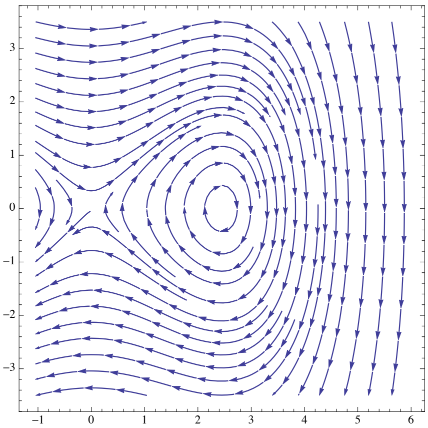
Example 3.3 (Hamiltonian Systems)**.** An important class of planar nonlinear ODEs are *Hamiltonian Systems.* Returning to consider the general case as in eq. (2.4)

$$\begin{aligned} \dot{x} &= f(x,y) \\ \dot{y} &= g(x,y) \end{aligned} \quad \text{or} \quad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix} =: F(x,y) \tag{3.6}$$

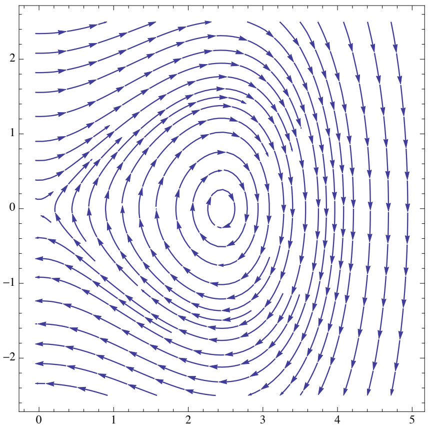
Equation (3.6) is called *a Hamiltonian system* if there exists a function $\mathcal{H}(x,y)$ such that

$$f(x,y) = \frac{\partial \mathcal{H}}{\partial y} \quad \text{and } g(x,y) = -\frac{\partial \mathcal{H}}{\partial x}.$$

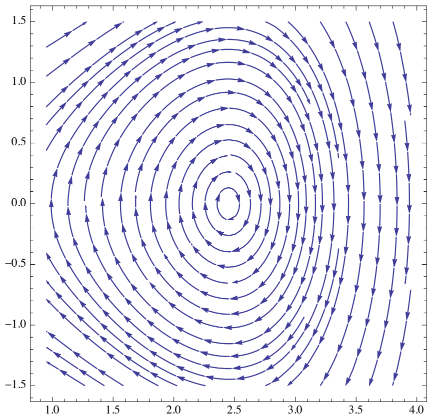
The function $\mathcal{H}$ is called the *Hamiltonian function* for the system. For example eq. (3.2) is a Hamiltonian system with Hamiltonian $\mathcal{H} = \frac{1}{2}G$ as in example 3.1. A necessary and



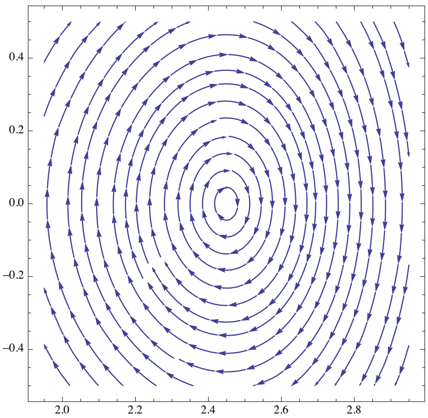
(A) The phase plane of the nonlinear ODE eq. (3.1) in the window $[\sqrt{6}-3.5, \sqrt{6}+3.5] \times [-3.5, 3.5]$.



(B) The phase plane of the nonlinear ODE eq. (3.1) in the window $[\sqrt{6}-2.5, \sqrt{6}+2.5] \times [-2.5, 2.5]$.



(C) The phase plane of the nonlinear ODE eq. (3.1) in the window $[\sqrt{6}-1.5, \sqrt{6}+1.5] \times [-1.5, 1.5]$.



(D) The phase plane of the nonlinear ODE eq. (3.1) in the window $[\sqrt{6}-0.5, \sqrt{6}+0.5] \times [-0.5, 0.5]$.

FIGURE 13. Zooming in on the critical point of eq. (3.1) at $(\sqrt{6}, 0)$. Note that the Hartman Grobman theorem does not predict this behaviour. The reason that the linearisation provides a 'good' approximation near this critical point is because the system is Hamiltonian.

sufficient condition for eq. (3.6) to be Hamiltonian is that

$$f_x + g_y = 0. \tag{3.7}$$

Along the phase paths $(x(t), y(t))$ we have that

$$\frac{d\mathcal{H}}{dt} = \frac{\partial \mathcal{H}}{\partial x}\frac{dx}{dt} + \frac{\partial \mathcal{H}}{\partial y}\frac{dy}{dt} = -f(x,y)g(x,y) + g(x,y)f(x,y) = 0$$

Thus we have that $\mathcal{H}$ is constant along the phase paths. Thus the phase paths are level curves or contours of $\mathcal{H}$, just as if we could solve $\frac{dy}{dx} = \frac{g(x,y)}{f(x,y)}$ (which we can in this case - think about why). Again, in this case we have that if (x_0, y_0) is a critical point of eq. (3.6) then it will also be a critical point of the function $\mathcal{H}$. We can use the second derivative test again to determine whether or not $\mathcal{H}(x,y)$ has a saddle at (x_0, y_0) or a maximum or a minimum there. The thing to notice is that the determinant of the Hessian of $\mathcal{H}$ at the critical point (x_0, y_0) is the same as the determinant of the Jacobian of the linearisation of eq. (3.6) at (x_0, y_0). To see this, notice that

$$\det D^2\mathcal{H} = \begin{vmatrix} \mathcal{H}_{xx} & \mathcal{H}_{xy} \\ \mathcal{H}_{xy} & \mathcal{H}_{yy} \end{vmatrix} = \mathcal{H}_{xx}\mathcal{H}_{yy} - (\mathcal{H}_{xy})^2$$

While the Jacobian at the critical point is

$$DF(x_0, y_0) = \begin{pmatrix} \mathcal{H}_{xy} & \mathcal{H}_{yy} \\ -\mathcal{H}_{xx} & -\mathcal{H}_{xy} \end{pmatrix}.$$

The linearisation thus predicts a saddle or a centre depending on the sign of the determinant of the Jacobian (provided it is nonzero). If it is a saddle, then the Hartman-Grobman theorem tells us that the nonlinear equation also has a saddle. If it is a centre, then the argument from above using the second derivative test states that the level sets of $\mathcal{H}$ will be closed paths in the phase plane around (x_0, y_0) and so we have that the critical point is a centre in this case as well. If the determinant of the Hessian of $\mathcal{H}$ or the Jacobian in linearisation of eq. (3.6) at (x_0, y_0) is 0, then local extrema of $\mathcal{H}$ will still be centres but more complicated types of saddle points are possible. (Evidently Hamiltonian systems only have saddles and centres).

4. The Lotka-Volterra / A Predator-Prey System

Next, we'll discuss a class of examples which have many applications - the Lotka-Volterra system, or a predator-prey system. Diving right in we have that the system is given by

$$\begin{aligned} \dot{N} &= aN - cNP \\ \dot{P} &= -bP + dNP \end{aligned} \tag{4.1}$$

where a, b, c, d are all positive constants.

A relatively straightforward derivation of the model is as follows: the dependent variables N and P represent two interacting species: N are the 'prey' and P are the 'predators'. In the absence of predators, the prey will 'live long and prosper' reproducing at a rate a (without bound). But there are predators in the habitat, which eat the prey at a rate proportional to the populations of both the predators and the prey (c). Further, we assume that if there are not enough prey to eat, the predators will die of at some rate, proportional to their population (b), while if they do have enough to eat, then they will be able to feed their baby predators, who will grow big and strong, increasing the predator population proportionally to the number of predators and prey (d).

Now we are ready to perform some analysis. First, we will concern ourselves only with the first quadrant (Q1), or those $P, N \geq 0$. This is to ensure that we stay in the realm where the model is representing something realistic. It is easy enough to see that there are two equilibrium points $(0, 0)$ and $(\frac{b}{d}, \frac{a}{c})$. Further we have that the Jacobian is given as

$$DF = \begin{pmatrix} a - cP & -cN \\ dP & -b + dN \end{pmatrix}.$$

Evaluating at the critical points gives

$$DF(0,0) = \begin{pmatrix} a & 0 \\ 0 & -b \end{pmatrix} \quad DF(\frac{d}{b}, \frac{a}{c}) = \begin{pmatrix} 0 & \frac{-cb}{d} \\ \frac{ad}{c} & 0 \end{pmatrix}.$$

At the origin we have that the critical point is a saddle, while at the point $(\frac{b}{d}, \frac{a}{c})$ we have that the linearisation is a centre. In Q1, we have that the phase curves must solve the

FIGURE 14. Phase curves of the Lotka-Volterra system in Q1. Here, for simplicity, we've chosen $a = b = c = d = 1$.

differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}N} = \frac{P(-b + dN)}{N(a - cP)}$$

which is separable, leading to the equation for the phase curves

$$a\log(P) - cP + b\log(N) - dN = C$$

where C is a constant. These will be closed curves around the point $(\frac{b}{d}, \frac{a}{c})$ (See Figure 14). By the previous analysis, we have that for these phase curves both $N(t)$ and $P(t)$ will be periodic solutions. You can then use your favourite computer software package (I used pplane8 in Matlab for this one) to get a plot of N and P versus t for any particular value that you want. The fact that our periodic solutions are not symmetric around the critical points means that there is an offset in the peaks of the populations of P and N. This makes sense if we refer to the derivation of the model - the prey population will grow, and only then can the predators eat them and then grow themselves, which in turn shrinks the prey population, which in turn will (at a later time) shrink the predator population (which causes the prey population to grow... etc). See Figure 15. It is worth noting that a change in the system, like a bad winter or something like that, will be reflected in a change in the constant C. That is, we will jump to a different phase curve, and then proceed from there. It is also worth noting that this model will not be able to model extinction, as every phase curve (in Q1) will be part of a periodic solution.

One more thing to point out is that this is really just the beginning for population dynamics. For example, if you were to change the assumptions on your model so that the

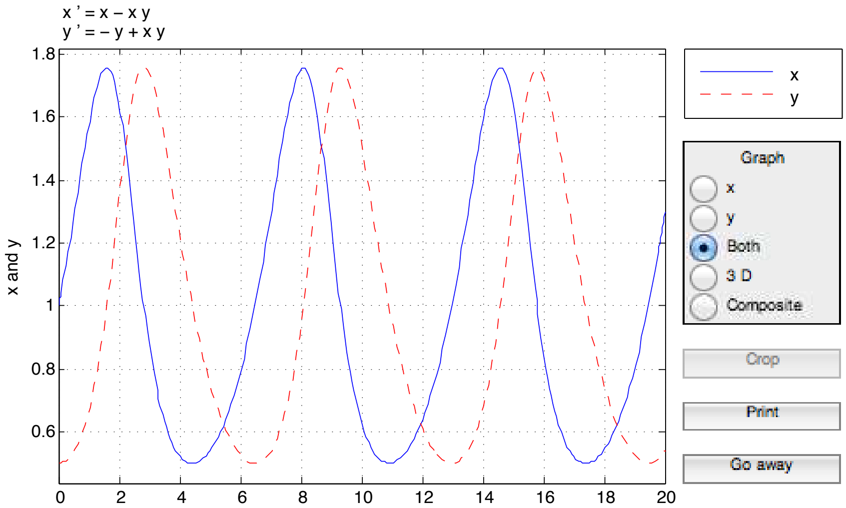


FIGURE 15. A plot of $N(=x)$ and $P(=y)$ versus t for one of the periodic solutions oscillating around the critical point $(1,1)$. One can interpret the delay in the peaks as how fast the P population 'reacts' to the N population.

prey would reproduce in a logistic way, then this is represented as

$$\begin{aligned}\dot{N} &= \frac{aN(k-N)}{k} - cNP \\ \dot{P} &= -bP + dNP\end{aligned} \tag{4.2}$$

where now k is the 'carrying capacity' of the environment for the prey. The steady state of this system are now $(0,0)$, $(k,0)$ and $(N_*, P_*) := (\frac{b}{d}, \frac{a}{c} - \frac{ab}{dck})$. You can verify yourself that at $(0,0)$ the linearisation is still a saddle, while $DF(k,0)$ is given as

$$DF(k,0) = \begin{pmatrix} -a & -ck \\ 0 & -b+dk \end{pmatrix}$$

and you can check for yourself that this will be a saddle, provided that k is 'reasonably' large (in this case $b - kd < 0$). Finally for the last critical point we have that

$$DF(N_*, P_*) = \begin{pmatrix} -\dfrac{ab}{dk} & -\dfrac{bc}{d} \\ -\dfrac{a(b-dk)}{ck} & 0 \end{pmatrix}.$$

This has determinant $= ab\left(1 - \frac{b}{dk}\right)$ which is positive if $b < dk$. The trace of $DF(N_*, P_*) < 0$ so we can use the Hartman-Grobman theorem to deduce that in the full, nonlinear system, the critical point 'looks like' a stable node or a spiral sink. To determine which, you need to evaluate the sign of $\operatorname{tr}(DF(N_*, P_*))^2 - 4 \det DF(N_*, P_*)$. A phase portrait for some specific values of the parameters is provided in Figure 16.

5. EPIDEMICS

In this section we'll continue with another large class of examples that arise in mathematical biology: modelling epidemics. In particular we are going look at a common class of models for infectious diseases called *SIR models*. There is a *vast* amount of literature relating to this model, and variations on it. As such, there is no way we can tell the whole story, and you should really consider this just the starting point for further study. In the SIR model, the total population is divided into three classes: S, I, and R. The

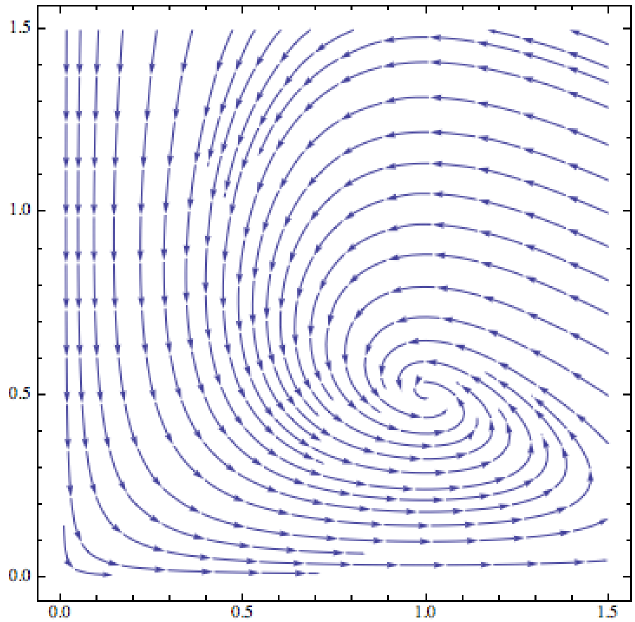


FIGURE 16. A portrait of eq. (4.2). The parameters chosen were $a = b = c = d = 1$ and $k = 2$.

S stands for *susceptible* - these are the members of the popluation who can catch the disease. The I stands for *infected* - these are the sick members of the population who can transmit the disease. The R stands for *recovered* or *removed*. These are the members of the population who have either had the disease and are immune, or are isolated from the susceptible and infected populations, or otherwise removed from the system. We denote the population of each class at time t by $S(t), I(t)$ and $R(t)$ respectively. We make the following assumptions about the model

(1) The infected population grows at a rate jointly proportional to the population of the infected and the population of the susceptibles, and the susceptible population decreases at this same rate (this is the $\pm r$ term in the first two equations of eq. (5.1)).
(2) The infected people recover at a constant rate (this is the $\pm aI$ term)
(3) The incubation period is short enough to be negligible (this says that we only have 3 classes)
(4) The total population stays the same $S + I + R = N$ a constant.
(5) We assume that every pair of individuals has an equal probability of coming into contact with each other.

Putting these all together we arrive at the (very famous) Kermack-McKendrick model from 1927.

$$\begin{aligned} \frac{dS}{dt} &= -rSI \\ \frac{dI}{dt} &= rSI - aI \\ \frac{dR}{dt} &= aI \end{aligned} \tag{5.1}$$

We will focus on values of S, I and R which are nonnegative and bounded above by the total population. And we remark here that assumption (4) is built into eq. (5.1) as

$$\frac{dS}{dt} + \frac{dI}{dt} + \frac{dR}{dt} = 0.$$

We consider the general initial conditions $S(0) = S_0$, $I(0) = I_0$ and $R(0) = 0$. The question is: given r, a, S_0 and I_0 will the infection spread? If so, how does it develop, and when, if ever will it decline?

We start by considering $\frac{dI}{dt}(0)$. We see that

$$\frac{dI}{dt}(0) = rS_0I_0 - aI_0 = I_0(rS_0 - a) \quad \begin{cases} > 0 \\ < 0 \end{cases} \quad \text{if} \quad S_0 \begin{cases} > \rho \\ < \rho \end{cases} \quad \rho = \frac{a}{r}. \tag{5.2}$$

If $S_0 < \frac{a}{r}$ then because $\frac{dS}{dt} = -SI$ we have that $S(t) < S_0$ for all $t > 0$, and so we can conclude that

$$\frac{dI}{dt} = I(rS - a) \leq 0 \quad \text{for all} \quad t \geq 0.$$

In which case $I_0 > I(t) \to 0$ as $t \to \infty$, and so the infection dies out. On the other hand, if $S_0 > \rho$, then $I(t)$ initially increases and we have what's called an *epidemic*. By *epidemic* we mean that $I(t) > I_0$ for some $t > 0$. We thus have what's called a *threshold phenomenon*. If S_0 is larger than ρ we have an epidemic, while if not, then we don't. What this is saying is that you need a large enough population for an epidemic to occur. The parameter ρ is sometimes called *the relative removal rate* and the reciprocal $\sigma = \frac{r}{a}$ is sometimes called the infection's *contact rate.*

Now, on the surface, this model seems like it is a 3-D model, but the constant population assumption really means that it can be reduced to a 2-D one. We consider the phase lines in the (S, I) plane. We have that the critical points all lie on the $I = 0$ axis. We also have

$$\frac{dI}{dS} = -\frac{(rS - a)I}{rSI} = -1 + \frac{\rho}{S}.$$

Integrating this gives the equation for the phase curves in the (S, I) plane:

$$I + S - \rho\log(S) = C = I_0 + S_0 - \rho\log S_0 = N - \rho\log(S_0) \tag{5.3}$$

We can then sketch the phase trajectories. See Figure 17.

If an epidemic occurs, one question is: how severe is it? We have that the maximum I, $I_{\max}$ will occur when $\frac{dI}{dt} = 0$. From eq. (5.3) we have that

$$\begin{aligned} I_{\max} &= \rho\log\rho - \rho + I_0 + S_0 - \rho\log S_0 \\ &= N - \rho + \rho\log\left(\frac{\rho}{S_0}\right). \end{aligned}$$

We also have that as $\frac{dS}{dt} < 0$ and from the second and third equations in eq. (5.1) $\frac{dS}{dR} = -\frac{S}{\rho}$ which means that

$$S = S_0e^{-R/\rho} \geq S_0e^{-N/\rho} > 0$$

or $0 < S(\infty) \leq N$. The reason that I point this out is that this means that the infection dies out because of a lack of infected individuals, and not because of a lack of susceptibles. This is a key (but subtle) feature in the model.

We illustrate one more point. We write:

$$R_0 = \frac{rS_0}{a}.$$

The parameter R_0 (don't confuse this with $R(0) \equiv 0$) is the called the *basic or intrinsic reproduction rate* of the infection. This is the number of secondary infections produced by one primary infection in a wholly susceptible population. If $R_0 > 1$ then we have an

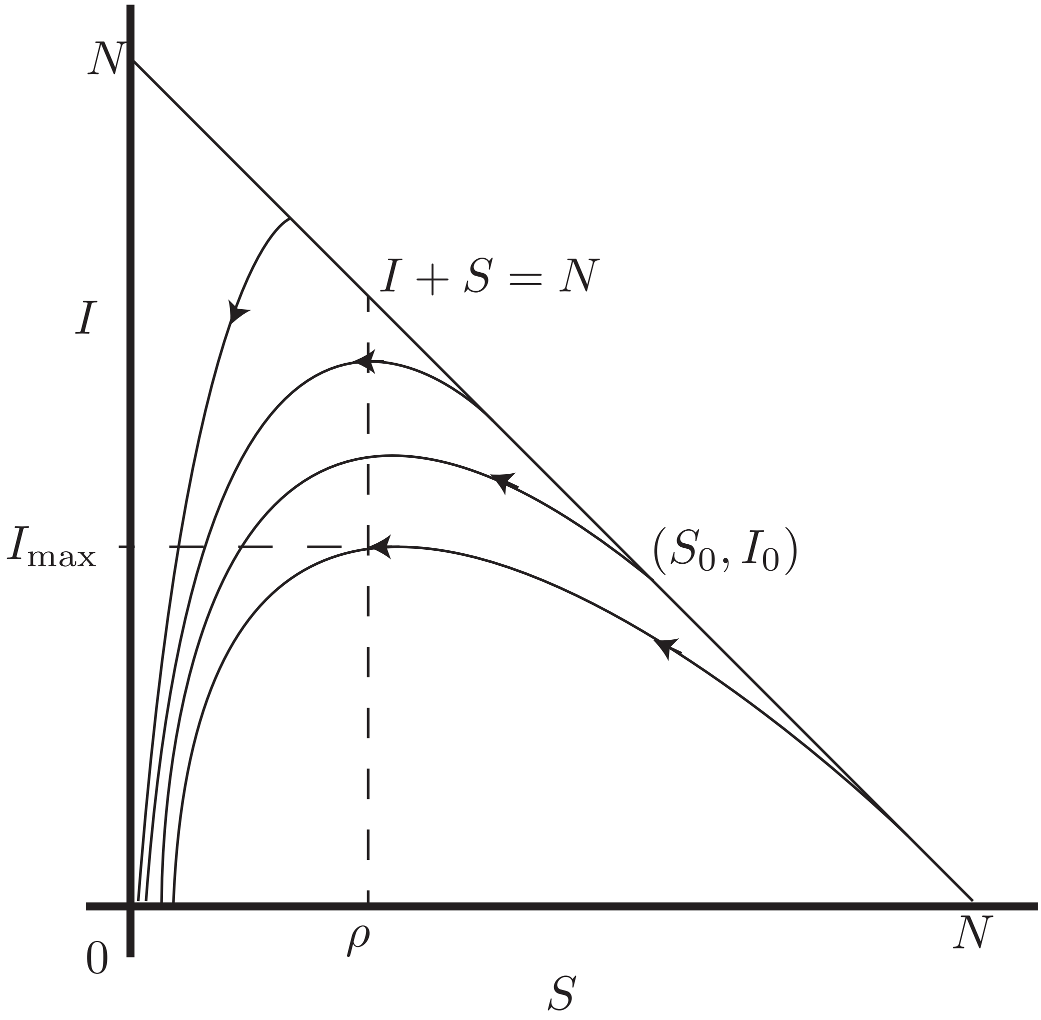


FIGURE 17. The phase trajectories for an SIR model in the (S, I) phase plane. The curves are determined by the initial conditions S_0 and I_0. All initial trajectories start on the line $S+I = N$ and remain within the triangle since $0 < S+I < N$. An epidemic occurs formally when $I(t) > I_0$ for some time t. This always occurs in this model if $S_0 > \rho$.

epidemic, while if $R_0 < 1$ then we don't (think about why this is so). The reason that this parameter is important is that it is measurable. So one can (and people do) try various methods to get $R_0 < 1$. One such method is by vaccination. If V is the percentage of the populace which is immunised against a disease, then $(1-V)$ is the percent not immunised. So the population 'participating' in the disease after an immunisation is $(1-V)N$. Thus, effectively, the intrinsic reproduction rate of disease after immunisation can be thought of as $\tilde{R}_0 = (1 - V)R_0$. The question then is what percent of the population do you need to vaccinate in order to successfully assure the eradication of the disease? We have

$$\tilde{R}_0 = (1 - V)R_0 < 1 \Rightarrow V > 1 - \frac{1}{R_0}.$$

As an example, of how this was practically used, it was determined that in developing countries, before the eradication of the smallpox virus, that the basic reproduction rate R_0 for the smallpox virus was approximately $3-5$. Thus it was estimated that approximately $70-80\%$ of the populations of these countries needed to be vaccinated in order to get rid of the disease.

Remark. In the example eq. (5.1) we were considering, the disease was modelled so that the the only steady state solutions were ones where $I = 0$. It is conceivable that one could, in another model write down different equations for the right hand side of eq. (5.1) which produced a steady state where $I \neq 0$. In this case the infection is called *endemic*.

6. Some Abstraction, Global Existence & Reparametrization

For the purposes of this section we'll consider a nonlinear ODE on some subset $E \subseteq \mathbb{R}^n$ (but you can think of it a subset of the plane if it helps):

$$\dot{x} = F(x), \tag{6.1}$$

where F is a C^1 function on some subset $E \subseteq \mathbb{R}^n$. We impose the C^1 restriction for convenience, and note that quite a few of these results can be extended even if F is not so nice everywhere though we 'recall' that from the existence and uniqueness theorem that we want F to be at least Lipschitz continuous. We also have from the existence and uniqueness theorems, that for each $t \in J$ (some domain of existence) we will have a solution map which was originally denoted as $u(t, y)$, but now we want to denote it as

$$\Phi_t : E \to E$$

to emphasise that we have a sequence of (differentiable) maps from E (the phase space) to itself which are parametrized by time. Further, we can think of Φ_t as giving us an evolution rule for our subset E - that is we can 'turn on time' and think of what happens to points in E under the motion of Φ_t as t increases (or decreases) from t_0 in J. Finally, we note that, we have that there is some maximal interval of existence J which is open (and sometimes is all of $\mathbb{R}$), on which these solution maps are defined. Putting these ideas together, we define the following:

Definition 6.1. A one parameter family of differentiable maps $\Phi_t : E \to E$ is called a *complete flow* (sometimes just a *flow*) if

1. Φ_t is defined for all $t \in \mathbb{R}$.
2. $\Phi_0(x) = x$ for all $x \in E$.
3. (The group property) $\Phi_t(\Phi_s(x)) = \Phi_{t+s}(x)$ for all $x \in E$ and all $s, t \in \mathbb{R}$.

The group property of a complete flow means that if we have a complete flow $\Phi_t^{-1} = \Phi_{-t}$, so we have a one parameter family of (nonlinear) maps which are (a) differentiable, (b) invertible and (c) whose inverse is also differentiable (such a map is called a diffeomorphism).

Next we observe that for each fixed $x \in E$, a complete flow $\Phi_t(x)$ defines a curve in E as t varies over $\mathbb{R}$. For the sake of convenience in what follows, we'll denote by $\mathbb{R}^+$ the nonnegative reals i.e. $t \geq 0$. Likewise $\mathbb{R}^-$ will denote the nonpositive reals.

Definition 6.2.

1. The *orbit* or *trajectory* of a point $x \in E$ is the set of all points that are in the image of $\Phi_t(x)$ for some $t \in \mathbb{R}$. This is the set
$$\Gamma_x = \{y \in E | \Phi_t(x) = y \text{ for some } t \in \mathbb{R}\}.$$
2. The *forward* (*backward*) orbit or trajectory of a point $x \in E$ is the set of all points in the image of $\Phi_t(x)$ for $t \geq 0$ ($t \leq 0$). This is the set
$$\Gamma_x^{\pm} = \left\{y \in E | \Phi_t(x) = y \text{ for some } t \in \mathbb{R}^{\pm}\right\}.$$

The simplest orbits are just critical points. These orbits are just the points themselves. A periodic orbit is a closed loop - these orbits correspond to periodic solutions, and we have that for some $T > 0$, $\Phi_{T+s}(x) = \Phi_s(x)$ for all $s \in \mathbb{R}$.

Definition 6.3. A set $D \subseteq E$ is called *invariant* if $\Phi_t(D) \subseteq D$ for all $x \in D, t \in \mathbb{R}$. A set is called *forward invariant* (*backward invariant*) if it is invariant for all $t \in \mathbb{R}^+$ ($\mathbb{R}^-$).

As a straightforward class of examples, we have that when we start with a linear constant coefficient ODE

$$\dot{x} = Ax.$$

The (complete) flow map is explicitly given $\Phi_t(x) = e^{At}x$. Further, we have already verified that we have a complete flow, that is that Φ_t satisfies properties (1), (2), and (3) in Definition 6.1.

Another way we've been looking at ODEs is that they arise from some *vector field*, given by the function $F(x)$. For every point $x \in E$, we put a little arrow, whose magnitude and direction are given by $F(x)$. From here, we simply stick an $\dot{x}$ on the left, and we have an autonomous ODE. Such an autonomous ODE can give rise to a complete flow, by solving the ODE. In the opposite direction, if some stranger walks up to you in an alley and hands you a complete flow $\Phi_t(x)$ according to Definition 6.1, then you can generate a vector field and subsequently an ODE. The vector field that you obtain from a flow is found by:

$$F(x) := \frac{\mathrm{d}}{\mathrm{d}t}\Phi_t(x)\bigg|_{t=0}$$

Even better, is that the ODE you generate will be solved by the flow you started with. This should be reasonably obvious.

In order for us to have a complete flow, we need to have global existence of solutions. When does this happen? We state without proof two important cases.

Theorem 6.1. *If $F : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz and bounded, then $\Phi_t(x)$, the solution to $\dot{x} = F(x)$ defines a complete flow.*

Theorem 6.2. *If $F : \mathbb{R}^n \to \mathbb{R}^n$ is globally Lipschitz, on $\mathbb{R}^n$ then $\Phi_t(x)$, the solution to $\dot{x} = F(x)$ defines a complete flow.*

Example 6.1. Consider the planar 2×2 system for the pendulum equation

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\sin(x) \end{aligned} \quad =: F(x, y)$$

We claim that F is globally Lipschitz, with Lipschitz constant 1. This is fairly straightforward to see. The Jacobian of F at any point (x, y) in the plane is given by $DF(x, y) = \begin{pmatrix} 0 & 1 \\ -\cos(x) & 0 \end{pmatrix}$. We have that the norm of this matrix (using the matrix norm defined in tutorial 1) is given as

$$\|DF\| = \sup_{p^2+q^2=1} \left| DF(x, y) \begin{pmatrix} p \\ q \end{pmatrix} \right| = q^2 + \cos^2(x)p^2 \leq p^2 + q^2 = 1.$$

We then appeal to the proof of Proposition 5.2 from Part II's lecture notes to see that F is Lipschitz on the entire plane with Lipschitz constant 1.

We'll remark here too that this is not the whole story - that a vector field can fail both of these criteria, and still define a complete flow. As an example, consider the vector field given by

$$\dot{x} = F(x) := \begin{pmatrix} -x_1 \\ -x_1 x_2 \end{pmatrix}.$$

You can check for yourself that the function

$$\Phi_t(x) = \begin{pmatrix} x_1 e^{-t} \\ x_2 e^{x_1(e^{-t}-1)} \end{pmatrix}$$

is defined for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^2$, is C^1,and satisfies the group property. It is also straightforward to see that $F(x)$ is only locally Lipschitz, and not bounded (i.e. it fails the hypotheses of both the previous theorems).

Next we introduce an important tool called *reparametrization* which will, without too much trouble, allow us to create complete flows from any locally Lipschitz vector field. Further these complete flows will have the same phase curves as our original one (which might not be defined for all time), though they will naturally not solve the same ODE. The basic reason that allows us to do this is that we know that we have a maximal interval of existence J for any solution to a vector field on some open neighbourhood, and that all open intervals are 'morally' the same as $(-\infty, +\infty)$. Essentially what we do is change our independent variable so that if $J = (\alpha, \beta)$ is our maximal interval of existence, then we set a new independent variable $\tau = \tau(t)$ such that $\tau(\alpha) = -\infty$ and $\tau(\beta) = +\infty$. Then we'll set our phase curves $y(\tau(t)) = x(t)$ and we'll have a new ODE $y'(\tau) = \tilde{F}(\tau)$ that y solves.

We can even explicitly write down the function $\tau(t)$, the reparametrization as follows:

$$\tau = \int_{t_0}^{t} 1 + |F(x(s))| \mathrm{d}s$$

This function gives us a one to one map from the interval $J = (\alpha, \beta)$ to $(-\infty, +\infty)$. Then we set $y(\tau(t)) = x(t)$, and we see that $y(\tau)$ satisfies the new ODE (we use $'$ to denote $\frac{\mathrm{d}}{\mathrm{d}\tau}$)

$$y'(\tau) = \frac{F(y)}{1 + |F(y)|},$$

which is locally Lipschitz if F is.

The point of all this is that from now on we can just assume that any locally Lipschitz vector field will give us a complete flow (by reparamterizing time) as necessary. Thus we will largely just drop the word 'complete' from the description of a flow and simply refer to the solution map of an ODE as a 'flow' though we will mean 'complete flow'.

7. Flows on $\mathbb{R}$

This section is really more of an extended example, but the idea is to get used to the terminology we've been introducing and discussing in the last weeks. In the end we will derive a full classification of the possible dynamics of *any* autonomous flow (linear, nonlinear, whatever) on the real line, and further we will see that the underlying geometry of the real line fundamentally determines the possible behaviour of a dynamical system (a flow, the solutions to an ODE) on it. We'll begin with an example.

Example 7.1. Consider the ODE

$$\dot{x} = f(x) = x^2 - 1 \tag{7.1}$$

where $x \in \mathbb{R}$. We want to sketch the 'phase line' of this ODE. That is we want to draw the 'phase portrait'. We see right away that there are two critical points $x = \pm 1$. We have that at these critical points, we have stationary solutions, i.e. $x(t) = \pm 1$ is a solution to the ODE, and it is constant. This means that under the 'flow of time' (i.e. if we were to consider the solutions to the ODE as a one parameter family of maps from $\mathbb{R}$ to itself), these points are 'fixed'. This motivates the following definition:

Definition 7.1. A critical point of a system of ODEs is called a *fixed point* or *equilibrium point* of the associated flow Φ_t.

Moving on, we have that the right hand side of our ODE $x^2 - 1 < 0$ when $x \in (-1, 1)$ and greater than 0 when $|x| > 1$. We interpret this in our sketch of the phase portrait as arrows moving to the *right* when $f(x) > 0$ and to the *left* when $f(x) < 0$. Putting this all together we have the following sketch of the phase line (see Figure 18).

FIGURE 18. A sketch of the phase line of eq. (7.1), $\dot{x} = x^2 - 1$

Example 7.2. Here we consider another example. Consider the ODE on $\mathbb{R}$ given by

$$\dot{x} = f(x) = x(x-1)(x-2) = x^3 - 3x^2 + 2x. \tag{7.2}$$

We have fixed points $x_* = 0, 1$ and 2. Moreover, we can apply the techniques from Section 2. The *linearisation* of our ODE at each of these critical points x_* is given by the system

$$\dot{y} = Df(x_*)y,$$

where $Df(x_*)$ is the 1×1 'matrix' of the Jacobian of f evaluated at $x_* = 0, 1$ and 2. That is the derivative of f, $f'(x)$, evaluated at $x_* = 0, 1$ and 2. That is the linearisation of the nonlinear ODE in eq. (7.2) at $x_* = 0, 1$, and 2 is given respectively by

$$\dot{y} = f'(0)y = 2y, \quad \dot{y} = -y, \quad \text{and} \quad \dot{y} = 2y.$$

Just as in the previous example we can pick points not equal to the critical points, and determine the sign of $f(x)$ there, and add our arrows accordingly. This produces the phase portrait seen in Figure 19.

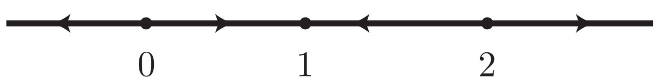


FIGURE 19. The phase portrait of the system given in eq. (7.2).

Now we're ready to do the general case. The secret is to *plot* $f(x)$ versus x. Then we note that the points where $f(x) = 0$ are the critical points, and then we can readily see which directions the arrows go by looking at the graph of $f(x)$. We can then put the arrows on the x-axis of our plot, and we have transformed it into the phase portrait of the flow corresponding to the solution to $\dot{x} = f(x)$ (see Figure 20).

We point out now that all of our examples thus far the orbits have all been *monotone* that is they either move towards a fixed point directly, or they move off to $\pm\infty$. This is not an accident. We have the following.

Theorem 7.1. *Periodic solutions don't exist in flows on the line.*

Proof. The proof is simply understanding what a periodic orbit means on the line. Given a flow on $\mathbb{R}$, $\dot{x} = f(x)$ we have that a periodic orbit must change direction and go back over itself at some point x_*. But this would mean that $\dot{x} = 0$ at this point, but this means that $f(x_*) = 0$ there, and so x_* is critical point, and so it can not be part of a periodic orbit. We conclude that periodic orbits do not exist for flows on the line. $\square$

Moving on, we can further exploit the geometry of $\mathbb{R}$ by using it to fully classify the types of (isolated) critical points that can occur. In Figure 20 we see an example of all the types of critical points that can occur. We either have flow moving in towards the point as in x_2, moving away from it as in x_4 or it is moving into and then away from as in x_1 and x_3. The first of these cases we call *stable*, the second is called *unstable*, and the final two are both called *semi-stable*.

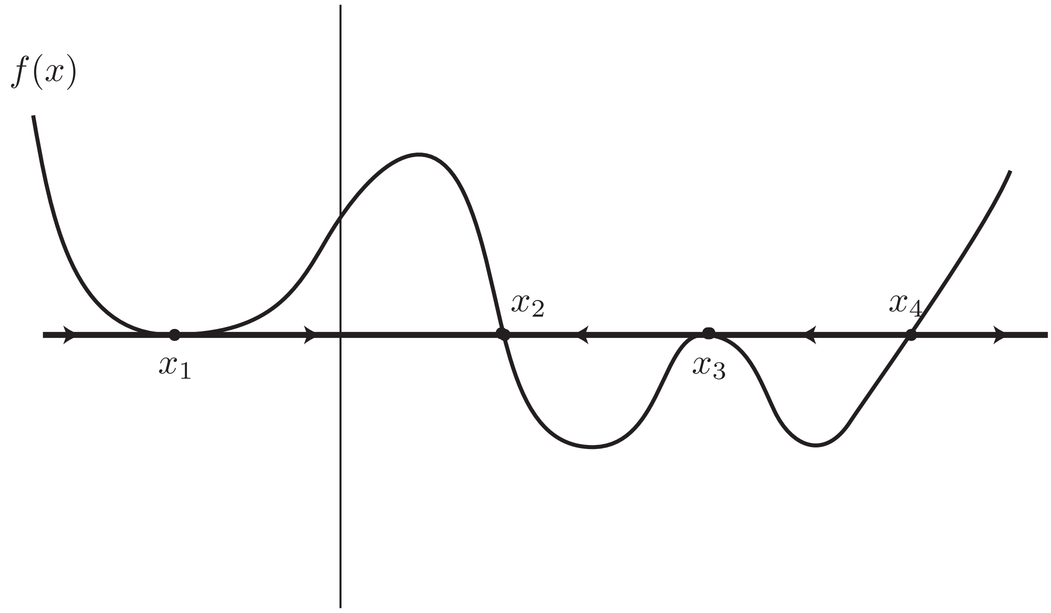


FIGURE 20. The general technique for drawing the phase portrait of an ODE on the line. The critical points are the roots of $f(x)$ denoted by x_i, and then the direction of the arrows in the phase line can be determined by whether $f(x)$ is positive or negative. This figure is also a classification of the types of critical points that can occur on a flow on a line x_1 and x_3 are semi-stable, while x_2 is stable, and x_4 is unstable.

Theorem 7.2. *Suppose the flow of*

$$\dot{x} = f(x)$$

Has only isolated critical points. Let x_* be a critical point of the flow. on $\mathbb{R}$. Then moving from left to right through x_*, the function $f(x)$ either goes from positive to negative, and x_* is of stable type, or $f(x)$ goes from negative to positive and x_* is of unstable type, or $f(x)$ has a turning point at x_* and x_* is of semi-stable type. Figure 20 shows visually the qualitative possibilities for critical points of flows on $\mathbb{R}$.

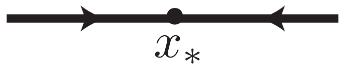
We remark that *orbits* of a flow on the line will be either critical points or the intervals between them, or perhaps interval going off to $\pm\infty$. We also remark that *hyperbolicity* of a critical point x_* say, of an ODE on $\mathbb{R}$ ($\dot{x} = f(x)$) is equivalent to $f'(x_*) \neq 0$.

Here is another example of how the *geometry* of the line $\mathbb{R}$ dictates what sort of qualitative behaviour is possible. Suppose that we have a flow on $\mathbb{R}$ which is the solution to the nonlinear ODE $\dot{x} = f(x)$, where $f(x)$ is some locally Lipschitz continuous function. Now suppose we had two critical points, x_1 and x_2, and no other critical points on the line. Further suppose that x_1 is of stable type. *Then x_2 can not be of stable type.* The reason for this is the intermediate value theorem (see Figure 22). Likewise if x_1 is unstable, then x_2 can not also be unstable.

Finally, we close this section with a 'full' classification of the types of flows that can occur on the line.

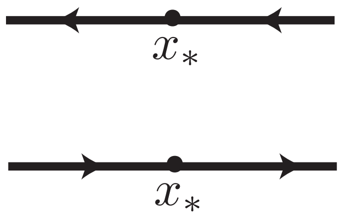
Theorem 7.3. *Suppose that you have a flow $\mathbb{R}$ which is the solution to the ODE:*

$$\dot{x} = f(x)$$



(A) A stable critical point

(B) An unstable critical point



(C) The two possibilities for critical points of semi stable type

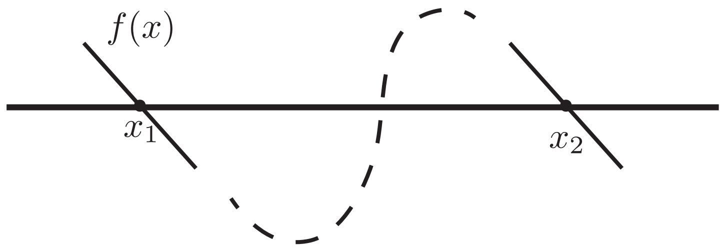


FIGURE 22. If x_1 is of stable type, then because of the intermediate value theorem, then the next critical point on the line x_2 can not be of stable type.

where $f(x)$ is a Lipshitz continuous function from $\mathbb{R}$ to $\mathbb{R}$. Suppose that you had only isolated critical points, and suppose that you knew the number and type of critical points as well as their relative order to each other. Then any other flow with the same characteristics 'looks like' (in the same sense as the Hartman-Grobman theorem) the given flow.

What this theorem says is that from a certain point of view (the qualitative one) knowing the number, type and order of the critical points is sufficient to fully qualitatively understand a flow on $\mathbb{R}$.

8. α and ω-Limit Sets

Another thing that we are interested in is what happens to a point, (or a collection of points) as time moves off to $\pm\infty$. In order to talk about this asymptotic behaviour, we introduce some more notions. Again for this section we let $\Phi_t(x)$ be the (now complete) flow associated with the vector field $\dot{x} = F(x)$.

Definition 8.1. A point y is called an *asymptotic limit point* of a forward orbit Γ_x^+ if there exists a sequence of t_js, $t_1 < t_2 < \cdots < t_k < \cdots$ such that $t_k \to \infty$ and $\Phi_t(x) \to y$ as $k \to \infty$.

Likewise we could define a *backward asymptotic limit point* by simply requiring that our sequence of t_js decrease to negative infinity.

Definition 8.2. The *ω-limit set* (read 'omega limit set') of a point x is the set of all asymptotic limit points of x. It is denoted $\omega(x)$.

In the opposite direction, we have *the α-limit set* which is defined as the set of all backwards asymptotic limit points of x and is denoted $\alpha(x)$.

This next example will not only give us a feel for what the ω-limit sets look like, but it will also introduce a very useful technique for analysing certain non-linear ODEs.

Example 8.1. Consider the planar autonomous (dynamical) system:

$$\begin{aligned} \dot{x} &= x(1-x^2-y^2) - y \\ \dot{y} &= y(1-x^2-y^2) + x. \end{aligned} \tag{8.1}$$

In order to get a better feel for what is going on, we need to look at this guy in polar coordinates. 'Recall' in polar coordinates we have:

$$\begin{aligned} r &= \sqrt{x^2+y^2} & x &= r\cos\theta \\ \theta &= \arctan\left(\frac{y}{x}\right) & y &= r\sin\theta \end{aligned}$$

By substituting in we get

$$\begin{aligned} \dot{r} &= \frac{1}{r}(x\dot{x}+y\dot{y}) = r(1-r^2) \quad \text{and} \\ \dot{\theta} &= \frac{1}{r^2}(x\dot{y}-y\dot{x}) = 1. \end{aligned}$$

So the dynamics in polar coordinates are the (now decoupled) equations

$$\begin{aligned} \dot{r} &= r(1-r^2) \\ \dot{\theta} &= 1 \end{aligned} \tag{8.2}$$

which we can explicitly solve for each variable (by separation of variables in the r variable.) The solution with initial conditions (r_0, θ_0) is given by:

$$\begin{aligned} r(t) &= \frac{e^t}{\sqrt{e^{2t} - 1 + \frac{1}{r_0^2}}} \\ \theta(t) &= t + \theta_0 \end{aligned}$$

It is straightforward to see from eq. (8.2) that the only fixed point is the origin (where $r = 0$ and θ isn't defined). The following straightforward linearisation argument shows that this is an unstable focus. We compute the linearisation of the original system (in terms of x and y!) at the origin, getting

$$\begin{pmatrix} \dot{p} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix},$$

and we see that the eigenvalues of the Jacobian at the origin are $\lambda_\pm = 1 \pm i$.

We can also see that $\dot{r} = 0$ when $r = 1$. What this means is that the radius of the orbit isn't increasing with t. So in particular, we have that as θ is simply increasing at a constant rate, then we have that there will be a closed orbit which will be the unit circle in the plane. Further we can see that as $\dot{r} > 0$ when $r < 1$ and $\dot{r} < 0$ when $r > 1$ we have that orbits will spiral in or out to the unit circle. Which means that the unit circle will be the ω-limit set of any point in the plane save the origin. (See Figure 23 for a Mathematica plot of the phase portrait of this system.)

Definition 8.3. A periodic orbit γ that is the ω (or α)-limit set of a point $x \notin \gamma$ is called a *limit cycle*.

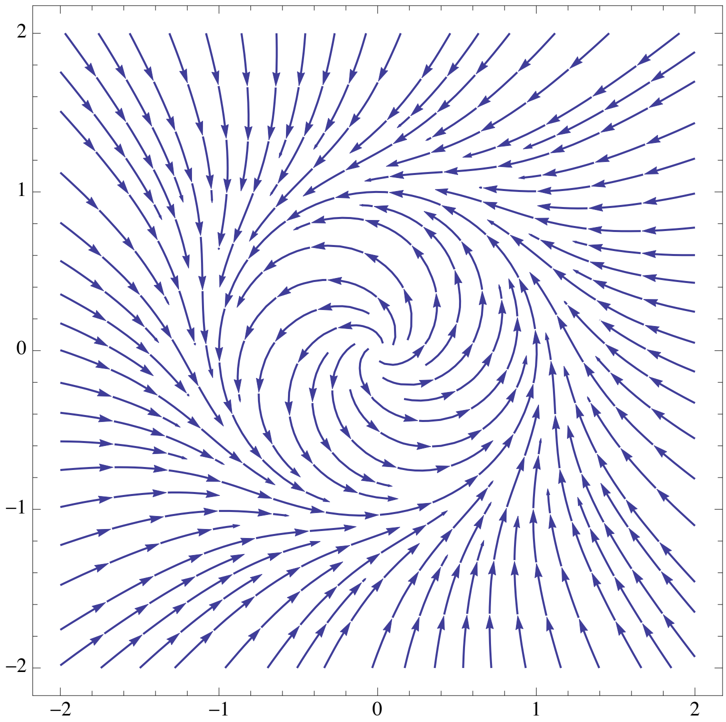


FIGURE 23. The phase portrait of the system in eq. (8.1).

Lemma 8.1. *An ω-limit set is closed. In fact*

$$\omega(x) = \bigcap_{T\geq 0} \overline{\Gamma}^{+}_{\Phi_T(x)},$$

where $\overline{\Gamma}^{+}_{\Phi_T(x)}$ is the closure of the forward orbit of $\Phi_T(x)$.

Proof. The proof is left as an exercise in the tutorials. □

Lemma 8.2. *An ω-limit set is invariant.*

Proof. If $y \in \omega(x)$ then there exists a sequence t_k such that $\Phi_{t_k}(x) \to y$ as $k \to \infty$. Further, the group property, and continuity means that for every $s \in \mathbb{R}$ we have that $\Phi_s(\Phi_{t_k}(x)) = \Phi_{s+t_k}(x) \to \Phi_s(y)$, so $\Phi_s(y) \in \omega(x)$ for every $s \in \mathbb{R}$, so $\omega(x)$ is invariant. □

We'll close this section out with some more vocabulary. You should familiarise yourself with the terminology so that we can talk about the limit points of dynamical systems with more ease.

Definition 8.4. A set N is called a *trapping region* if it is compact (closed and bounded) and $\Phi_t(N) \subseteq \text{Int}\,(N)$, where $\text{Int}\,(N)$ means the *interior* of N, or all the points $x \in N$ such that there is an $r > 0$ with $B_r(x) \subseteq N$.

If you have a compact set N and can show that everywhere on the boundary, the vector field points inward, then you have a trapping region.

Definition 8.5. A set Λ is called an *attracting set* if there is a (compact) trapping region N, with $\Lambda \subseteq N$ so that

$$\Lambda = \bigcap_{t>0} \Phi_t(N).$$

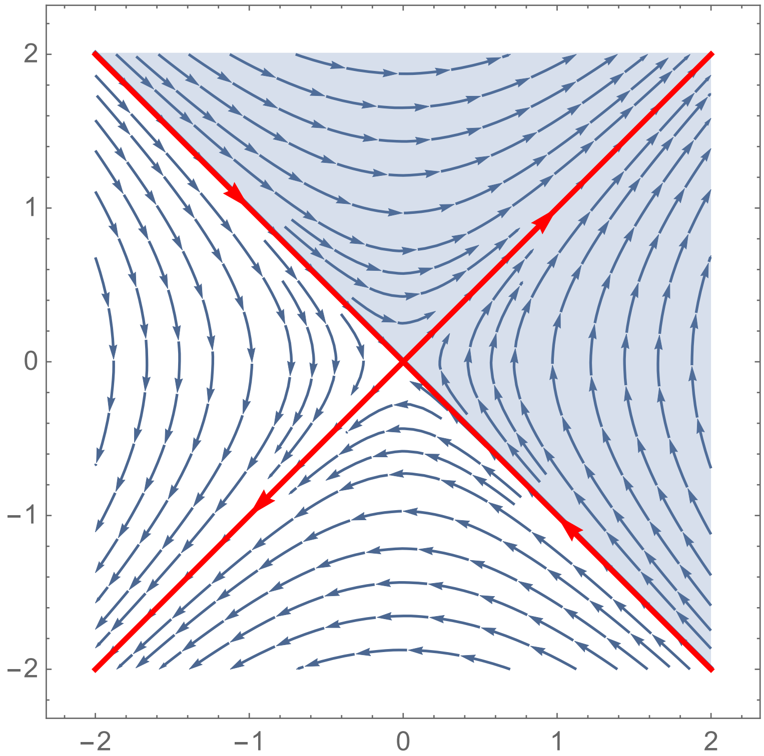


FIGURE 24. The shaded region represents $\mathcal{H}$, the basin of attraction for the unstable eigenspace in the right half plane.

Definition 8.6. The *basin of attraction* of an invariant set Λ which we'll write as $W^s(\Lambda)$ is the set of all points x such that the distance between $\Phi_t(x)$ and Λ:

$$d(\Phi_t(x), \Lambda) := \inf_{y\in\Lambda} |\Phi_t(x) - y| \to 0 \quad \text{as } t \to \infty$$

Remark. As a remark, we note that *any* invariant set can have a basin of attraction. Not just attracting sets. As an example, consider the linear system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{8.3}$$

The unstable eigenspace in this system is

$$\mathbb{E}^u = \{(k,k) | k \in \mathbb{R}\}$$

i.e. space spanned by $(1,1)$. The half line $k > 0$ (i.e. the unstable eigenspace in the right half plane), is invariant, and it is reasonably straightforward to see that the basin of attraction for this set is the half plane (see Figure 24):

$$\mathcal{H} = \left\{(x,y) \in \mathbb{R}^2 | y \geq -x\right\}.$$

It is worth noting that this includes the stable subspace, as well as the point 0. This is because in the definition of the basin of attraction, you are allowed to approach any point on the invariant set. So while most points tend to the line $y = x$ for $x \gg 0$, the points on the stable subspace get arbitrarily close to the origin under the flow.

If Λ is an attracting set with trapping region N, then

$$W^s(\Lambda) = \bigcup_{t\leq 0} \Phi_t(N)$$

or the backward flow of the trapping region N.

Definition 8.7. A set Λ is an *attractor* if it is an attracting set and there is an x such that $\Lambda = \omega(x)$.

Now let's look at some examples that can tease out some more information from these definitions.

Example 8.2. Consider the dynamical system in polar coordinates

$$\begin{aligned}\dot{r} &= r(1-r) \\ \dot{\theta} &= 1\end{aligned} \tag{8.4}$$

with the phase diagram shown in Figure 25.

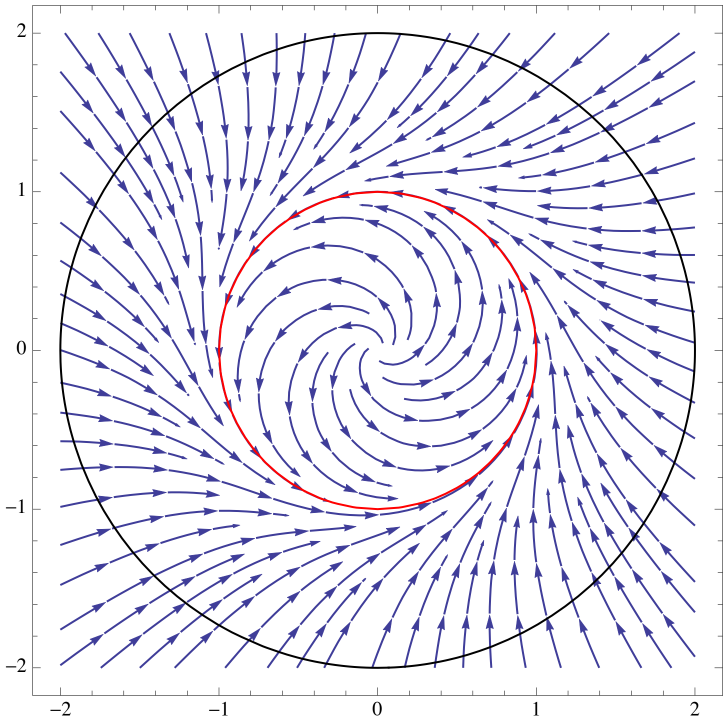


FIGURE 25. The phase portrait of the system in eq. (8.4).

We claim that the disc of radius 2 about the origin $D_2 = \{(r,\theta)|r \leq 2\}$ is a trapping region. On the boundary (the black circle) we have that $\dot{r} < 0$ so the vector feild is pointing inward everywhere. The point is that you can have quite complicated behaviour and lots of interesting dynamics within a trapping region, specifically more than just a critical point. It is also relatively straightforward to see that the set $D_1 = \{(r,\theta)|r \leq 1\}$ is an attracting set, with trapping region D_2. The basin of attraction of D_1 is the set of all points in $\mathbb{R}$. Finally, we point out that D_1 is *not* an attractor by the definitions we've just introduced, for the entire thing is not an ω-limit set of any point. We do have though that the unit circle is an attractor, with trapping region the annulus $R = \{(r,\theta)|\frac{1}{2} \leq r \leq \frac{3}{2}\}$. (We could have chosen any closed annulus around the unit circle).

9. A more complicated example

In this section we'll explore some of the more complicated asymptotic behaviour that can occur in planar 2-D systems. Limit cycles mean that periodic orbits can occur as ω-limit sets of nonlinear ODEs in the plane. We know from our analysis of linear 2-D systems points can also be ω-limit sets. This example will show that other possibilities can occur as well. (We will delay for the time being the full description of what is possible for ω-limit sets of nonlinear ODEs in the plane). Consider the system

$$\begin{aligned} \dot{x} &= y + x^2 - \frac{1}{4}x(y - 1 + 2x^2) \\ \dot{y} &= -2(1+y)x. \end{aligned} \tag{9.1}$$

With a little bit of work, we have the following table describing the critical points, and the dynamics of their linearisation (notice that they are all hyperbolic).

Critical Point	Jacobian	Eigenvalues	Classification
$(-1,-1)$	$\begin{pmatrix} -3 & \frac{5}{4} \\ 0 & 2 \end{pmatrix}$	$-3, 2$	Saddle
$(0,0)$	$\begin{pmatrix} \frac{1}{4} & 1 \\ -2 & 0 \end{pmatrix}$	$\frac{1}{8}\left(1 \pm i\sqrt{127}\right)$	Unstable Focus
$(1,-1)$	$\begin{pmatrix} 1 & \frac{3}{4} \\ 0 & -2 \end{pmatrix}$	$-2, 1$	Saddle
$(2,-1)$	$\begin{pmatrix} -\frac{3}{2} & \frac{1}{2} \\ 0 & -4 \end{pmatrix}$	$-4, -\frac{3}{2}$	Stable Node

It is also straight forward to see that the line $y = -1$ is invariant (this is because $\dot{y} = 0$ along it. It is maybe not so straightforward (but true) to see that the parabola $y = -2x^2 + 1$ is also invariant. To see this, observe that

$$\left.\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right|_{y=-2x^2+1} = \left.\frac{-2(1+y)x}{y + x^2 - \frac{x}{4}(y - 1 + 2x^2)}\right|_{y=-2x^2+1} = \frac{-2x(2-2x^2)}{-2x^2+1+x^2} = -4x.$$

So on this parabola, the vector field is pointing in the direction $-4x$. We also have that the tangent line of the curve $y = -2x^2 + 1$ is $-4x$. So the vector field is pointing in a direction parallel to the tangent line of this curve along it. This means that the curve is a phase curve, and is hence invariant.

The flow along the parabola $y = -2x^2+1$ above the line $y = -1$ goes from the saddle at $(-1,-1)$ to the saddle at $(1,-1)$, 'around' the unstable focus at $(0,0)$. Since we know that trajectories can not cross (we're in an autonomous system) this creates a region around the origin where the orbits are 'trapped', thus if we choose an initial condition near $(0,0)$ (say $x_0 = (0.1, 0.1)$, we see that we have the flow spirals outward getting closer and closer to the edge of this region, but never hitting either the parabola or the line $y = -1$ between -1 and 1. We can thus conclude that the $\omega-$limit set of this point is the *union* of the line segment $[-1,1] \times \{0\}$, (this is the set of points $\{(x,y) \in \mathbb{R}^2 | |x| \leq 1, y = 0\}$, and the parabola $P = \{(x, -2x^2+1) | |x| \leq 1\}$. Call this set $\mathcal{S}$. We have that the ω-limit set of all the points inside the region bounded by $\mathcal{S}$ is $\mathcal{S}$, while the ω-limit set for the points on $\mathcal{S}$ itself is either the point $(-1,1)$ or the point $(1,-1)$. For completeness, we have that the $\alpha-$limit set of the points inside $\mathcal{S}$ is the origin, while for those points on $\mathcal{S}$ it is again the saddle points on the line $y = -1$. See Figure 26 for a sketch of the phase portrait.

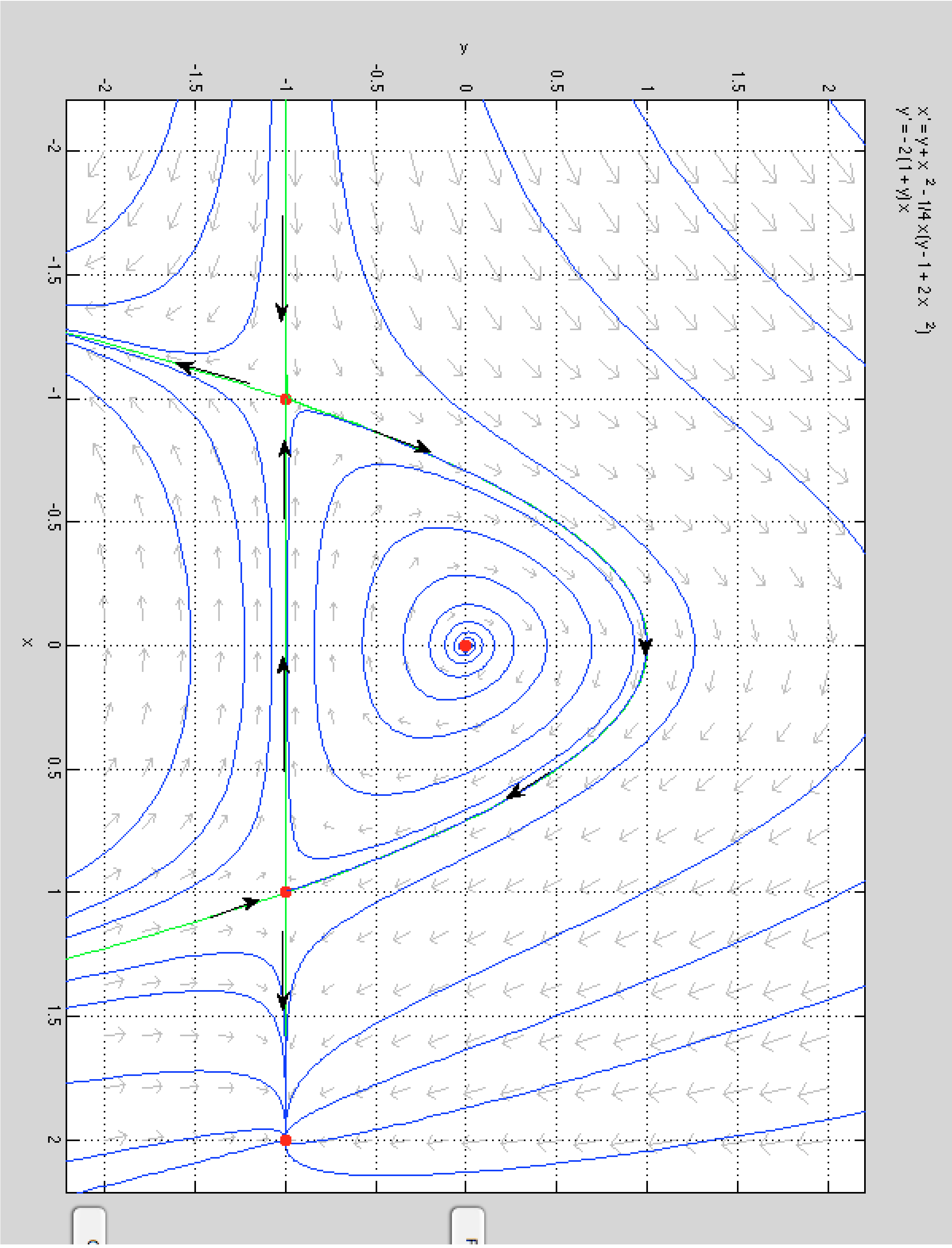


FIGURE 26. The phase portrait (rotated 90 degrees) for eq. (9.1). We have that the parabola $y = -2x^2+1$ and the line $y = -1$ are both invariant. The presence of an unstable equilibrium point inside these two curves, means that the set $\mathcal{S}$, boundary of the region that they enclose, is the $\omega-$limit set for all the points inside it. Red dots represent equilibrium points, while the green curves represent invariant sets. The arrows tell you which direction along the invariant sets solutions will travel, and the blue curves are solutions (found using pplane).

10. Stability

We've discussed linear stability - and the linearised stability of a critical or equilibrium point of a nonlinear system. In this section we're going to move onto nonlinear stability. In what follows, we'll be considering a dynamical system (a vector field, a system of ODEs) which is given by $\dot{x} = F(x)$ and a (complete) flow map $\Phi_t(x)$ which we will think of as the solution. In this section, we will denote a critical point of our flow by x_* (hopefully the context makes it clear what's what).

Definition 10.1. We say that x_* is *Lyapunov stable* (or *stable in the sense of Lyapunov*) if for every neighbourhood N around x_* (that is for every $r > 0$ there is a ball of radius r containing x_*) - for every neighbourhood N there is another neighbourhood M contained in N such that if $x \in M$ then $\Phi_t(x) \in N$ for all $t \geq 0$.

This says that if you start 'near' a critical point x_* then you stay 'near' it. You are allowed to leave the neighbourhood M, but for every N there must be a neighbourhood M that will be 'trapped' so to speak by the flow.

Example 10.1. A simple example of this is the following linear system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -1 & 10 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{10.1}$$

It is easy enough to see that solutions are given by

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^{-t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} 10 \\ 1 \end{pmatrix},$$

where $C_{1,2}$ are constants. In this example, the origin is asymptotically linearly stable as all the eigenvalues are negative, but, depending on where we start, we won't just go directly to the origin. For example if we start on the inner ellipse in Figure 27, the solution with such an initial condition might leave the region enclosed by the outer ellipse, but it will always stay within the outer ellipse (and eventually the solution will tend towards the origin).

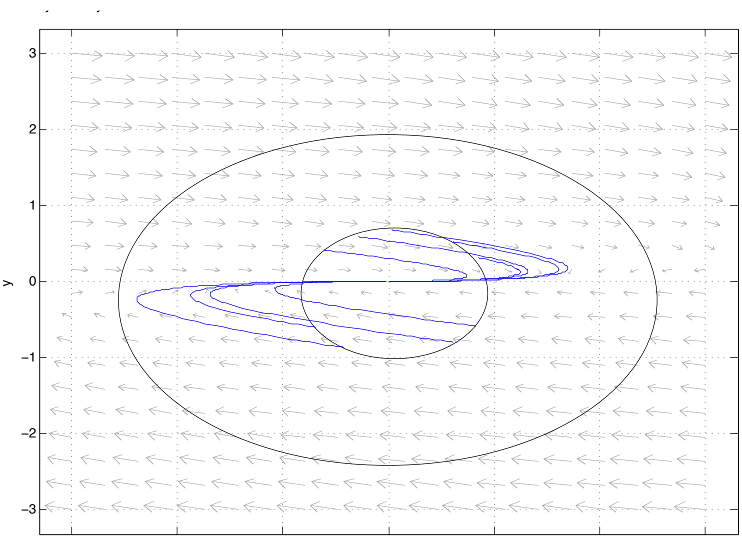


FIGURE 27. The phase portrait near the origin of eq. (10.1)

The phrase 'eventually tend towards the origin' that was used in the last example is a motivation for the extension of the concept of asymptotic linear stability to the current non-linear regime, however, we need to be a bit careful, as things can get a bit funny in some of these non-linear examples. To illustrate what I mean, here is another example:

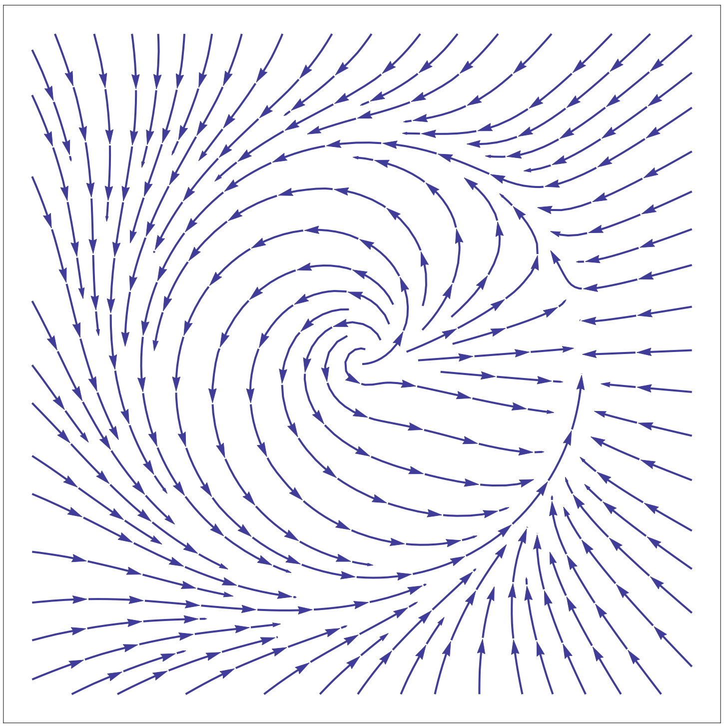


FIGURE 28. The phase portrait near the origin of eq. (10.2)

Example 10.2. Consider the following nonlinear system in polar coordinates:

$$\dot{r} = r(1-r) \quad \text{and} \quad \dot{\theta} = \sin^2\left(\frac{\theta}{2}\right). \tag{10.2}$$

It's easy enough to see that there are exactly two critical points, one at (0,0) and one at (1,0). Further, just as with the example motivating the definition of a limit cycle, it is easy enough to see that for all $r > 0$ we have that $r \to 1$. We also have that $\dot{\theta} > 0$, so we have that our solutions will be winding around the origin. What we can see from Figure 28 is that all solutions will 'eventually tend towards' the point $(1,0)$, but if we were to start just above it, but still on the unit circle, we would travel all the way around the unit circle before settling back (eventually) to the point $(1,0)$. That is, we can find a neighbourhood N containing the point $(1,0)$, such that there is no such neighbourhood M so that all the initial conditions in M will stay in N. As an example, simply choose the ball of radius $\frac{1}{100}$ about the point $(1,0)$. In other words the point $(1,0)$, even though it 'attracts' (eventually) all the nearby points - is not Lyapunov stable. Evidently this is bad, and so we will need to rethink our intuition - and change our definition of (nonlinear) asymptotic stability.

Definition 10.2. A critical point of a flow x_* is *asymptotically stable* if it is

(1) Lyapunov stable, and
(2) there is a neighbourhood $\tilde{N}$ of x_* such that all $x \in \tilde{N}$, $\Phi_t(x) \to x_*$ as $t \to \infty$.

Note that (1) is included to get rid of examples of the previous type and (2) is our original intuition of asymptotic stability. Further, it should be noted that the neighbourhoods used to prove asymptotic stability may have nothing in common with the neighbourhoods used to establish Lyapunov stability (hence denoting them with a ˜).

So, as has been intoned before, when we have a critical point, and the linearised system is hyperbolic, the behaviour of the linear system should be 'close' to the behaviour of the non-linear system. A nice way to make this precise is the following:

Theorem 10.1 (Asymptotic linear stability implies asymptotic stability)**.** *Suppose that $F(x)$ is a C^1 vector field, with a critical point at x_*. Further, suppose that the linearisation is asymptotically linearly stable (that is all the eigenvalues of $DF(x_*)$ have negative real part). Then x_* is asymptotically stable (in the full non-linear system).*

Remark. We will prove this theorem directly, but first we remark that this is not strictly necessary. We have the Hartman-Grobman theorem, (Theorem 2.1) which tells us that in a neighbourhood of a hyperbolic critical point our linear system is a 'good' approximation of the non-linear system. Hence if our linearisation is asymptotically linearly stable (i.e. if the constant coefficient matrix of the Jacobian has only eigenvalues with negative real part), it is hyperbolic, and in this neighbourhood we have non-linear asymptotic stability (we haven't proven this, but you should think about why the origin is a Lyapunov stable critical point in a constant coefficient linear system where all the eigenvalues of the matrix have negative real part).

The point is that we haven't proven the Hartman-Grobman theorem rigorously (nor will we), but we can prove this partial result without appealing to it, and moreover we get a feel for the types of arguments used in nonlinear dynamical systems.

Proof. First, we can Taylor expand $F(x)$ near the critical point. Set $y := x - x_*$, and $A := DF(x_*)$ and we have

$$\begin{aligned} F(x) &= F(x_*) + DF(x_*)(x - x_*) + g(x - x_*) \\ &= Ay + g(y) \end{aligned} \tag{10.3}$$

where $g(y)$ is the higher order terms in the Taylor expansion. Further, the fact that F is C^1 means that

$$\lim_{y\to 0} \frac{|g(y)|}{|y|} \to 0$$

(this is called being $o(y)$ - read 'little-oh' of y). This follows from the definition of differentiability of F and the definitions of $g(y)$ and y. Differentiability of F means that

$$\lim_{x\to x_*} \frac{|F(x) - DF(x_*)|}{|x - x_*|} = 0$$

but this is just

$$\lim_{y\to 0} \frac{|g(y)|}{|y|} = 0.$$

Another way of saying this is that for every $\varepsilon > 0$ there is a δ such that if $|y| < \delta$ then $|g(y)| < \varepsilon|y|$.

Returning, we have that y must solve the differential equation

$$\dot{y} = Ay + g(y). \tag{10.4}$$

In order to solve this equation, we use the technique of variation of parameters. We write $y(t) = e^{At}\eta(t)$ where η is some function of t that we need to determine. Differentiating the expression for $y(t)$ and using (10.4) we have that $\eta(t)$ must satisfy the differential equation

$$\dot{\eta}(t) = e^{-At}g(y). \tag{10.5}$$

Formally integrating gives

$$\eta(t) - \eta(0) = \int_0^t e^{-As}g(y(s))\mathrm{d}s \tag{10.6}$$

and substituting in for η and isolating the $y(t)$ function gives an integral equation for $y(t)$

$$y(t) = e^{At}y_0 + \int_0^t e^{A(t-s)}g(y(s))\mathrm{d}s. \tag{10.7}$$

The next step is to try and bound the right hand side of equation (10.7). The first term, we have a bound from (a generalisation of) question 3 from the week 5 tutorial, we have that since A has only negative eigenvalues by hypothesis, then there is a constant K and a real $\alpha > 0$ so that $|e^{At}y_0| \leq Ke^{-\alpha t}|y_0|$. Now the fact that $g(y)$ is $o(y)$ means that for any ε there is a δ such that if $|y| < K\delta$ then $|g(y)| < \varepsilon|y|$. Putting everything together we have

$$|y(t)| \leq Ke^{-\alpha t}|y_0| + K\varepsilon \int_0^t e^{-\alpha(t-s)}|y(s)|\mathrm{d}s \tag{10.8}$$

or (why not?)

$$e^{\alpha t}|y(t)| \leq K|y_0| + K\varepsilon \int_0^t e^{\alpha s}|y(s)|\mathrm{d}s. \tag{10.9}$$

Now we assume that $|y_0| \leq \delta$ and set $\xi(t) = e^{\alpha t}|y(t)|$. Then

$$\xi(t) \leq K\delta + K\varepsilon \int_0^t \xi(s)\mathrm{d}s$$

which means that we can use Grönwall's lemma to say that

$$\xi(t) \leq K\delta e^{K\varepsilon t} \Rightarrow |y(t)| \leq K\delta e^{-(\alpha - K\varepsilon)t}.$$

So, if $K\varepsilon < \alpha$, then $|y| \to 0$ and stays bounded below $K\delta$ for all $t \geq 0$. What this says is that if M is the ball of radius δ, then N is the ball of radius $K\delta$, and so we have Lyapunov stability. The fact that $|y| \to 0$ then means that we also have asymptotic stability. □

As per the discussion before the proof, this theorem is directly implied by Theorem 2.1. But, we don't need to introduce extra terminology (like topologically conjugacy) in order to prove it. Also the theorem itself predates the Hartman-Grobman theorem by about 50 years. Indeed this theorem is sometimes called Lyapunov's first method. In some sense this theorem should reinforce the idea that it is much easier to get qualitative (asymptotic at least) behaviour of a nonlinear system near a hyperbolic critical point, than one which is not hyperbolic (hyperbolic - good, nonhyperbolic - bad).

11. Lyapunov functions

Another (very nice) way to show stability is through what's called a Lyapunov function. Lyapunov functions, are great - with the caveat that you may not always be able to find them. Again, for this section, we'll suppose that x_* is a critical point of a flow $\Phi_t(x)$ coming from a vector field $\dot{x} = F(x)$.

Definition 11.1. Suppose we can find a function $L : \mathbb{R}^n \to \mathbb{R}$ such that the following hold:

(1) $L(x_*) = 0$,
(2) $L(x) > 0$ for all $x \neq x_*$ in some neighbourhood N of x_* (i.e. some open ball of radius r about the critical point).
(3) $\frac{\mathrm{d}}{\mathrm{d}t}L(x) = \nabla L(x) \cdot \dot{x} = \nabla L(x) \cdot F(x) \leq 0$ (or < 0) for all $x \in N \setminus \{x_*\}$ and $t > 0$.

Then we call L a weak (or strong if < 0) *Lyapunov function.*

Lyapunov functions are important for the following reason:

Theorem 11.1.

(1) If there exists a weak Lyapunov function with respect to the point x_*, then the critical point is Lyapunov stable.
(2) If there exists a strong Lyapunov function with respect to the point x_*, then the critical point is asymptotically stable.

We're not going to prove this theorem, but I want to give you a couple of ideas about intuitively what is going on.

So the first thing to notice is that the first two conditions in the definition of a Lyapunov function, mean that $L(x)$ has a local minimum at the critical point. In this sense, Lyapunov functions have sometimes been described as 'energy' - or a generalisation of the concept of energy - level sets of $L(x)$ will be sort of like 'energy surfaces'. The next thing to point out is that the third condition can be thought of as the following: Make a new (in $n+1$) dimensional 'surface' that consist of the points $(x, L(x))$. The first two conditions means that you have a local minimum at x_*. The third condition says that if you were to 'lift' the flow from $\mathbb{R}^n$ to this surface, then the last coordinate would always be negative, that is, the flow is 'running down' the basin created by the Lyapunov function.

Another way of thinking of these guys is that in 2-dimensions, the level curves of $L(x)$ do what's called 'foliate' the neighbourhood N. That is you can think of N as being the disjoint union of the level sets of $L(x)$ (or parametrized by $L(x) = C$). The third condition says that along the level sets of the Lyapunov function, the arrows of the vector field are always pointing in to the critical point (in the case of a strong Lyapunov function). (None of these are complete proofs, by the way, they are just to give you an idea of what is going on). Let's do some examples.

Example 11.1 (Van der Pol equation)**.** The nonlinear second order ODE:

$$\ddot{x} - \beta(x^2 - 1)\dot{x} + x = 0 \qquad \beta > 0 \tag{11.1}$$

can be written as a system of first order equations by setting $\dot{x} = y$, and rewriting as:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \beta(x^2 - 1)y - x. \end{aligned} \tag{11.2}$$

You can see for yourself that $(0, 0)$ is the only critical point. Linearising, we have that the Jacobian of the van der Pol equation at the origin is

$$DF(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & -\beta \end{pmatrix}$$

which has eigenvalues

$$\lambda_{\pm} = \frac{1}{2}\left(-\beta \pm \sqrt{\beta^2 - 4}\right)$$

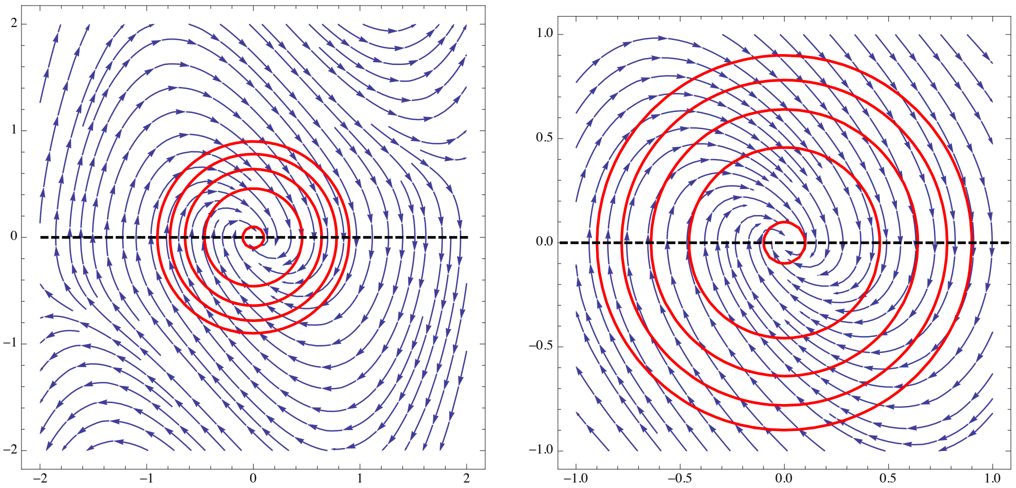
which will always have real part < 0. In this case, we can appeal to Theorem 10.1 to know that because the linearised system is asymptotically linearly stable, the origin in this case is asymptotically stable. Can we find a Lyapunov function? The answer is yes, and in this case it is very straightforward.

$$L(x, y) = x^2 + y^2$$

Clearly this has a local minimum at $(0, 0)$ and we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}L(x) = 2x\dot{x} + 2y\dot{y} = 2\beta(x^2 - 1)y^2 \leq 0$$

for x, y in a small enough neighbourhood of the origin.



(I) Some level sets (red) of a weak Lyapunov function. The vector field points inwards on the level sets, except at the line $y = 0$ (black, dashed).

(II) Zooming in on the origin, you can clearly see that the vector field points inwards on the level sets, except at the points where $y = 0$. This makes this a weak Lyapunov function.

FIGURE 29. The phase portrait of eq. (11.2).

Example 11.2. Let's do another example that isn't quite as straightforward. Consider the 2-D dynamical system

$$\begin{aligned} \dot{x} &= -x - 2y^2 \\ \dot{y} &= xy - y^3. \end{aligned} \tag{11.3}$$

Again we have that the origin is a fixed point, but this time at $(0, 0)$ we have that

$$DF(0,0) = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$$

so we can't say that the origin is asymptotically linearly stable, and we can't appeal to Theorem 10.1. In fact, this system isn't even hyperbolic. However, we can without too much trouble find a Lyapunov function. Let

$$L(x, y) = x^2 + 2y^2.$$

Then again, it is clear that $L(x)$ has a local minimum at $(0, 0)$. Further we have that

$$\frac{\mathrm{d}}{\mathrm{d}t} L(x) = -2(x^2 + 2y^4) < 0$$

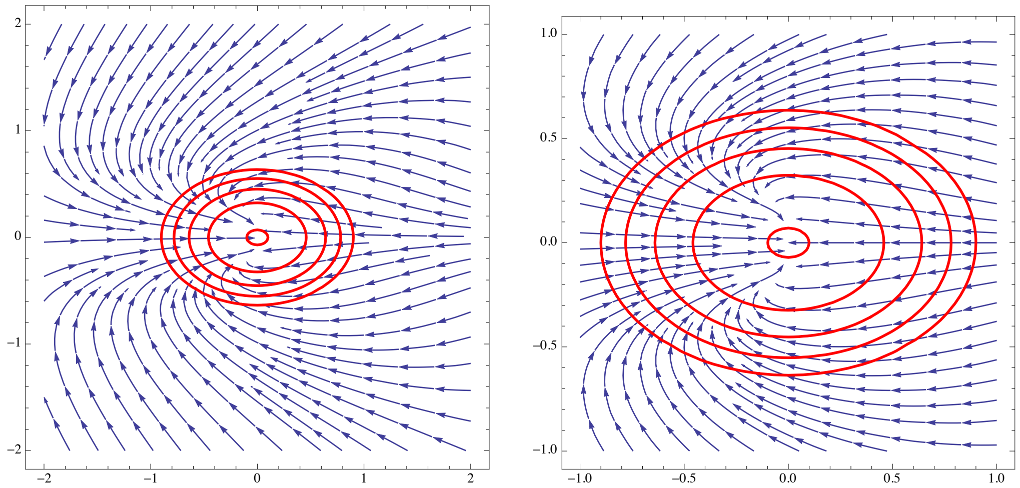
for all $x \neq (0, 0)$, so we have strong Lyapunov function, and the origin is asymptotically stable (nonlinearly - even though it is *not* asymptotically linearly stable).

12. A test for instability

So far we've mostly been talking about whether a critical point x_* is stable or not. Now we're going to talk about a similar test to determine whether a critical point is unstable.

Definition 12.1. A critical point which is not Lyapunov stable is called *unstable*.

If the critical point is hyperbolic, then it is enough that the linearisation be unstable to conclude that the critical point is unstable. This is the content of the Hartman-Grobman theorem. If the critical point is not hyperbolic, it can still fail to be stable, as we have already shown in a couple of examples so far. Now we introduce a new technique (sort



(I) Some level sets (red) of a strong Lyapunov function. The vector field points inwards on the level sets.

(II) Zooming in on the origin, you can clearly see that the vector field points inwards on the level sets.

FIGURE 30. The phase portrait of eq. (11.3).

of the inverse of the Lyapunov function method) to determine whether or not our critical point is stable.

Theorem 12.1. *Suppose that we can find a function $U : \mathbb{R}^n \to \mathbb{R}$ (which is differentiable) such that the following hold (again, $\dot{x} = F(x)$ is our C^1 vector field with critical point x_*):*

(1) $U(x_*) = 0$,

(2) In every neighbourhood N, of x_* (in every $B_r(x_*)$) there is at least one point at which $U(x) > 0$,

(3) $\dot{U}(x) > 0$ for all $x \neq x_*$ in some neighbourhood M of x_*.

If we can find such a function U, then x_* is an unstable critical point.

Again, we're going to skip the proof of this theorem, but let's do an example:

Example 12.1. Consider the ODE:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= x - y \sin x. \end{aligned} \tag{12.1}$$

We have that $(0, 0)$ is a critical point and the Jacobian at the origin is given by

$$DF(0,0) = \begin{pmatrix} 0 & 1 \\ 1 - y\cos x & \sin x \end{pmatrix}\Bigg|_{(x,y)=(0,0)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

So we have a saddle. Again, we can, from the Hartman-Grobman theorem conclude that we have an unstable critical point at the origin. Let's construct a function satisfying hypotheses (1)–(3) of Theorem 12.1 anyway. Consider the function

$$U(x, y) = xy.$$

Then $U(x, y)$ satisfies (1) and (2) right away, since along the line $y = x$ we have that $U(x, y) = x^2 > 0$. Now we check the third condition

$$\dot{U} = y\dot{x} + x\dot{y} = x^2 + y^2 - xy \sin x.$$

We claim that this function

$$G(x, y) = x^2 + y^2 - xy \sin x$$

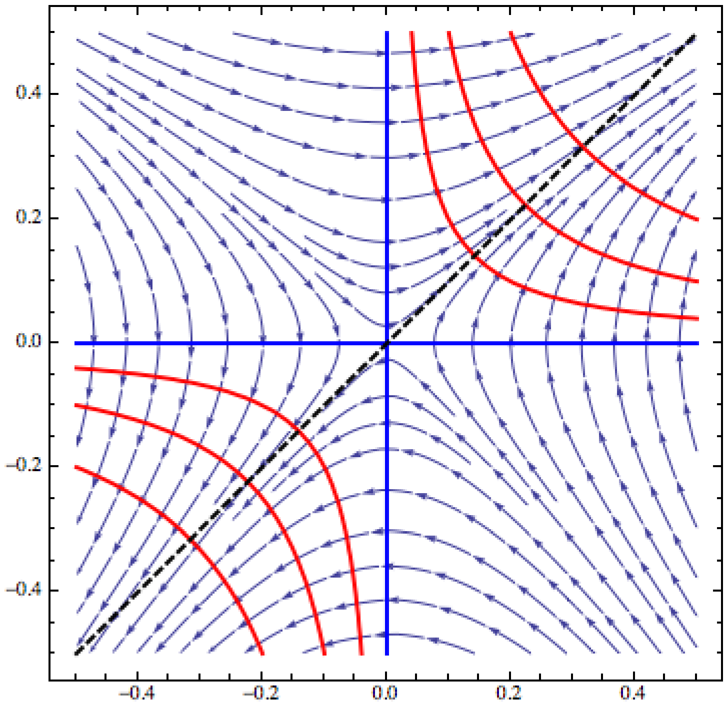


FIGURE 31. A plot of the vector field given in eq. (12.1). The red lines are the contour lines of $U(x, y) = xy$. The black dashed is where $U(x, y)$ is always positive.

has a local minimum at $(0, 0)$. To see this heuristically, Taylor expand $\sin x = x - \frac{x^3}{6} + h.o.t.$ Thus for small x, y we have that $G(x, y)$ will be dominated by the quadratic terms $x^2 + y^2$ and we can neglect the terms to third order (the rest). To see this formally, you can observe that $\nabla G(0, 0) = (0, 0)$ and the second derivative of G at the origin is given as

$$DG(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

and so we can use the second derivative test to conclude that G has a local minimum of 0 at the origin, and so $\dot{U} = G$ must be positive in some neighbourhood of the origin. See Figure 31 for a plot of the vector field of eq. (12.1) as well as the level sets of the function $U(x, y)$.

Example 12.2. In this next example, we will need the function $U(x, y)$ to establish the instability of the critical point. Consider the ODE $\ddot{x} = x^3$. Writing this as a system we have

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= x^3. \end{aligned} \tag{12.2}$$

It is clear that $(0, 0)$ is a critical point, but the linearisation at the origin is not helpful

$$DF(0,0) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

So we again consider the function $U(x, y) = xy$. We have that

$$\dot{U} = y\dot{x} + x\dot{y} = y^2 + x^4 > 0$$

for all $(x, y) \neq (0, 0)$. So even though we can't say anything with the linearisation, we have that the critical point is unstable.

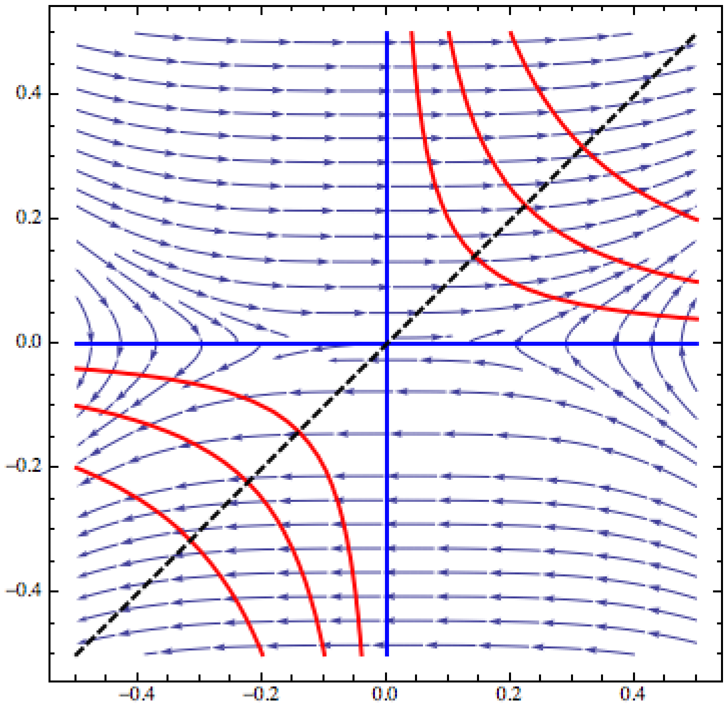


Figure 32. A plot of the vector field given in eq. (12.2). The red lines are the contour lines of $U(x,y) = xy$. The black dashed is where $U(x,y)$ is always positive.

13. LaSalle's Invariance Principle

This next example will give us some idea of what to do when we only have a weak Lyapunov function, as well as lead us up to the statement of a relatively famous theorem. Not to mention the fact that it is a fairly ubiquitous dynamical system in biological settings.

Example 13.1 (Logistic growth with delay)**.** The logistic growth equation is used to model the growth of a population when the environment is of finite size. We have

$$\dot{x} = rx(1-x).$$

Here r is the unabated growth rate or reproduction rate of the species, and 1 represents the (normalised) carrying capacity of the environment. Now suppose that we didn't have an instantaneous reproduction, that is, there was a little bit of delay - so that really $\dot{x} = rx(1-y)$ where $y(t)$ is $x(t)$ at some previous time. Then if we suppose that y will increase proportionally to the 'new' population we have that our system becomes

$$\begin{aligned} \dot{x} &= rx(1-y) \\ \dot{y} &= b(x-y). \end{aligned} \tag{13.1}$$

In this set up, b represents the 'birth rate' of the population, and r = births - deaths and so we have $0 < r < b$. We see that in eq. (13.1) we have two critical points, $(0,0)$ and $(1,1)$. Computing the Jacobian gives:

$$DF = \begin{pmatrix} r(1-y) & -rx \\ b & -b \end{pmatrix}$$

and so at $(0,0)$ we have that $DF(0,0) = \begin{pmatrix} r & 0 \\ b & -b \end{pmatrix}$ which is lower triangular, and has one positive and one negative eigenvalue, so this means we have a saddle at the origin.

Likewise, we have $DF(1,1) = \begin{pmatrix} 0 & -r \\ b & -b \end{pmatrix}$, which has eigenvalues $\frac{1}{2}\left(-b \pm \sqrt{b^2 - 4rb}\right)$ which will both have negative real parts, so we can use Theorem 10.1 to infer that there is some neighbourhood about $(1,1)$ and the critical point is asymptotically stable with respect to such a neighbourhood. We would like to know if this neighbourhood includes 'realistic' initial conditions. In particular, suppose that our initial conditions were $x_0 > 0$ and $y_0 > 0$, then we'd like to know if $\Phi_t(x_0, y_0) \to (1,1)$ as $t \to \infty$.

First, notice that the line $x = 0$ is an invariant set, so solutions that start with $x > 0$ can't cross it. Second, notice that along the line $y = 0$ in Q1, we have that $\dot{y} > 0$, so an orbit can't leave Q1 once it has entered it. That is to say the set $\mathrm{Q1} = \{(x,y) | x > 0, y > 0\}$ is invariant. To rephrase our original question, we have that $(1,1)$ is asymptotically stable, but we want to know if the *entire* set Q1 ends up there (i.e. is $\omega(Q1) = (1,1)$?).

Let's introduce the function (why not?):

$$L(x,y) = (x-1) - \log(x) + \frac{r}{2b}(y-1)^2. \tag{13.2}$$

We claim that this is a weak Lyapunov function for the point $(1,1)$ and for all of Q1. You can verify the first two parts yourselves, and now we look at

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} L(x,y) &= \dot{x} - \frac{\dot{x}}{x} + \frac{r}{b}(y-1)\dot{y} \\ &= -r(y-1)^2 \leq 0. \end{aligned} \tag{13.3}$$

And moreover we have that $\frac{\mathrm{d}}{\mathrm{d}t} L(x,y) < 0$ for all of Q1 except on the line $y = 1$. We are interested in computing $\lim_{t\to\infty} \Phi_t(x_0, y_0)$ for an arbitrary initial condition in Q1. We have that either $\Phi_t(x_0, y_0) \to \infty$ or $\omega(x_0, y_0)$. Since we have weak Lyapunov function on all of Q1, we can not have the former. Further if we take any sequence $0 < t_1 < t_2 < \cdots < t_k < \cdots$ we have that because L is decreasing in t we must have

$$L(x_0, y_0) \geq L(\Phi_{t_1}(x_0, y_0)) \geq \ldots > L(\Phi_{t_k}(x_0, y_0)) \geq \ldots \tag{13.4}$$

Does the sequence of values of L terminate? The answer is yes, because L has a lower bound in $Q1$ (it has a minimum at $(1,1)$). But this means that we must have that $\lim_{t\to\infty} \Phi_t(x_0, y_0)$ be contained in the set (call it Z) where $\frac{d}{dt}L = 0$. But it must also be invariant (ω-limit sets are invariant), and we can see that the only invariant set entirely contained in the line $y = 1$ is the critical point $(1,1)$. So what we've just shown is that *no matter what initial condition* (x_0, y_0) *we choose in Q1, we end up at the fixed point* $(1,1)$.

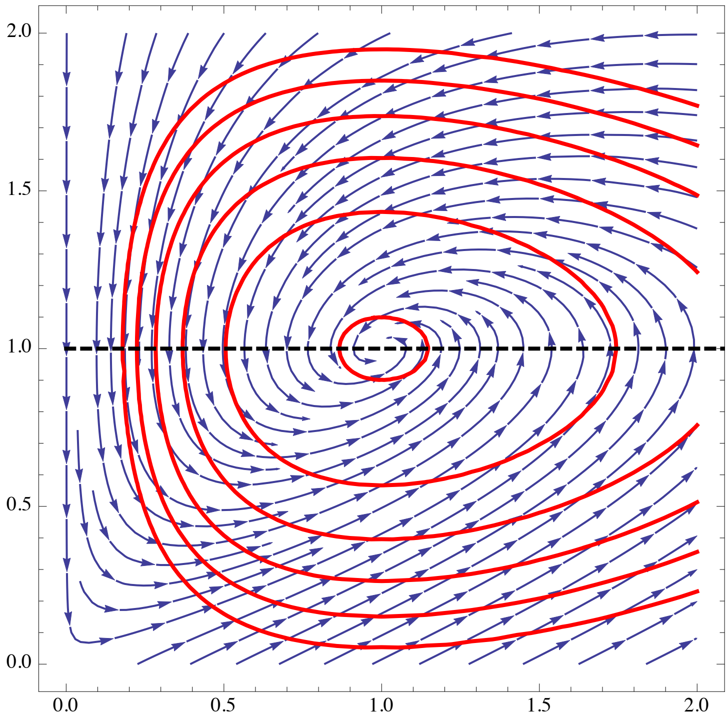


FIGURE 33. Some of the level sets (red) of the weak Lyapunov function given in eq. (13.2). The black dashed line is the set Z where $\frac{d}{dt}L = 0$ and we have that any ω-limit set must be contained in Z.

Restating this what we have just proved is the following

Theorem 13.1 (LaSalle's Invariance Principle)**.** *Suppoe that you have a weak Lyapunov function on some set U that is forward invariant. Let Z be the union of all complete orbits entirely contained in the set where $\frac{d}{dt}L(x) = 0$. Then for each $x \in U$ we have that $\omega(x) \subseteq Z$.*

In the previous example, we used some independent analysis to deduce that the whole first quadrant was forward invariant. From there we were able to apply LaSalle's invariance principle to gain some qualitative information regarding the ω-limit sets of the points in $Q1$. However often times it is not so obvious what to choose as a forward invariant set. But, if we have a weak Lyapunov function, then we can oftentimes use this to get a little bit more qualitative information than just Lyapunov stability. We have the following theorem.

Theorem 13.2. *Suppose that (x_*, y_*) is a critical point of a planar flow, and suppose that $L(x_*, y_*)$ is a weak Lyapunov function for (x_*, y_*) in the neighbourhood*

$$N_k = \{(x, y)|L(x, y) \leq k\}.$$

Suppose further that there are no other critical points in the neighbourhood N_k, and that the curves

$$N_a = \{L(x, y) = a|0 < a \leq k\}$$

are not phase curves of the flow. Then (x_*, y_*) is asymptotically stable.

Proof. The idea is to show that the set N_k is forward invariant. Then we appeal to LaSalle's invariance principle. Because none of the level sets of the Lyapunov function are complete orbits, we have that the only point in the set Z defined in the statement of

LaSalle's invariance principle is the point (x_*, y_*). So we know that it must be the ω-limit set of any point in N_k, i.e. it is asymptotically stable.

To see that N_k is forward invariant, choose a point $(x_1, y_1) \in N_k$. Suppose that $L(x_1, y_1) = a$. The forward orbit from (x_1, y_1) is not a phase path, and so it must either separate from the curve $L(x, y) = a$ immediately, or after following it for some time, but before completing a cycle and forming a closed curve. Because we have a weak Lyapunov function, when $\Gamma^+_{(x_1,y_1)}$ departs from the curve $L(x, y) = a$, it must move to the interior of N_a. But this means that the set N_k is forward invariant. $\square$

14. Hilbert's 16th Problem and the Poincaré–Bendixson Theorem

We have already classified all of the possible (qualitative) behaviours that we are going to see for flows on $\mathbb{R}$. Now, we'd like to do the same thing for flows in the plane. This is much harder, and in fact, a full classification of flows on the plane has not really been discovered yet. Actually, the problem is much worse than that. In 1900, at the International Congress of Mathematicians, one of the most 'famous' mathematicians of all time, David Hilbert (you might know him from Hilbert spaces), gave one of the most 'famous' mathematical speeches of all time, in which posed to the mathematics community, 23 problems that he thought would 'define mathematics for the 20th century'. One of these was the Riemann Hypothesis, and one was the one that we are going to talk about below. Of the 23 problems, 10 have seen resolutions that are accepted by consensus. Some have a partial resolution, and a few of the problems were formulated in too vague a way to ever have a real resolution. However three of the problems remain totally unsolved - these are Problems 8 (the Riemann Hypothesis), Problem 12, and Problem 16. The 16th problem is formulated in a couple of ways, we're (for obvious reasons) going to focus on the dynamical systems formulation.

Question (Hilbert's 16th Problem)**.** Consider the planar dynamical system

$$\begin{aligned}\dot{x} &= f(x, y)\\ \dot{y} &= g(x, y)\end{aligned}$$

where $f(x, y)$ and $g(x, y)$ are polynomials of degree d. Find an upper limit for the number of limit cycles in the system.

This problem has resisted the test of time, and in fact, there hasn't even been progress made (in terms of a proof) for when $f(x, y)$ and $g(x, y)$ are quadratic! The solution in this case is thought to be 4, but no proof or counterexample is known.

Okay, so classifying all the behaviour for planar ODEs is unknown, however, there is a very powerful theorem which enables us to determine what the *asymptotic* behaviour of such ODEs is. That is, we know what the ω-limit sets of flows in the plane can be. So while we aren't going to be able to fully determine all of the qualitative behaviours of a planar nonlinear ODE in the same way we did for the line, we will be able to determine asymptotically (i.e. as $t \to \pm\infty$) what is possible.

Theorem 14.1 (Poincaré-Bendixson)**.** *There's no chaos in 2 dimensions.*

This is nicely put, but a bit glib (and vague) for us, so what we'll actually prove is the following:

Theorem 14.2 (Poincaré-Bendixson)**.** *Suppose that Ω is a nonempty, closed and bounded ω-limit set of a complete flow in $\mathbb{R}^2$. Suppose that Ω does not contain any critical points of the flow. Then Ω is a periodic orbit.*

It is worth noting, that this type of theorem (or even more - a full classification of behaviours see the next theorem) in for nonlinear flow in $\mathbb{R}^3$ is essentially impossible. The reason for this is that chaotic dynamical system exist (and are even fairly prevalent). You may have heard of the 'strange' attractor in a Lorenz system. You know what attractor means, what 'strange' means in this case (among other things) is that the attractor doesn't have an integer dimension! - examples like this really put an analogue of the Poincaré–Bendisxon theorem for $3D$ flows out of reach.

Before we begin with the proof of the Poincaré–Bendixson theorem, let's look at some consequences.

Theorem 14.3. *Let Φ_t be a complete flow associated with a vector field $\dot{x} = F(x)$ on the plane $\mathbb{R}^2$. Suppose that $\mathcal{R}$ is a positively invariant set with compact closure (i.e. it is bounded). If $p \in \mathcal{R}$,and there are at most a finitely many number of critical points of Φ_t then $\omega(p)$ is one of the following:*

(1) an equilibrium point
(2) a periodic orbit
(3) a union of finitely many critical points and perhaps a some connecting orbits.

That last theorem is essentially a full classification of the types of limiting behaviour of flows in $\mathbb{R}^2$. As a final consequence of the Poincaré-Bendixson theorem before moving on to the proof, we have the following.

Theorem 14.4. *If a flow in $\mathbb{R}^2$ has a positively invariant, bounded annular region (that is a region $\mathcal{R}$ with only one hole in it) and there are no equilibrium points of the flow in $\mathcal{R}$, then $\mathcal{R}$ contains at least one periodic orbit. If in addition, there is a point on the boundary of $\mathcal{R}$ that gets mapped under the flow to a point in the interior, $\mathcal{R}$ contains a limit cycle.*

15. Proof of the Poincaré-Bendixson Theorem

Before we get to the proof of the Poincaré-Bendixson theorem, it is necessary to introduce some terminology that will help us later on.

Definition 15.1. A set X is said to be *disconnected* if there exists a pair A, B of nonempty disjoint open subsets of X such that their union is all of X. That is we have

(1) $X \subseteq A \cup B$
(2) $X \cap A \neq \emptyset$ and $X \cap B \neq \emptyset$
(3) $A \cap B = \emptyset$.

The sets A and B are called *a disconnecting pair* or *a separating pair* of X. If a set is not disconnected, it is *connected*.

A simple fact that we'll need later on is the following:

Proposition 15.1. *The closure of the forward orbit of a point $\overline{\Gamma}_x^+$ is connected.*

Proof. Suppose you could find a disconnecting pair A, B. Then suppose that $x \in A$. Then suppose that for some other $t > 0$ we had $\Phi_t(x) \in B$. Then as the flow is continuous, the intermediate value theorem means that $\Phi_t(x)$ would take every value along its path. This means that it would have to leave A at some point, but this means that there would be some point in $\overline{\Gamma}_x^+$ which is not in $A \cup B$, so we must have that the whole of $\overline{\Gamma}_x^+ \subset A$ but then this means that $\overline{\Gamma}_x^+ \cap B = \emptyset$, a contradiction. □

Finally, we'll include the statement of a very famous theorem about the plane that we won't prove, but that we'll need later on.

Theorem 15.1 (Jordan Curve Theorem). *A simple (i.e. no crossings) closed curve $\mathcal{C}$ in the plane $\mathbb{R}^2$ separates $\mathbb{R}^2$ into two connected components, both with boundary $\mathcal{C}$. One is bounded, and called* the inside *of $\mathcal{C}$ and the other is unbounded and called* the outside.

In order to prove Theorem 14.2, we will use four lemmata and the Jordan curve theorem. The first of these is the following

Lemma 15.1. *Suppsose that we have a complete flow Φ_t and suppose that $\overline{\Gamma_x^+}$ (the closure of the forward orbit of x) is bounded. Then $\omega(x)$ is nonempty, closed and bounded and connected.*

Proof. We already have that

$$\omega(x) = \bigcap_{t\geq 0} \overline{\Gamma}^+_{\Phi_t(x)}$$

which says that $\omega(x)$ is the intersection of a collection of closed nonempty nested, connected sets. This implies that $\omega(x)$ is nonempty, and because it is closed and it is contained in a compact set, we have that $\omega(x)$ is also compact. Suppose that $\omega(x)$ were disconnected. Then we could find a pair of disconnecting sets A and B. Consider the following:

$$\bigcap_{t\geq 0} \overline{\Gamma}^+_{\Phi_t(x)} \setminus (A \cup B)\,.$$

Now this set is empty since $\omega(x) \subseteq A \cup B$, but this means that there is some T such that $\overline{\Gamma}^+_{\Phi_T(x)} \subseteq (A \cup B)$, but this would mean that A and B were a disconnecting set for $\overline{\Gamma}^+_{\Phi_T(x)}$, a connected set by Proposition 15.1, a contradiction. □

The next lemma involves an important object in the further study of dynamical systems.

Definition 15.2. A curve segment Σ in the plane is called a *transversal* for the vector field $\dot{x} = F(x)$ if for every point $p \in \Sigma$ we have that the pair $(F(p), T_p\Sigma)$ spans $\mathbb{R}^2$. The line $T_p\Sigma$ is the tangent line to the curve Σ at the point p.

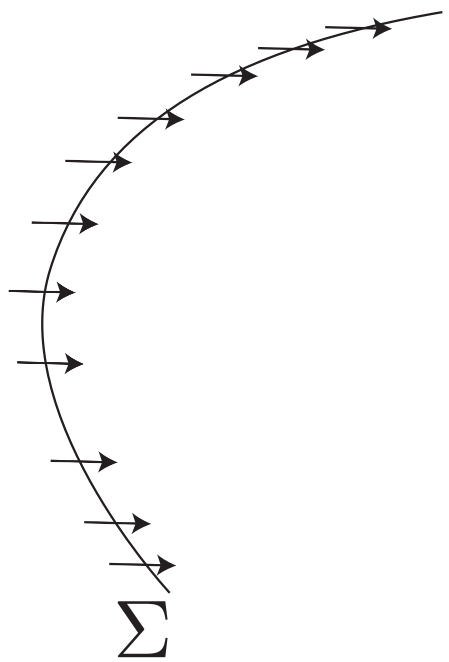
What this says is that the tangent line to Σ is never in the same direction of the flow. See Figure 34. We note here that we can't have any equilibria of our flow on Σ if Σ is to be a transversal. There are other properties that one can infer as well, but for now, let's press on to the next lemma.

Lemma 15.2. *If Σ is a transversal for the flow Φ_t and if $p \in \Sigma$, then the forward orbit through the point p intersects Σ monotonically. That is if $\Phi_{t_1}(p), \Phi_{t_2}(p)$ and $\Phi_{t_3}(p)$ are all also on Σ, and if $t_1 < t_2 < t_3$, then $\Phi_{t_2}(p)$ lies between $\Phi_{t_1}(p)$ and $\Phi_{t_3}(p)$ on Σ.*

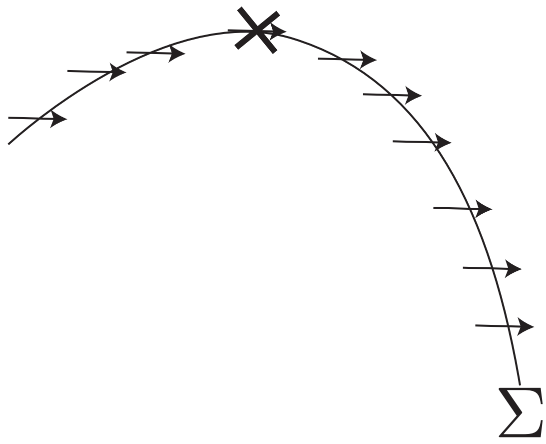
Proof. Suppose that the orbit of p intersects Σ more than once, say at the point $x_1 = \Phi_{t_1}(p)$. Then form the simple closed curve consisting of the orbit of p from $t \in [0, t_1]$ and the part of Σ between p and x_1. This curve divides the plane into two regions. Let's call $\mathcal{R}$ the interior. Because there are no equilibria on Σ and because the flow lines can not cross, the orbit from x_1 on must either enter or leave the region $\mathcal{R}$ along Σ. If the flow leaves $\mathcal{R}$, then for the same reason it can't enter it again, so if there is another point of intersection with Σ, it must be beyond x_1. Likewise if the flow enters $\mathcal{R}$, then if it hits Σ again it must do so on the opposite side of x_1 as p. For an illustration see Figure 35 □

We will use the monotonicity of intersections with a transversal to establish the following.

Lemma 15.3. *Let Σ be a transversal to a planar flow, and let $p \in \Sigma$. Then $\omega(p)$ can intersect Σ in at most one point.*



(A) A transversal Σ. The flow is never tangent to the curve segment.



(B) The curve Σ is not a transversal in this case because the flow is tangent to the curve.

FIGURE 34

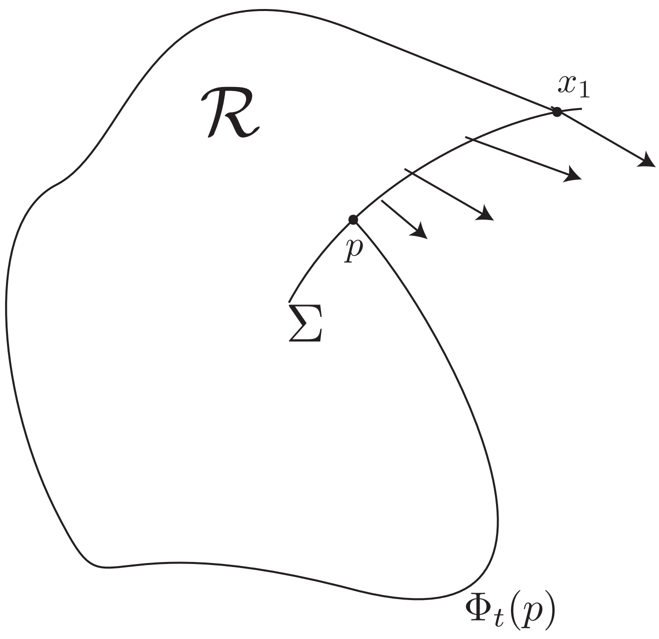


FIGURE 35. The forward orbit from p is leaving the region $\mathcal{R}$ at the point x_1. Since trajectories can't intersect, and because of the Jordan Curve theorem, if the forward orbit from x_1 hits Σ again, it must do so on the opposite side of Σ from p.

Proof. Let $z \in \omega(p) \cap \Sigma$. Because $z \in \omega(p)$ we can choose an infinite sequence of times $t_k \to \infty$ with k such that $\lim_{k\to\infty} \Phi_{t_k}(p) = z$. The claim is that we can choose these t_k such that $\Phi_{t_k}(p) \in \Sigma$. To see this, observe that since the $\Phi_{t_j}(p)$s converge to z they have to get within some (any) neighbourhood of it. But since $F(z) \neq 0$ and F is continuous, there must be some neighbourhood N of z where all $x \in N$ are such that $F(x) \neq 0$. Within this neighbourhood, all of the orbits must cross Σ. Thus we can flow our times forward so that $\Phi_{t_k}(p) \in \Sigma$. Then by appealing to Lemma 15.2 we have that these points form a monotone sequence which converge to a unique limit. But we already know this limit, the point z. $\square$

The final lemma we need is the following

Lemma 15.4. *If $\omega(p)$ is connected and contains a periodic orbit γ, then $\omega(p) = \gamma$.*

Proof. Let $y \in \gamma$ and construct a transversal Σ through y. Since $\omega(p)$ is closed and connected, if there is a point in $\omega(p)$ that is not in γ, then by connectedness, there must be a sequence of points $y_n \in \omega(p) \setminus \gamma$ such that the limit as $n \to \infty = y$. A point y_n must have an orbit that intersects Σ, however this contradicts Lemma 15.3 which says $\omega(p)$ intersects Σ exactly once. $\square$

We are now ready to prove the Poincaré-Bendixson theorem.

Proof of Poincaré-Bendixson Theorem. Suppose that $\omega(p)$ is closed and bounded and contains no equilibria. By lemma 15.1 $\omega(p)$ is also connected. Choose a $q \in \omega(p)$, and observe that $\omega(q) \subseteq \omega(p)$. Let $x \in \omega(q)$. Since x is not a critical point we can make a transversal through x, and a sequence of points in Γ_q^+ which converge to x. Those points are in $\omega(p)$ because $q \in \omega(p)$ and $\omega(p)$ is invariant. Lemma 15.3 says that the only point in this sequence can be x itself, which means that q is on a periodic orbit say Γ, and in particular we have that $\Gamma \subseteq \omega(p)$. Lemma 15.4 means that $\omega(p) = \Gamma$, and this completes the proof of the Poincaré-Bendixson theorem. $\square$

16. Applications and Related Results

In this last section, we'll show how the Poincaré-Bendixson theorem can be used to show the existence of a limit cycle in quite a few general results. We'll start with an example

Example 16.1. Consider the second order nonlinear ODE

$$\ddot{x} + (x^2 + \dot{x}^2 - 4)\dot{x} + x^3 = 0$$

or the system which is equivalent to it

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -(x^2 + y^2 - 4)y - x^3. \end{aligned} \tag{16.1}$$

We have a single critical point at the origin, and as usual, denoting the right hand side of eq. (16.1) by $F(x, y)$ we have that the Jacobian of F at the origin is given by

$$DF(0,0) = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

which isn't hyperbolic, so we don't know anything about it. However, this section is about periodic orbits, and by putting together what we've been studying for the last week, we are going to deduce the qualitative behaviour of this system, namely, we're going to find a limit cycle. For no really good reason right now, consider the function

$$\mathcal{E}(x, y) = \frac{1}{2}y^2 + \frac{1}{4}x^4.$$

It is easy enough to see that $\mathcal{E}(x,y)$ has a local minimum at the origin and is monotonically increasing. We have a family of level sets $\mathcal{E}(x,y)=k$ for all $k\geq 0$. Let's denote by $\mathcal{C}_k$ the level set of $\mathcal{E}$ whose value is k

$$\mathcal{C}_k := \left\{(x,y) \;\middle|\; \mathcal{E}(x,y)=k\right\}.$$

These $\mathcal{C}_k$ are simple, closed curves. Now, let's look at $\frac{d\mathcal{E}}{dt}(x,y)=\nabla\mathcal{E}\cdot F(x,y)$. We have

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(x,y) &= \dot{y}y+\dot{x}x^3\\ &= -y^2(x^2+y^2-4)-x^3y+x^3y\\ &= -y^2(x^2+y^2-4).\end{aligned}$$

Now we notice that inside the circle of radius 2 about the origin, $\mathcal{S}_2=\{(x,y)|x^2+y^2=4\}$, we have that $\dot{\mathcal{E}}\geq 0$ while outside it we have that $\dot{\mathcal{E}}\leq 0$ and with equality only when $y=0$. The idea now is to choose a k_1 so that the curve $\mathcal{C}_{k_1}$ is entirely contained inside the interior of $\mathcal{S}_2$, and a k_2 so that C_{k_2} is entirely contained *outside* of $\mathcal{S}_2$. For example, choose $k_1=\frac{1}{2}$ and $k_2=5$ as in Figure 36. Now, except for the points $y=0$, we have that

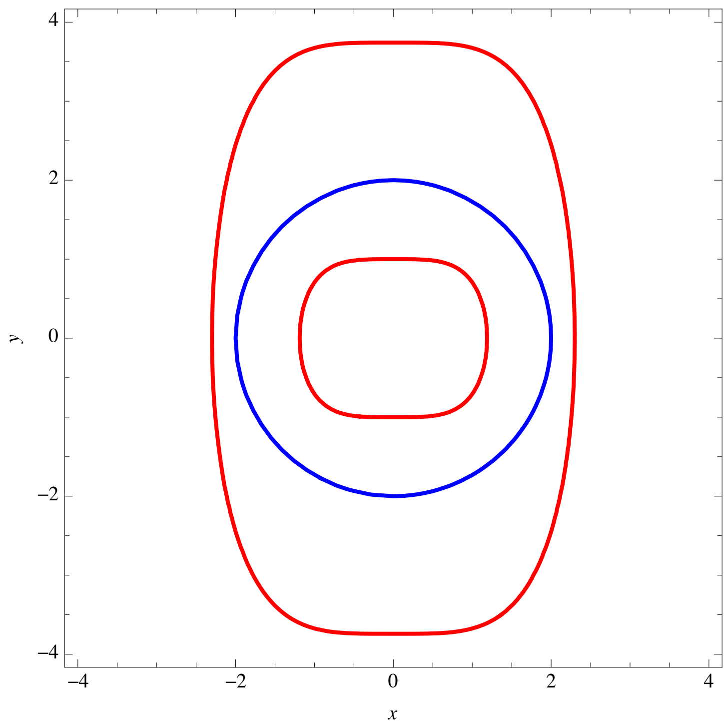
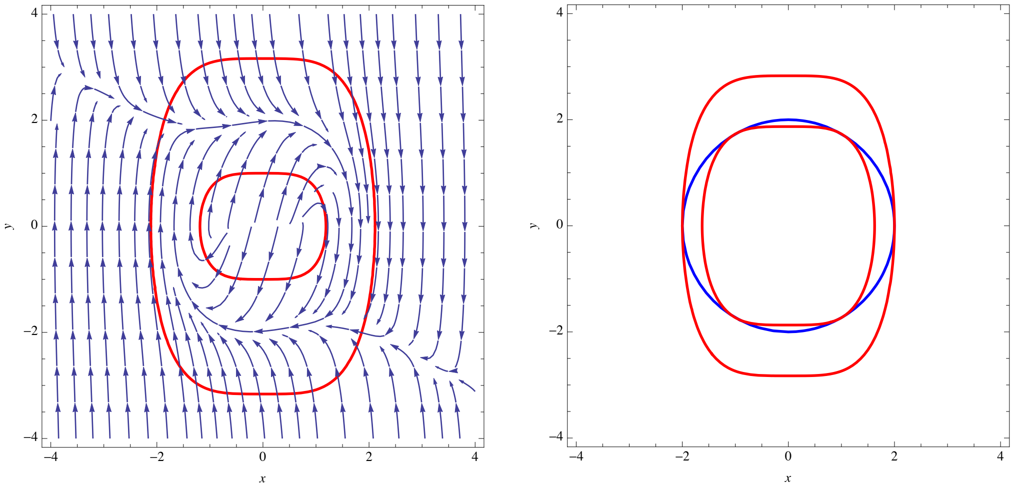


FIGURE 36. The blue curve is the circle $\mathcal{S}_2$ while the red curves are the level sets $\mathcal{E}(x,y)=k_{1,2}$ with $k_1=\frac{1}{2}$ (inside) and $k_2=5$ (outside).

$\dot{\mathcal{E}}(x,y)>0$. So choose a point p not on this set, and consider the forward orbit from p, Γ_p^+. We have that $\Phi_t(p)$ for $t>0$ must leave $\mathcal{C}_{k_1}$ moving outward from it, and it is clear that it can never return to the region inside $\mathcal{C}_{k_1}$ because $\dot{\mathcal{E}}(x,y)\geq 0$ on $\mathcal{C}_{k_1}$. Moreover, for the same argument, we know it can never leave the area bounded by $\mathcal{C}_{k_2}$ because $\dot{\mathcal{E}}(x,y)\leq 0$ on $\mathcal{C}_{k_2}$. Thus we have that the forward orbit is trapped in the region $\mathcal{R}$ between the curves $\mathcal{C}_{k_1}$ and $\mathcal{C}_{k_2}$, and since there are no critical points of our flow there, there must be a limit cycle by the Poincaré-Bendixson Theorem. Further, we can 'squeeze' the location of this limit cycle by a judicious choice of $\mathcal{C}_{k1}$ and $\mathcal{C}_{k2}$. We need to choose k_1 and k_2 so that $\mathcal{E}(x,y)$ and $\mathcal{S}_2$ are tangent to each other, both on the inside and on the outside. In this way we can minimise the location of the limit cycle. For what it's worth, it turns out that $k_1 = \frac{7}{4}$ and $k_2 = 4$ are the values that do this (these can be found using the method of Lagrange multipliers). See Figure 37 for a more detailed picture of what is going on.



(A) The contours $\mathcal{E}(x, y) = k_{1,2}$ on a numerical plot of the phase plane. You can clearly see that there is a limit cycle in between them.

(B) The red curves are $\mathcal{E}(x, y) = \frac{7}{4}$ and $\mathcal{E}(x, y) = 4$. These curves minimise the area where the limit cycle can be and are tangent to the circle S_2.

FIGURE 37

This example generalises really nicely to what's known as *Lienard's (first) theorem.*

Theorem 16.1 (Lienard's Theorem I). *The differential equation*

$$\ddot{x} + f(x, \dot{x})\dot{x} + g(x) = 0$$

or equivalently, the system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -f(x, y)y - g(x) \end{aligned}$$

where f and g are C^1 functions has at least one periodic solution under the following conditions

(1) There is an $a > 0$ so that $f(x, y) > 0$ for $\sqrt{x^2 + y^2} > a$.
(2) $f(0, 0) < 0$ (hence $f(x, y) < 0$ in a neighbourhood of the origin)
(3) $g(0) = 0$ and $g(x) > 0$ when $x > 0$ and $g(x) < 0$ when $x < 0$
(4) $G(x) = \displaystyle\int_0^x g(u)du \to \infty$ as $x \to \infty$.

The idea is exactly the same as in the previous example, only now you want to choose $\mathcal{E}(x, y) = \frac{1}{2}y^2 + G(x)$ and instead of $\mathcal{S}_2$ you want to use $\mathcal{S}_a$, the circle of radius a. (You can also use any curve $\mathcal{C}$ where you can find that $\dot{\mathcal{E}}$ changes sign on either side of it).

We'll close this week with a test to determine when there are *not* any periodic orbits in a region.

Theorem 16.2 (Bendixson's negative criterion). *Suppose that in the C^2 (twice continuously differentiable) system of planar ODEs*

$$\begin{aligned} \dot{x} &= f(x, y) \\ \dot{y} &= g(x, y) \end{aligned}$$

we can find a region $\mathcal{R}$ in the plane with no holes in it, and such that on $\mathcal{R}$ the quantity

$$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$$

is of one sign. Then there are no periodic orbits in $\mathcal{R}$.

Proof. A periodic orbit means that we have a closed path $\mathcal{C}$. Suppose that there were one, and let $\mathcal{S}$ be the interior of $\mathcal{C}$. Defining $\mathbf{n}$ to be the outward pointing normal along $\mathcal{C}$, we have that the divergence theorem in the plane says

$$\iint_{\mathcal{S}} \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \right) dxdy = \int_{\mathcal{C}} (f, g) \cdot \mathbf{n} ds.$$

Since on $\mathcal{C}$, (f, g) is perpendicular to $\mathbf{n}$, we have that the integral on the right is zero. But since the integrand on the left is of one sign, its integral can not be zero. Therefore we can not have any closed paths $\mathcal{C}$. $\square$

Example 16.2. As an example of where theorem 16.2 can be applied let's consider the Duffing oscillator with damping

$$\ddot{x} + \delta\dot{x} - x + x^3 = 0$$

where $\delta > 0$, or equivalently

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\delta y + x - x^3. \end{aligned}$$

When $\delta = 0$, the system is Hamiltonian, and we have periodic orbits almost everywhere. In fact all orbits are periodic with the exception of three limit points and two others (the lobes of the 'bowtie'). Now if $\delta \neq 0$ we have that

$$f_x + g_y = -\delta \neq 0$$

and so we have from Bendixson's negative criterion that there are no periodic orbits anywhere in all of $\mathbb{R}^2$. So we've gone from lots and lots of periodic orbits to none at all by simply 'switching on' the parameter δ. This type of quantitative behaviour is going to be the focus of our next module, Bifurcation Theory.