MATH 3963
NONLINEAR ODES WITH APPLICATIONS

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Contents

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1. The First Examples

In this section we are going to explore how an ODE with a parameter can change as we change the parameter. Specifically, we are going to look at how a critical point can undergo changes in stability and type as we move around our parameter(s). For the most part, I want to concentrate on examples of 'famous' bifurcations that you will encounter. Working through them is a very good way to get a handle on how the theory as a whole behaves.

Example 1.1. Let's begin with a simple example. Consider the one dimensional ODE

$$\dot{x} = x^2 + r, \tag{1.1}$$

where we have one independent variable, x and single parameter $r \in \mathbb{R}$. Since we're in the case of a one dimensional ODE we want to plot the right hand side of eq. (1.1), and we'll do this for various values of r. Further, we can draw the phase diagram for the one dimensional system on the x-axis, and we have that wherever the right hand side crosses the x axis, we have a critical point of our one dimensional flow. Thus we see that for $r > 0$ we have no (real) critical points, while for $r = 0$ we have a single, semi-stable one, and for $r < 0$ we have a pair of critical points at $x_*^{\pm} = \pm\sqrt{-r}$ and we can see that $x_*^+ = \sqrt{-r}$ is unstable, while $x_*^- = -\sqrt{-r}$ is stable. This is illustrated in Figure 1. It is also helpful to sketch what's called the *bifurcation diagram*. This is a graphical representation of the location of the critical points (x_* in this example) versus the bifurcation parameter (r in this example), taking into account stability. The convention is to mark where the equilibria are stable with a solid line and where they're unstable with a dashed line. Sometimes, it is also convenient to mark some of the phase lines in the diagram as well. For this example we have the bifurcation diagram in Figure 2.

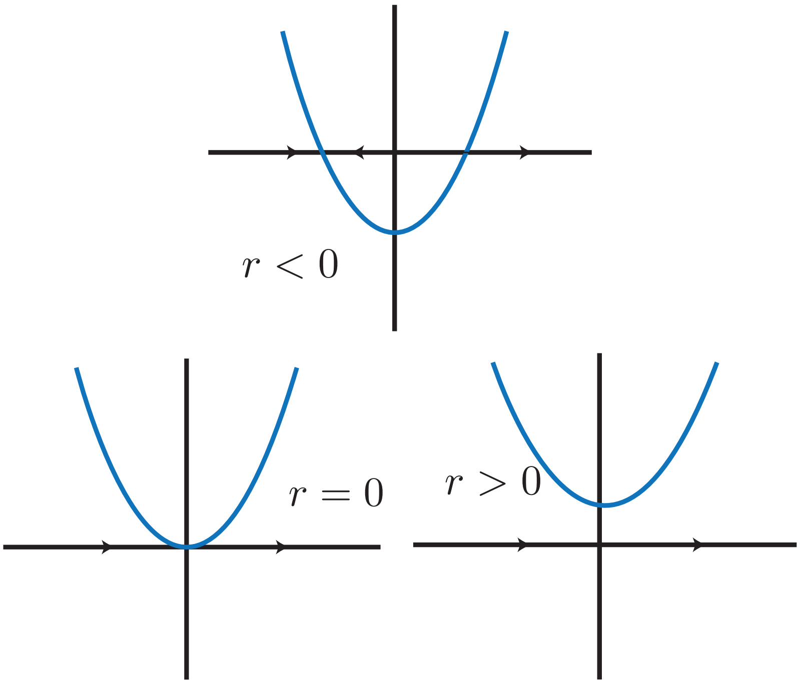


Figure 1. A plot of $f(x) = x^2 + r$ for various values of r. The phase diagram of the ODE $\dot{x} = f(x)$ is drawn on the x-axis. We have that for $r < 0$ there are two critical points, one stable and one unstable, while for $r = 0$ there is a single semi-stable critical point, and $r > 0$ there are no real critical points.

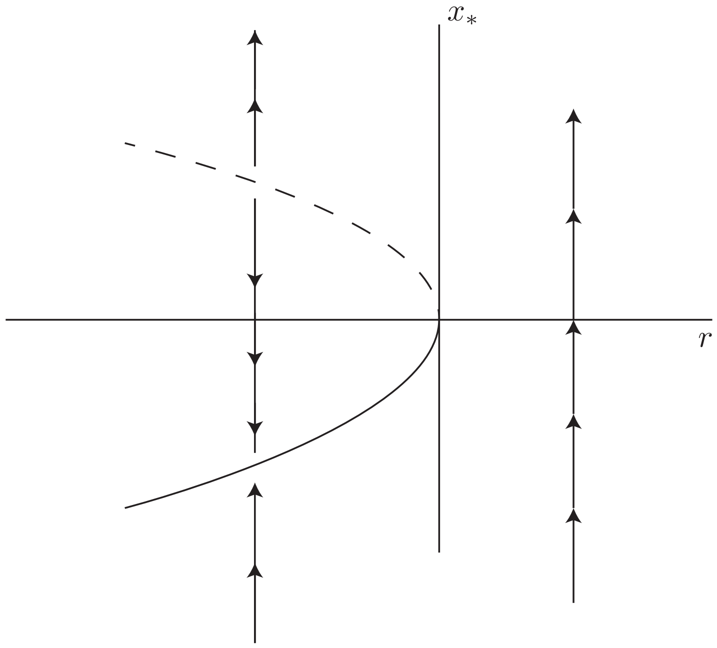


Figure 2. The bifurcation diagram for the ODE in eq. (1.1). Phase lines for a pair of values of the parameter r are also sketched.

Example 1.2. Moving on, we want to consider another simple ODE in one independent variable, logistic growth:

$$\dot{x} = rx(1-x) - h, \tag{1.2}$$

where $r > 0$ is the rate of logistic growth, and $h \in \mathbb{R}$ is a harvesting component, say the amount of fishing allowed in a lake if h is positive or the amount of stocked fish added to the lake per year if h is negative. The critical points will be those x_* such that the right hand side of eq. (1.2) is 0. That is

$$rx_*^2 - rx_* + h = 0.$$

Using the quadratic formula, we get that

$$x_*^{\pm} = \frac{r \pm \sqrt{r^2 - 4rh}}{2r} = \frac{1}{2}\left(1 \pm \sqrt{1 - \frac{4h}{r}}\right).$$

Letting $\mu = \frac{4h}{r}$ we can then write $x_*^{\pm} = \frac{1}{2}\left(1 \pm \sqrt{1-\mu}\right)$ and our expression for x_*, the critical points of our ODE, depends only on the single parameter μ. Now we notice that when $\mu < 1$ we have two equilibrium points, when $\mu = 1$ we have only one, and when $\mu > 1$ there are no equilibrium points at all (we need our equilibria to be real). Now we want to draw the bifurcation diagram. In this case to determine stability, we need to look at the derivative of the right hand side of eq. (1.2). Using the fact that

$$\frac{d}{dx}f(x; r, h) = -2rx + r = -r(2x - 1)$$

we have that

$$\frac{d}{dx}f(x_*^{\pm}; r, h) = -r\left(1 \pm \sqrt{1 - \frac{4h}{r}} - 1\right) = \mp r\sqrt{1-\mu},$$

and we have that as $r > 0$, when $\mu < 1$ we have that the larger of the two equilibria (x_*^+) is stable, while the smaller one (x_*^-) is unstable. As μ is increased towards 1 the two equilibria move towards each other, eventually colliding and annihilating each other. This point in the (μ, x_*) plane $(1, \frac{1}{2})$ is called the *bifurcation point*. Finally, because we are in such low dimensions, we can draw, on the bifurcation diagram a qualitative sketch of the phase portrait for each of the relevant values of μ (both less than 1 and greater than 1). See Figure 3.

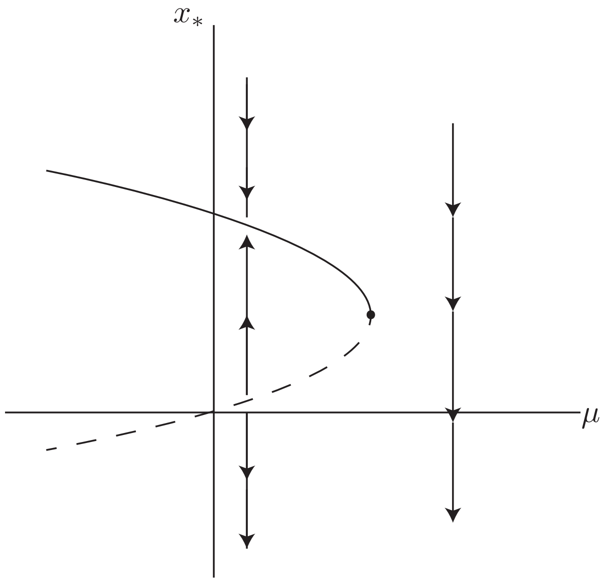


FIGURE 3. The bifurcation diagram for the ODE in eq. (1.2).

It is worth noting that the bifurcation diagrams for these first two examples are the same (more or less). In some sense bifurcations are 'classified' by their diagrams. That is, familiar diagrams are given a name, and if you are investigating a particular problem that has a bifurcation in it, and you can draw a diagram that you recognise, then you can call that bifurcation by that name. This bifurcation has a name, but it isn't exactly intuitive where it comes from yet, so before we name it, let's move on to a 2×2 example.

Example 1.3 (Saddle-Node Bifurcation)**.** Another, related example is the following. Consider the 2×2 system of ODE

$$\dot{x} = y, \qquad \dot{y} = x^2 - y - \mu$$

where μ is a real parameter. We have that equilibrium points occur when $y = 0$ and when $x^2 - \mu = 0$. When $\mu > 0$ we have two critical points, when $\mu = 0$ there is one, and when $\mu < 0$ there are none. If there are equilibria, they are at the points $(\pm\sqrt{\mu}, 0)$ in the (x, y) plane. Linearising gives $DF(\pm\sqrt{\mu}, 0) = \begin{pmatrix} 0 & 1 \\ \pm 2\sqrt{\mu} & -1 \end{pmatrix}$. When the critical point is $(\sqrt{\mu}, 0)$ we have that $\det(DF) < 0$, and we are in the case of a saddle. When the critical point is $(-\sqrt{\mu}, 0)$, the eigenvalues of the Jacobian are given by $\lambda = \frac{1}{2}\left(-1 \pm \sqrt{1 - 8\sqrt{\mu}}\right)$. Thus if $0 < \mu \leq \frac{1}{64}$ we are in the case of a stable node, while if $\frac{1}{64} < \mu$ we are in the case of a stable focus. Either way, we are stable though. When $\mu = 0$ we see that the Jacobian has a 0 eigenvalue, it loses hyperbolicity (this will be important later on).

Since the critical points always occur when $y = 0$ we can plot the location of the x coordinate of the critical point in the (x, μ) plane and draw the bifurcation diagram with respect to these variables. We get the picture in Figure 4.

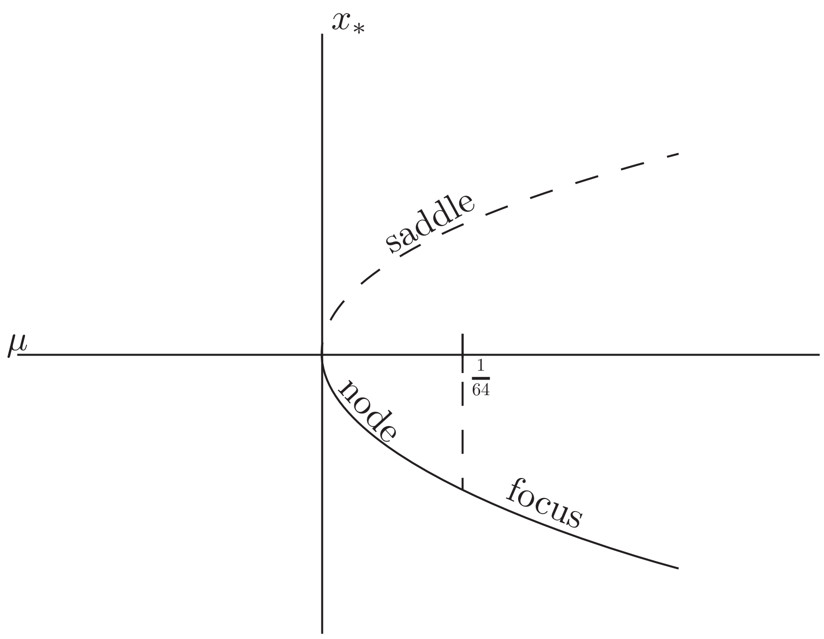


FIGURE 4. The bifurcation diagram for the saddle-node bifurcation. This system is where this bifurcation gets its name.

Not surprisingly, this type of bifurcation is called a *saddle-node bifurcation.* The point where the bifurcation occurs $(0,0)$ in (μ, x) space is the bifurcation point. As in the last two examples, the bifurcation diagram we get is qualitatively the same. In particular we are going to call all three of these saddle-node bifurcations, even though we need 2 dimensions to explicitly see a saddle and a node bifurcate out of nothing. In Figure 5 using Mathematica's StreamPlot function a few values of μ near the bifurcation have been plotted.

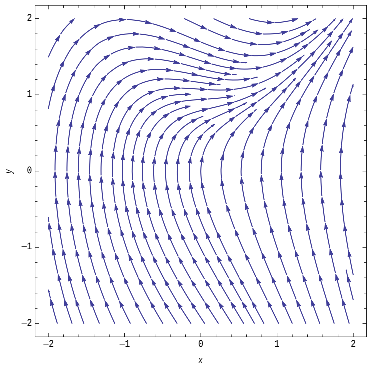
Example 1.4 (Transcritical Bifurcation)**.** For our next example, we look at the following 1-D system of ODEs:

$$\dot{x} = x^2 - \mu x = x(x - \mu). \tag{1.3}$$

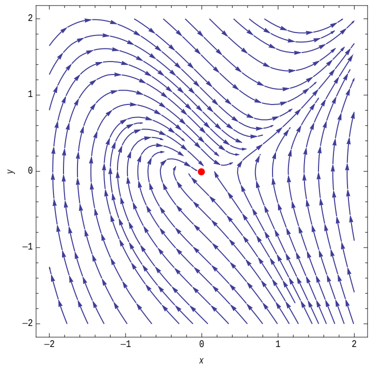
The equilibria are at $x_* = 0$ and $x_* = \mu$ for all $\mu \in \mathbb{R}$. So right away we see that this is different from the saddle-node bifurcation. Namely, we don't have two equilibria colliding and annihilating each other. As the right hand side of eq. (1.3) is reasonably straightforward, we can plot the graph of $f(x;\mu)$ versus x and get a feel for how this system behaves. See Figure 6. We see that when $\mu < 0$, $f(x)$ has two equilibria, and the one at $x_* = \mu$ is stable while the one at $x_* = 0$ is unstable. Then as μ increases to zero, the two critical points collide and we have a semi-stable equilibrium point. Then as μ increases further and becomes positive, we see that we still have two critical points, one at $x_* = 0$ and one at $x_* = \mu$ only this time, we have that the stability has interchanged, now the one at $x_* = \mu$ is unstable, while the one at $x_* = 0$ is stable. It is pretty straightforward to draw the bifurcation diagram in this case, and in this case we see that we have a different picture. Thus we have a new type of bifurcation. This is called a *transcritical bifurcation*, and the bifurcation diagram is shown in Figure 7).

Example 1.5 (Pitchfork Bifurcation)**.** In this next example we consider a one dimensional ODE:

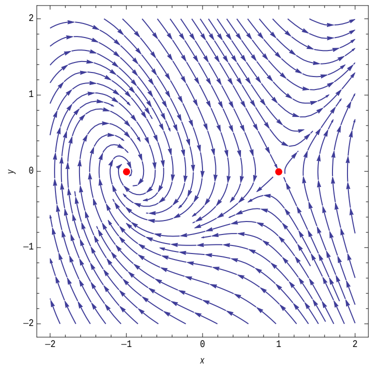
$$\dot{x} = f(x) = \mu x - x^3,$$



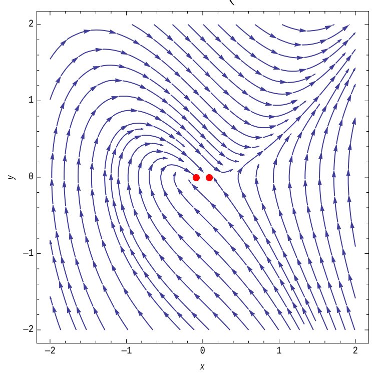
(A) $\mu = -1$. There are no critical points



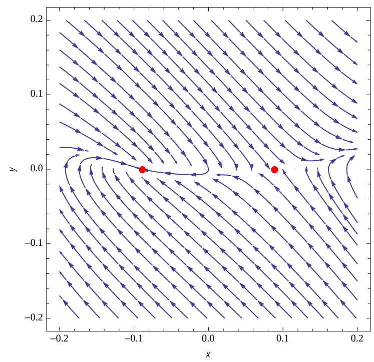
(B) $\mu = 0$. There is a single nonhyperbolic critical point (the red dot).



(C) $\mu = 1$. There are two critical points, a saddle and a stable focus (the red dots).



(D) $\mu = \frac{1}{128}$. There are two critical points (red dots), a saddle and a stable node. They are a bit close together though so it is tough to see them in this frame.



(E) $\mu = \frac{1}{128}$. This picture is zoomed in on the critical points (red dots). There are two critical points, a saddle and a stable node.

FIGURE 5. A sequence of plots showing the emergence of a saddle-node bifurcation

where μ is (again) a real parameter. The critical points will occur when $f(x) = 0$, so at $x_* = 0$ and $x_*^2 = \mu$. As we are looking for real critical points, we will have that $x_* = 0$ will be a critical point of the system for all real values of μ. When $\mu \leq 0$, this will be the only critical point. When $\mu > 0$ however, we will have that there will be three critical points $x_* = \pm\sqrt{\mu}$ and $x_* = 0$. Now we need to analyse stability. The critical points will be stable provided $f'(x_*) < 0$ and unstable if $f'(x_*) > 0$ (c.f. finding the sign of the real parts of the eigenvalues of the Jacobian in the 2×2 case). We have that $f'(x) = \mu - 3x^2$ and so we have that $f'(0) = \mu$, which will be stable if $\mu < 0$ and unstable if $\mu > 0$. Next, when $\mu > 0$ we have that $f'(\pm\sqrt{\mu}) = -2\mu < 0$ and so the other two equilibria will both be stable. In Figure 8 we've shown the bifurcation diagram. Something to note is that we now have a new picture! This means that we have a new bifurcation. This particular example is called a *supercritical pitchfork bifurcation* - the 'pitchfork' bit should be obvious, while the 'supercritical' maybe not so much. The thing to remember about this guy is that 'supercritical' means that the equilibria branching out from the bifurcation point ($(0, 0)$ in the (μ, x) plane) are stable equilibria. The direction the pitchfork is facing does not matter, what matters is the stability of the equilibria branching out of it. If the equilibria

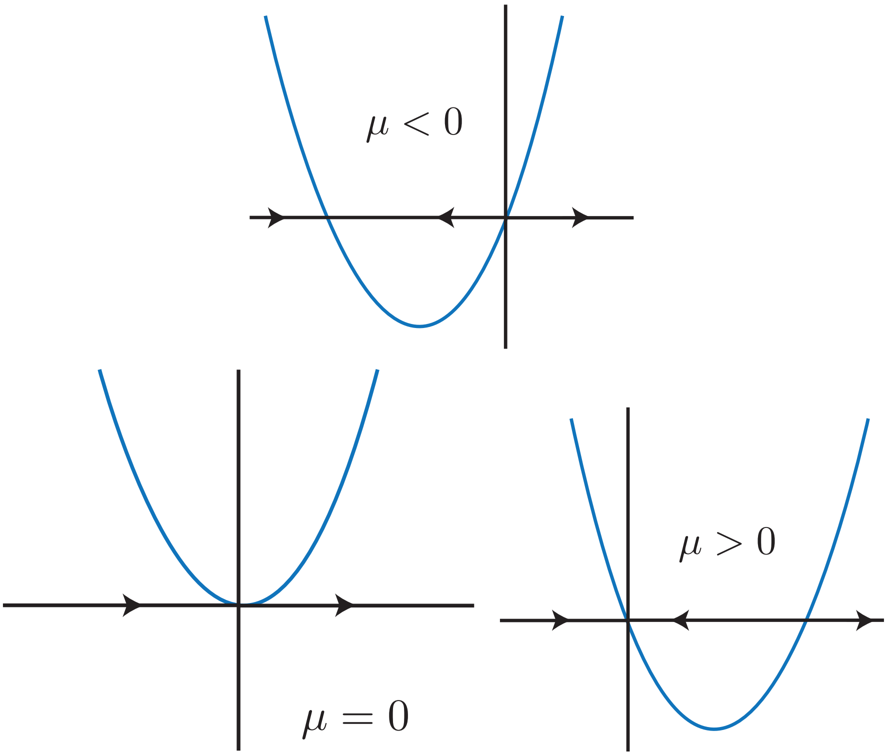


FIGURE 6. A plot of $f(x) = x^2 - \mu x$ for various values of μ. The phase diagram of the ODE $\dot{x} = f(x)$ is drawn on the x-axis. We have that for $\mu < 0$ the critical point at $x_* = 0$ is unstable, while the one at $x_*\mu$ is stable. When μ is increased through zero however the stability of the two critical points interchanges so that 0 is now an unstable critical point while $x_* = \mu$ is now an unstable critical point.

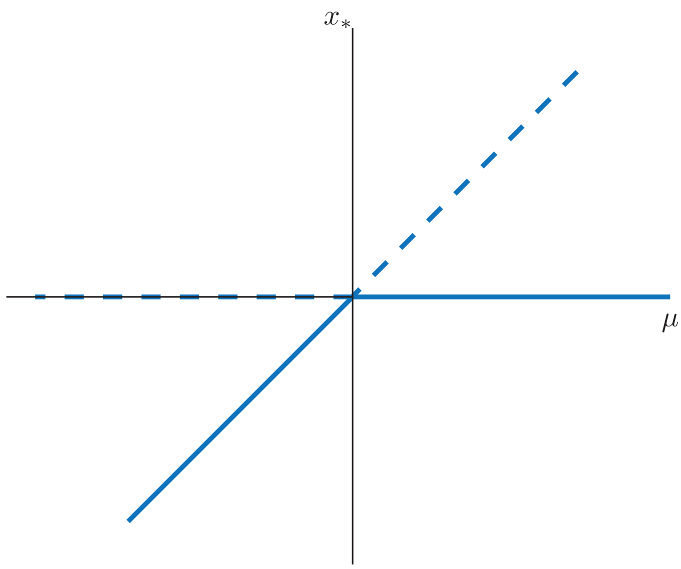


FIGURE 7. The bifurcation diagram for the transcritical bifurcation in eq. (1.3).

were unstable (as in the tutorial question), then it would be called a *subcritical pitchfork bifurcation.*

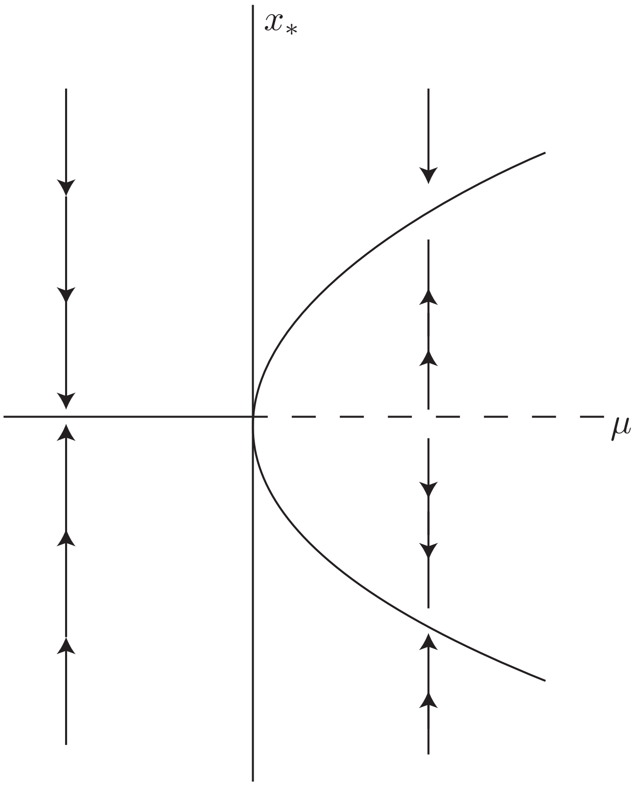


FIGURE 8. The bifurcation diagram for a supercritical pitchfork bifurcation

This next example considers what happens when you might have two or more bifurcations in your system.

Example 1.6. Consider the nonlinear ODE

$$\dot{x} = f(x; r) = 1 + xr + x^2. \tag{1.4}$$

We have that the critical points are the x_* where $f(x_*) = 0$ so this means that

$$x_*^{\pm} = \frac{1}{2}\left(-r \pm \sqrt{r^2 - 4}\right).$$

So, in particular, if $|r| > 2$ we have two critical points, if $|r| = 2$ we have one and if $|r| < 2$ we have none. Stability analysis means that we need to take the derivative of the right hand side of eq. (1.4) and so we have $f'(x) = r + 2x$ and so evaluating at the critical points gives

$$f'(x_*) = r + (-r \pm \sqrt{r^2 - 4}) = \pm\sqrt{r^2 - 4},$$

which says that for all values of r such that $|r| > 2$ we have one stable critical point and one unstable critical point. Let's try to get a feel for the bifurcation diagram. Suppose that $r = 3$. From the above computation we know that we have one stable critical point and that it is located at $x_*^- = \frac{1}{2}(-r - \sqrt{r^2 - 4}) = \frac{-3-\sqrt{5}}{2}$. We also have one unstable critical point, located at $x_*^+ = \frac{1}{2}(-r + \sqrt{r^2 - 4}) = \frac{-3+\sqrt{5}}{2}$. Now as we lower r down to two, these critical points will collide at $r = 2$, and then we'll have a single critical point $x_* = -1$. Then they will disappear. This suggests that we have a saddle node bifurcation here. Further, as we increase $r \to \infty$, we see that $x_*^+ \to 0$, but never quite gets there while $x_*^- \to -\infty$. It turns out that the exact same thing happens as r increases through -2, and again this time $x_*^+ = \frac{1}{2}(-r + \sqrt{r^2 - 4})$ is the unstable critical point, but this time we have $x_*^+ \to \infty$ as $r \to -\infty$. Meanwhile x_*^- is the stable critical point, but this time it approaches 0 as $r \to -\infty$.. The bifurcation diagram is plotted in Figure 9. It is also a very good idea to plot the graph of $f(x)$ for various values of r and to try and deduce the bifurcation diagram from there. Use the x-axis as the phase portrait, the roots of $f(x;r)$ correspond to the equilibrium points of the ODE, and the sign of $f(x)$ tells you which direction the arrows are pointing. Either to the right if $f(x)$ is positive, or to the left if $f(x) < 0$. Then you can vary r and you can see how the equilibria change. This is qualitatively done in Figure 10.

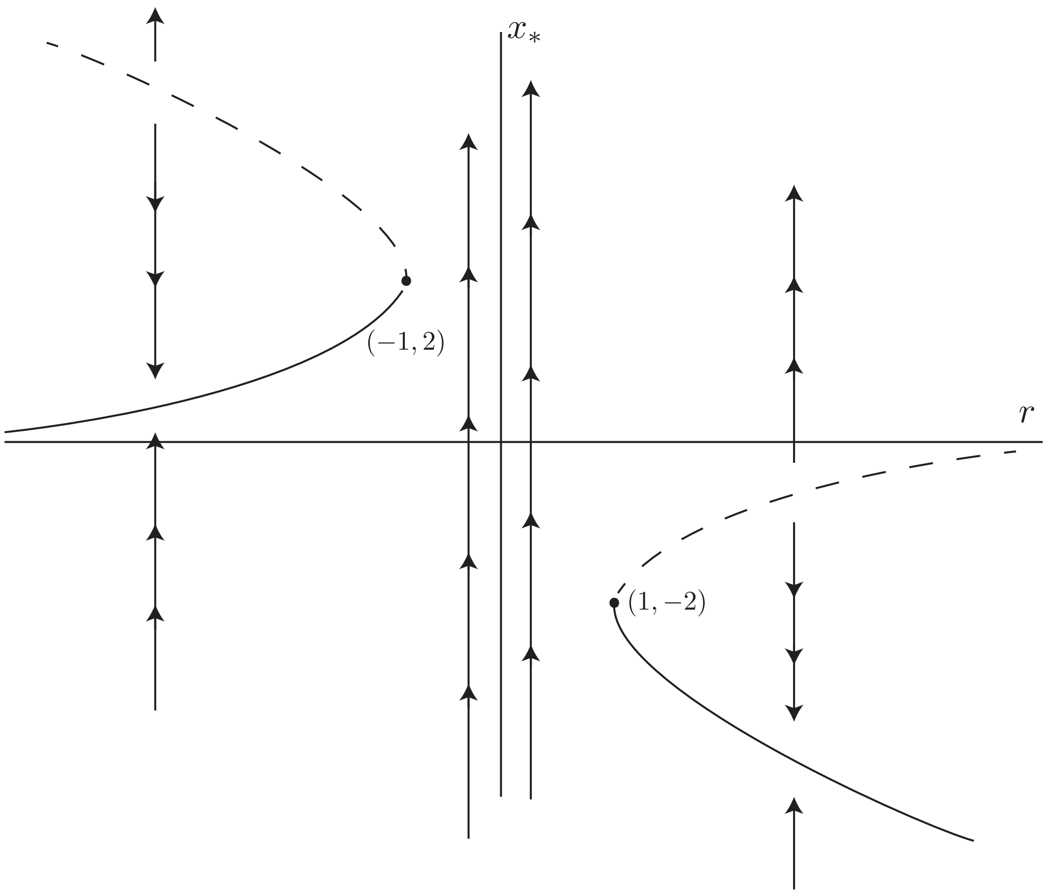


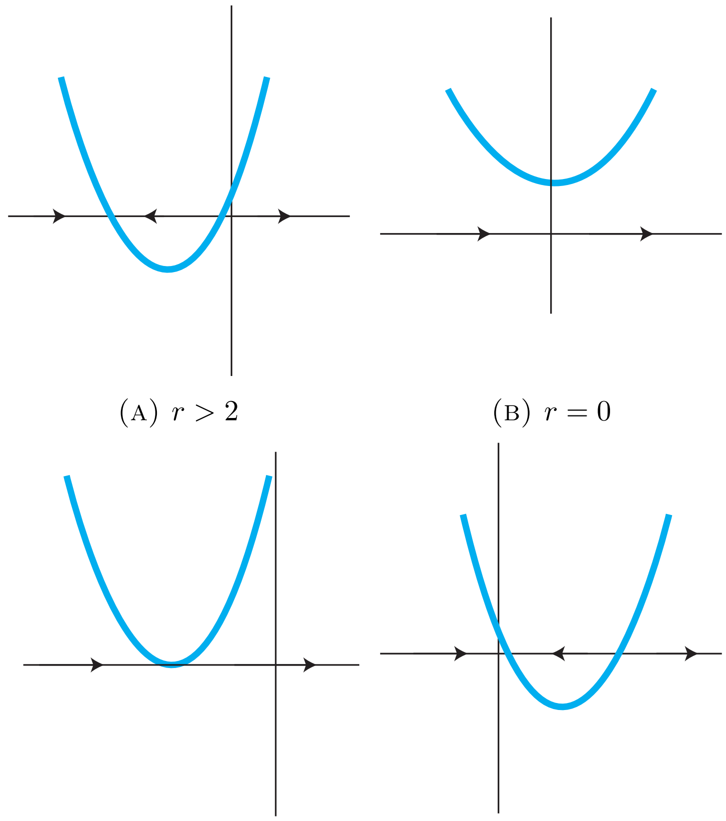
FIGURE 9. The bifurcation diagram for eq. (1.4).

2. The Hopf bifurcation

So far all of our bifurcations have occurred when an eigenvalue has crossed 0. There is one more type of bifurcation that can occur when we have a pair of complex conjugate eigenvalues crossing from the left hand to the right hand side of the complex plane.

Definition 2.1. A *Hopf bifurcation* is a bifurcation that occurs when a complex conjugate pair of eigenvalues $\lambda_\pm$ crosses from the $\mathrm{Re}(\lambda_\pm) < 0$ (the left half plane) to the $\mathrm{Re}(\lambda_\pm) > 0$ (right half plane) *and* a limit cycle emerges from the critical point. The emergence of a stable limit cycle is called a *supercritical* Hopf bifurcation while if the limit cycle that emerges is unstable, we have what's called a *subcritical* Hopf bifurcation (just like the sub- and supercritical pitchfork).

This definition is a bit wordy, but that is because I want to focus on the qualitative aspects of Hopf bifurcations rather than a quantitative one. The two main ideas to take home from it are the following: a Hopf bifurcation occurs when:



(A) $r > 2$ (B) $r = 0$

(C) $r = 2$ (D) $r < -2$

FIGURE 10. Some qualitative plots of $x^2 + rx + 1$ for various values of r.

- We have a change in stability of a critical point from one type of focus to another
- Along with the change in stability, a limit cycle (a periodic orbit which is the α or ω-limit set of a point not in it) emerges in the phase plane.

Example 2.1. An example of a Hopf bifurcation is as follows. Consider the system

$$\begin{aligned} \dot{x} &= \mu x + y - x(x^2 + y^2) \\ \dot{y} &= -x + \mu y - y(x^2 + y^2). \end{aligned} \tag{2.1}$$

In polar coordinates this is

$$\begin{aligned} \dot{r} &= r(\mu - r^2) \\ \dot{\theta} &= -1. \end{aligned} \tag{2.2}$$

The Jacobian at the origin (in the original coordinates!) can be seen to be

$$DF(0,0) = \begin{pmatrix} \mu & 1 \\ -1 & \mu \end{pmatrix},$$

which has a pair of complex eigenvalues $\lambda_{\pm} = \mu \pm i$. Thus when $\mu < 0$, we have that the origin is a stable focus, while when $\mu > 0$ we have that the origin is an unstable focus. Using polar coordinates, we see in eq. (2.2) that when $\mu < 0$ there are no limit cycles, while when $\mu > 0$ we have an attracting limit cycle at $r = \sqrt{\mu}$. Thus we have that an attracting limit cycle bifurcates from the origin (in the (x, y) plane) at the point $\mu = 0$. A sketch of the bifurcation diagram is in Figure 11. For a few relevant phase portraits, see Figure 12.

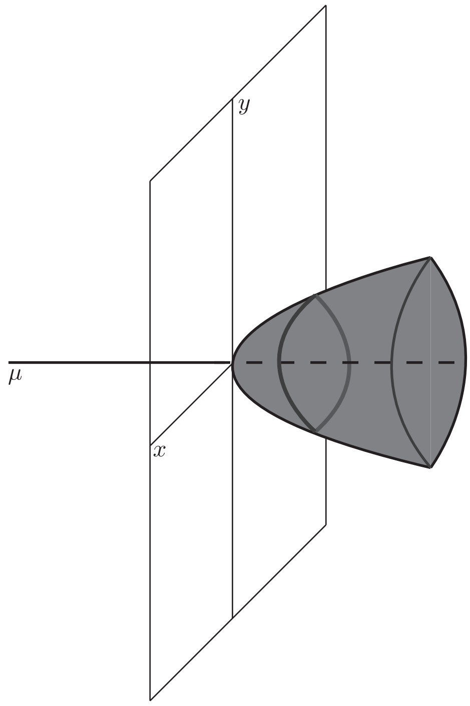


FIGURE 11. The bifurcation diagram for a supercritical Hopf bifurcation. A stable limit cycle emerges from the bifurcation point, while the fixed point switches stability.

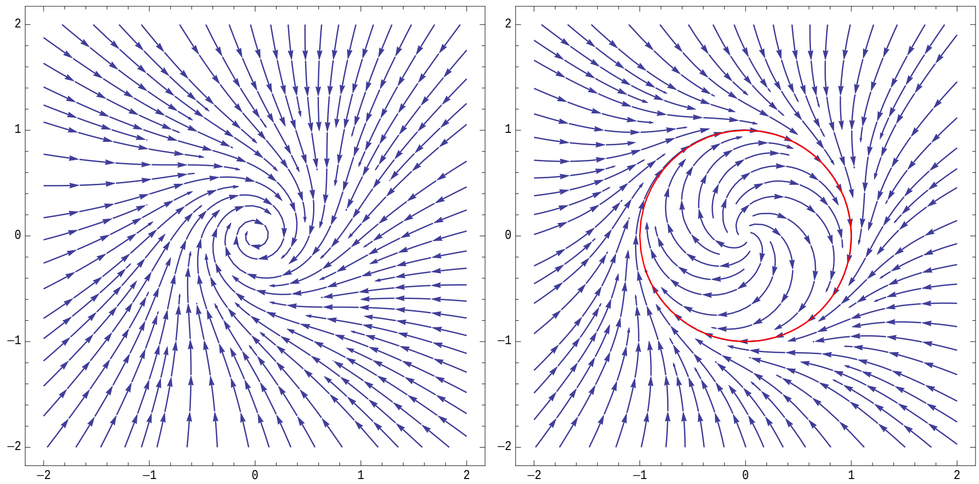


FIGURE 12. The left figure is the phase portrait of eq. (2.1) when the value of $\mu = -0.2$. We can see that the origin is a stable focus, while on the right $\mu = 1$, and the origin is an unstable focus. In this case too, we have that the circle $r = 1$ is a stable limit cycle (red), and so we have a Hopf bifurcation. Because the limit cycle is attracting, this is a supercritical Hopf bifurcation.

Example 2.2 (A degenerate Hopf bifurcation)**.** It is important to note that there are *two* requirements in order for a Hopf bifurcation to occur. The first is that a pair of complex conjugate eigenvalues cross the imaginary axis so we have that a focus changes stability. The second requirement is just as important - we must have the emergence of a limit cycle. This doesn't always happen, and if it doesn't, the resulting bifurcation is called a *degenerate Hopf bifurcation*. That is the focus of this example. 'Recall' the F-KPP ODE, 'F' for Fisher, and KPP for Kolmogorov, Petrovsky and Piscunov,

$$\ddot{u} + c\dot{u} + u(1-u) = 0, \tag{2.3}$$

or equivalently

$$\begin{aligned} \dot{u} &= v \\ \dot{v} &= u^2 - u - cv. \end{aligned} \tag{2.4}$$

This ODE (or system) was originally proposed by both Fisher, and KPP independently in 1937 to describe the propagation of an advantageous characteristic throughout a population. We can see that in the system formulation there are two critical points. One at $(0,0)$ and one at $(1,0)$. The Jacobian is given as $DF = \begin{pmatrix} 0 & 1 \\ 2u-1 & -c \end{pmatrix}$. When $(u,v) = (1,0)$ this has determinant $= -1$ and so is a saddle for all values of c. Thus, no bifurcations occur here. At the origin we have that $DF(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & -c \end{pmatrix}$ which is always going to have positive determinant. The trace of the Jacobian at the origin is $-c$ so we have that when $c > 0$ this fixed point is stable, while for $c < 0$ this fixed point is unstable. So we have that the eigenvalues cross the imaginary axis, and the fixed point switches stability. However, *no limit cycle emerges!* You can see this yourself by using Bendixson's negative criterion. In this case, the quantity is $-c$ which is of one sign on all of $\mathbb{R}^2$ for any value of $c \neq 0$. Thus there is no limit cycle for non-zero values of c. This is called (for various technical reasons that we're not going to go into) a *degenerate Hopf bifurcation*. What happens (which is sort of typical in these cases), is that when $c = 0$ the system is Hamiltonian (and hence conservative), with Hamiltonian $H(u,v) = \frac{v^2}{2} + \frac{u^2}{2} - \frac{u^3}{3}$. Heuristically (morally if you will) what you can think of is that the level sets of the Hamiltonian are all closed circles, so you can sort of see that all of the limit cycles are 'collapsed' onto a single plane at the point $c = 0$ in the parameter space. For a few relevant phase portraits see Figure 13.

3. Bifurcation Theory: More Complicated Examples

In these next examples, we will examine some more of the types of problems that you are likely to encounter.

The first example emphasises the fact that in some sense all of the bifurcations we're looking at are *local*, that is if we can find simpler local approximations for our ODE near the bifurcation, we can use these simpler equations to investigate the bifurcation that is happening.

Example 3.1. Consider the ODE

$$\dot{x} = x(1-x^2) - a(1-e^{-bx}), \tag{3.1}$$

where a and b are given parameters. Let's investigate the type of bifurcation that occurs at $x_* = 0$ and find an approximate formula for the fixed point that bifurcates off $x = 0$. First we observe that $x_* = 0$ is a fixed point of eq. (3.1) for all values of a and b. Moreover, by computing the derivative of the RHS of eq. (3.1) we have

$$f'(x) = 1 - 3x^2 - abe^{-bx}\Big|_{x=0} = 1 - ab,$$

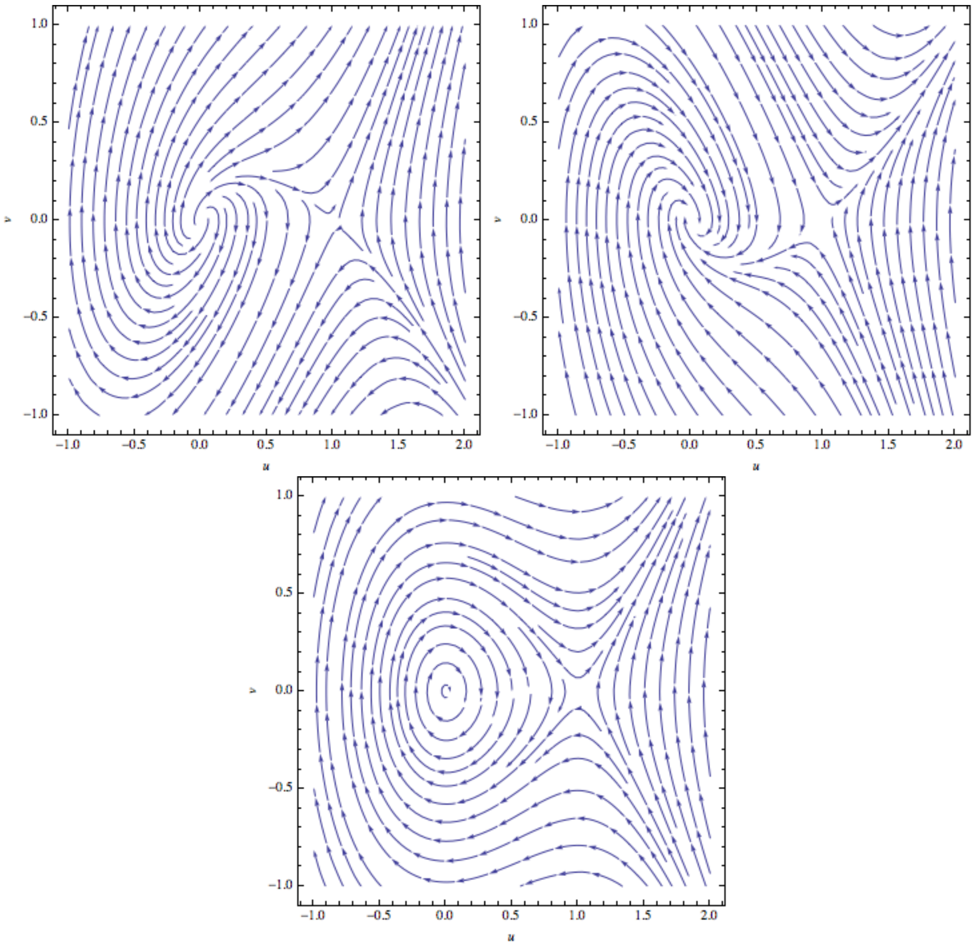


FIGURE 13. The phase portraits of eq. (2.4) for a few relevant values of the parameter c. The top left has $c = -1$ the top right $c = 1$ and the bottom figure shows the phase portrait when $c = 0$. For this value of c the system is Hamiltonian. You can visually see that no periodic orbit emerges, so this is not a Hopf bifurcation. We do have a pair of complex eigenvalues crossing the imaginary axis as c is varied through 0 though, so we call this a *degenerate Hopf bifurcation*.

and so we see that as the quantity ab increases through 1 the sign of the derivative will change, and we have that $x_* = 0$ will go from being an unstable fixed point to a stable one.

Next, in principle, we'd like to find another critical point of eq. (3.1), but this is pretty much impossible to do in some kind of closed form. Even Mathematica gives up. However, since we're interested in the bifurcation near $x_* = 0$, we can use a Taylor series approximation. Namely we have that for small x

$$\begin{aligned} 1 - e^{-bx} &= 1 - (1 - bx + \frac{1}{2}b^2x^2 + \mathcal{O}(x^3)) \\ &= bx - \frac{1}{2}b^2x^2 + \mathcal{O}(x^3) \end{aligned}$$

and so

$$\begin{aligned} \dot{x} &= x - a(bx - \frac{1}{2}b^2x^2) + \mathcal{O}(x^3) \\ &= (1 - ab)x + \frac{1}{2}ab^2x^2 + \mathcal{O}(x^3). \end{aligned}$$

We *can* solve for the lower order terms here and we get an approximate value for another fixed point:

$$x_* \approx 2\frac{(ab-1)}{ab^2}$$

Now if we assume too that ab is near 1 because we're interested in the bifurcating behaviour near $x_* = 0$, we see that as ab passes through 1 we have that the critical point x_* passes through 0. Moreover, by investigating the stability for the critical point $x_* \approx 2\frac{(ab-1)}{ab^2}$ as ab passes through one, we have

$$\begin{aligned} f'(x_*) &\approx (1-ab) + ab^2 x + \mathcal{O}(x^2)\Big|_{x=x_*} \\ &= 1 - ab + ab^2 2\frac{(ab-1)}{ab^2} + h.o.t \\ &= ab - 1 + \mathcal{O}((1-ab)^2) \end{aligned} \tag{3.2}$$

So we can see that the critical point at x_* goes from being *stable* when $ab < 1$ to *unstable* when ab is greater than 1. Thus we can see that near the curve $ab = 1$ in the parameter space, the system will undergo a *transcritical* bifurcation. A partial sketch of the bifurcation diagram is provided below. It is important to note that there may be other bifurcations elsewhere in this system (in this guy in particular, numerical evidence suggests a saddle node bifurcation for a curve in parameter space too), so we have just plotted the *local* bifurcation diagram near $x_* = 0$ and $ab = 1$. The curve $ab = 1$ in the parameter space is called a *bifurcation curve*.

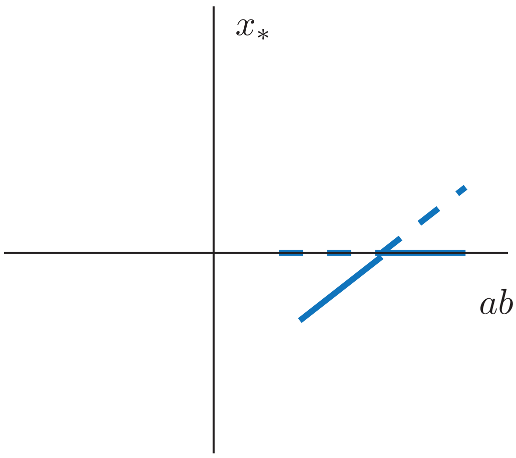


FIGURE 14. The local bifurcation diagram for eq. (3.1) near $x_* = 0$, and near the curve $ab = 1$ in the parameter space.

Example 3.2. For another example, let's name and classify all the bifurcations that occur in the 1-D system below and sketch the bifurcation diagram as well:

$$\dot{x} = f(x;\mu) = \left(x^2 - \mu^2 + \frac{\mu^4}{2}\right)(x + \mu - 1)\,. \tag{3.3}$$

Critical points are the points in (μ, x_*) space where $f(x_*;\mu) = 0$. These are just the locus of points where the either of the two parts of the product of $f(x;\mu) = 0$. That is the curves defined by

$$x_*^2 - \mu^2 + \frac{\mu^4}{2} = 0, \quad \text{or} \quad x_* + \mu - 1 = 0.$$

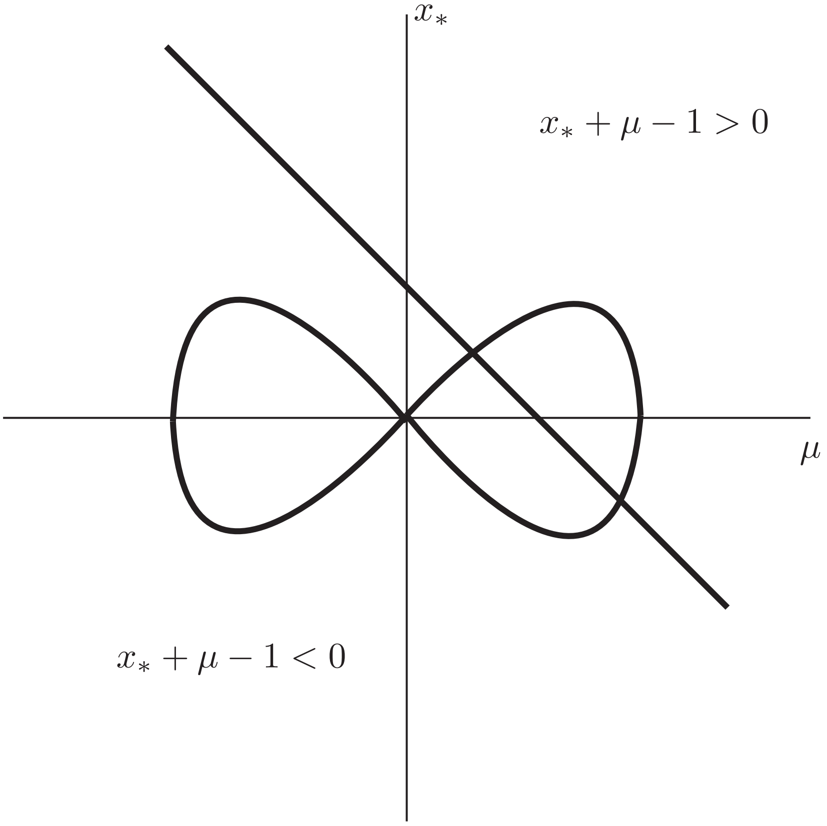


FIGURE 15. The bifurcation diagram for eq. (3.3) without any stability data.

Both of these curves are readily recognisable. The first is the 'bow-tie' orbit from the undamped Duffing oscillator - formally, the shape is called a lemniscate. The second one is a straight line. A plot of the curves is in Figure 15. Now all we need to do is compute the stability data. To compute the stability data, we use the derivative. We have that

$$\frac{d}{dx}f(x;\mu) = 2x(x+\mu-1) + \left(x^2 - \mu^2 + \frac{\mu^4}{2}\right). \tag{3.4}$$

When $x_* = 1-\mu$ we are on the line, and the first part of the derivative in eq. (3.4) is 0. Thus, on this curve we have that

$$f'(x;\mu) = x^2 - \mu^2 + \frac{\mu^4}{2}. \tag{3.5}$$

Plugging in $x_* = 1-\mu$ gives

$$f'(x_*;\mu) = 1 - 2\mu + \frac{\mu^4}{2}.$$

We are interested in the sign of this quantity for $\mu \in \mathbb{R}$. When $|\mu| \gg 1$ we have that this quantity is positive, so far out on the line $x_* = 1-\mu$ we have that the critical points are unstable. We know that the sign of the derivative can only change when going through 0, which happens exactly when the line intersects the lemniscate (c.f. eq. (3.5)). Thus we have that the critical point is stable inside the lemniscate and unstable outside it.

To see what happens for critical points on the lemniscate, we note first that to the right of the line in Figure 15 we have that $x_* + \mu - 1 > 0$ while to the left of it $x_* + \mu - 1 < 0$. This can be seen by simply evaluating the formula for the line at $(0,0)$ in the (μ, x_*) plane and then observing that the quantity will only change sign crossing the line.

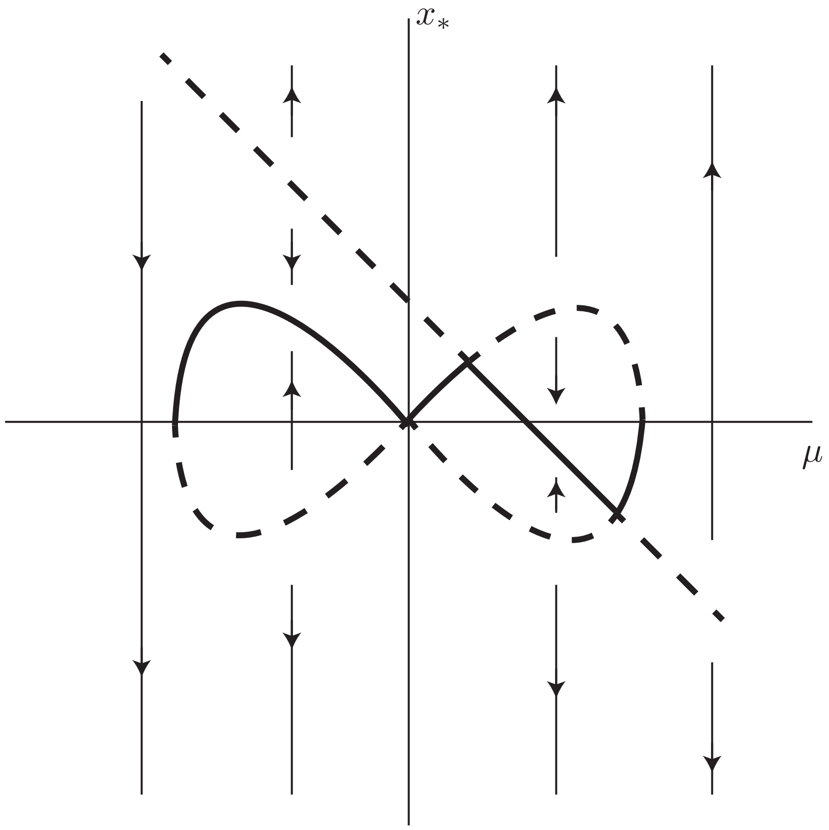


Figure 16. The bifurcation diagram for eq. (3.3).

Returning to determining the stability data for the bifurcation diagram, we have that the derivative of the right hand side of eq. (3.3) with respect to x on the lemniscate is

$$f'(x_*;\mu) = 2x_*(x_* + \mu - 1).$$

Thus it is a simple matter to determine the sign of this quantity by simply observing the signs of x_* and $x_* + \mu - 1$ respectively. The full bifurcation diagram is given in Figure 16.

Example 3.3. In this example, we explore how a bifurcation in polar coordinates can be interpreted in the Cartesian coordinate system. Consider the ODE in polar coordinates

$$\begin{aligned} \dot{r} &= (\mu r + r^3 - r^5) \\ \dot{\theta} &= 1. \end{aligned} \tag{3.6}$$

In Cartesian coordinates we have that the origin is a fixed point of this system, and moreover as $x = r\cos(\theta)$ we have

$$\dot{x} = \dot{r}\cos(\theta) - r\dot{\theta}\sin(\theta) = \mu x - y + \mathcal{O}(3).$$

Similarly, we have that $\dot{y} = x + \mu y + \mathcal{O}(3)$. Thus the Jacobian at the origin is given as

$$DF(0,0) = \begin{pmatrix} \mu & -1 \\ 1 & \mu \end{pmatrix},$$

which we see has eigenvalues $\mu \pm i$. We have that the origin changes stability as μ increases through zero and we expect some kind of Hopf bifurcation (sub or supercritical or degenerate) at the origin. To understand what is going on, we need to consider the

1-D system $\dot{r} = \mu r + r^3 - r^5 =: f(r;\mu)$ and understand the bifurcations there, and then remember that we are working in polar coordinates and interpret the context accordingly in Cartesian coordinates. The roots of $f(r;\mu)$ are

$$r_* = 0 \quad \text{and} \quad r_* = \sqrt{\frac{1}{2} \pm \frac{\sqrt{1+4\mu}}{2}}.$$

Recalling that r_* is a polar radius, we are only interested in those critical points where r_* is real and positive. We have that when $\mu < -\frac{1}{4}$ there is one critical point $r_* = 0$. When $-\frac{1}{4} < \mu < 0$ we have three critical r_*s and when $\mu > 0$ there are two: $r_* = 0$ and $r_* = \sqrt{\frac{1}{2} \pm \frac{\sqrt{1+4\mu}}{2}}$. Putting this together, we have the following table:

μ	critical points
$\mu < -\frac{1}{4}$	$r_* = 0$
$-\frac{1}{4} < \mu < 0$	$r_* = 0, \quad \sqrt{\frac{1}{2} \pm \frac{\sqrt{1+4\mu}}{2}}$
$0 < \mu$	$r_* = 0, \quad \sqrt{\frac{1}{2} + \frac{\sqrt{1+4\mu}}{2}}$

Now we need to determine the stability data. Again we take the derivative. We have that

$$f'(r;\mu) = \mu + 3r^2 - 5r^4.$$

Plugging in $r_* = 0$ reveals that the critical point at the origin goes from stable to unstable as μ increases through zero (which confirms what we already know). Substituting $r_*^2 = 1$ and $\mu = 0$ into the equation for $f'(r_*;\mu)$ we see that the upper branch of the bifurcation curve in the (μ, r) plane is stable. Since stability can only change at bifurcation points, we conclude that the lower branch must be unstable (i.e r_* with the $-$ in front of the square root). This branch collides with $r_* = 0$ when $\mu = 0$ and with the upper branch when $\mu = -\frac{1}{4}$ and so we conclude that we have a sub-critical pitchfork bifurcation at $(\mu, r_*) = (0, 0)$ and a saddle-node bifurcation in the (μ, r_*) plane at the point $(-\frac{1}{4}, \frac{1}{2})$. See figure Figure 17.

The next step is to interpret this in terms of the 'original' Cartesian coordinates. In particular - since r_* represents a limit cycle (or a periodic orbit) we have that there is a subcritical Hopf bifurcation at $\mu = 0$ and a *saddle-node bifurcation of periodic orbits* when $\mu = -\frac{1}{4}$. This is a new type of bifurcation. What this means is that as we pass through the bifurcation point a stable limit cycle and an unstable limit cycle are born - or merge and annihilate each other.

Finally, note that if we increase μ through 0, you pass through the Hopf bifurcation, and your solutions will 'jump' to the upper, stable branch of the bifurcation curve. However, lowering your μ down below 0 does *not* restore you to your previous state. Now you will stay near the stable limit cycle until you lower μ below $-\frac{1}{4}$. This phenomena is called *hysteresis* and we'll discuss it further later on.

4. Fireflies and Flows on the circle

Example 4.1 (Fireflies and the nonuniform oscillator)**.** We have considered flows (and bifurcations) on the line

$$\dot{x} = f(x).$$

In this section we are going to consider flows (and bifurcations) on the circle S^1. By S^1, I mean the complex numbers of the form $e^{i\theta}$ for $\theta \in \mathbb{R}$ (or really any interval of length 2π). To express this we write out the vector field on S^1 as

$$\dot{\theta} = f(\theta),$$

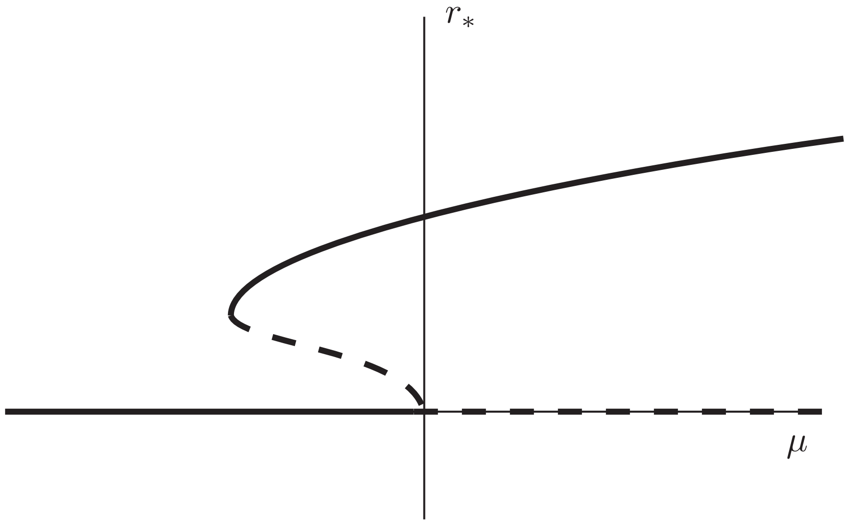


FIGURE 17. The bifurcation diagram for the radial coordinate of eq. (3.6). The sub-critical pitchfork bifurcation means we have a sub-critical Hopf bifurcation in Cartesian coordinates. The saddle-node bifurcation in the radial coordinate means that we have a saddle-node bifurcation of periodic orbits in this system.

where $f(\theta)$ is some function. One thing to notice is that it is now possible to have periodic solutions. For example if $\dot{\theta} = \omega$, a constant frequency, then the solution is just $\theta(t) = \omega t + C$ which is just a constant oscillation around the circle. So we have a periodic solution with a constant frequency. Another thing to notice is that not all $f(\theta)$s are allowed. We need $f(\theta)$ to be *periodic* in θ, otherwise the conditions of the theorem for existence and uniqueness will not hold. As an example of what I mean, the ODE $\dot{\theta} = \theta$ doesn't really make sense on the circle, as the vector field won't be continuous at the endpoints (for example $f(0) \neq f(2\pi)$). So we need to be a little careful with our right hand side.

We consider the so-called *non-uniform oscillator*, which is a flow on S^1 defined by

$$\dot{\theta} = \omega + a\sin(\theta) \qquad \omega, a \in \mathbb{R}^+. \tag{4.1}$$

If $a = 0$ then $\dot{\theta} = \omega$ then this is what is called the *uniform oscillator*, and flows just move around the circle at a constant frequency. Let's assume that $\omega > a > 0$ for the time being. In this instance, you can see from the plot of $f(\theta)$ vs θ, (see Figure 18) there are no critical points, so the flow just moves around the circle. It doesn't move at a uniform speed though (hence the name of this example), but instead goes around faster when $f(\theta)$ is larger and slower when $f(\theta)$ is smaller. It's going the fastest when $\theta = \frac{\pi}{2}$ and the slowest when $\theta = -\frac{\pi}{2}$. When $\omega = a$ we have a semi-stable fixed point, solutions all travel around the circle and end up at $\theta = -\frac{\pi}{2}$. As a is increased further, we see that two fixed points appear in a saddle-node bifurcation. One is stable and one is unstable. See Figures 19 and 20. Stability analysis on the fixed points is done just in the same way as stability analysis on the line - take derivatives and examine the sign of $f'(\theta_*)$, for θ_* a fixed point.

You can see that if θ_* is such that $f(\theta_*) = 0$ (i.e. θ_* is a fixed point) then

$$\sin(\theta_*) = -\frac{\omega}{a} \qquad \text{and,} \qquad \cos(\theta_*) = \pm\sqrt{1 - \left(\frac{\omega}{a}\right)^2}.$$

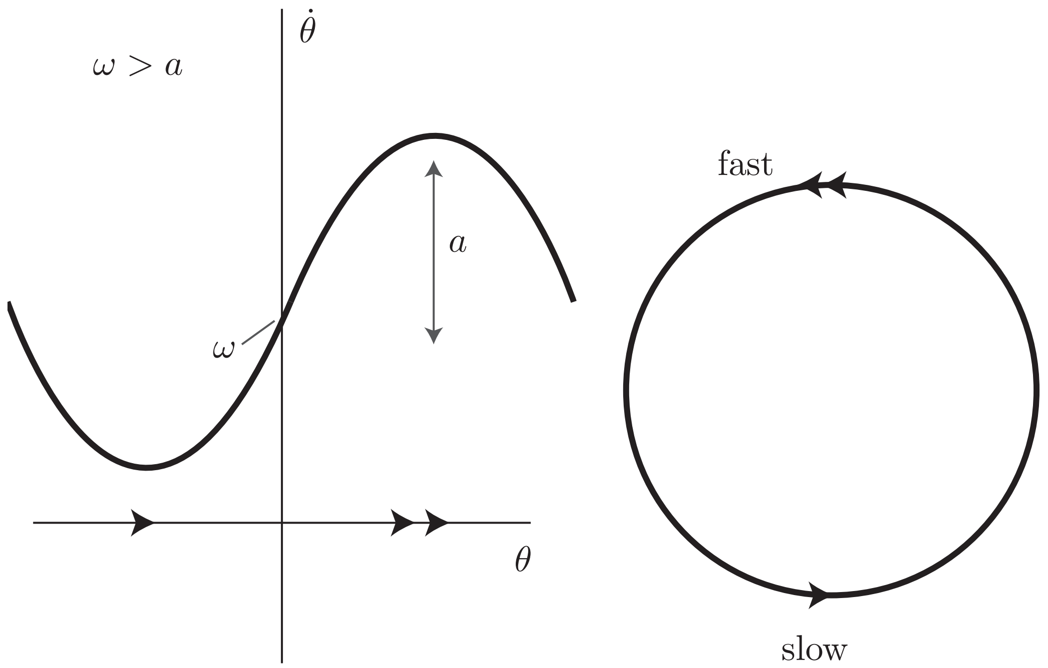


Figure 18. $\omega > a > 0$. There are no fixed points of the flow on the circle.

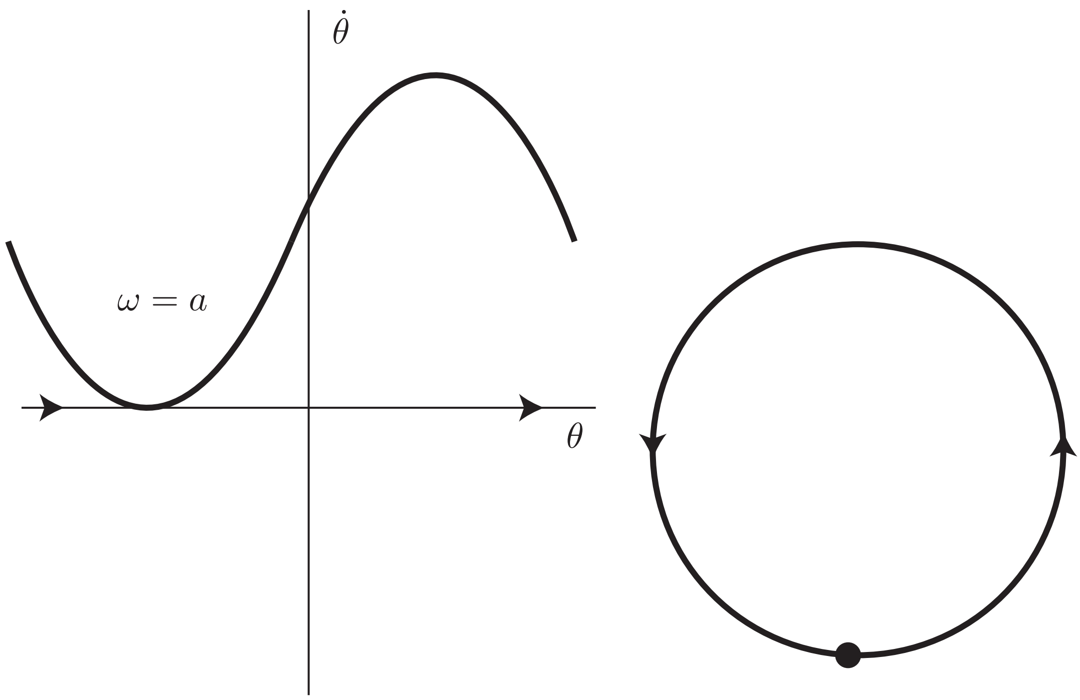


Figure 19. $\omega = a$. There is a semi-stable fixed point of the flow on the circle.

Then we have that

$$f'(\theta_*) = -a\cos(\theta_*) = \mp a\sqrt{1 - \left(\frac{\omega}{a}\right)^2},$$

and so we have that the fixed point with $\cos(\theta_*) > 0$ is stable ($f'(\theta) < 0$) and the fixed point with $\cos(\theta_*) < 0$ is unstable.

One of the most striking applications of flows on the circle involves fireflies. For a great video of this effect, check out David Attenborough, and the BBC program 'Trials of Life'

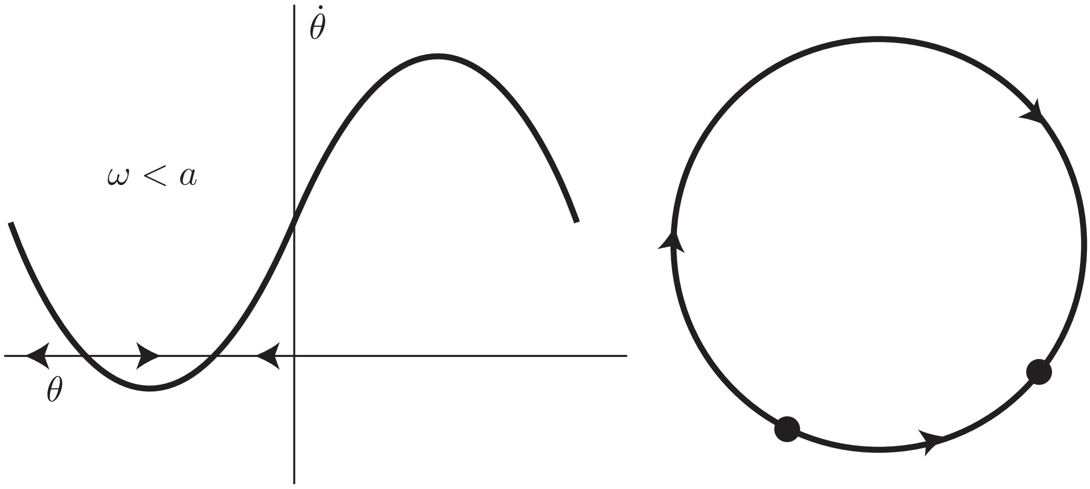


FIGURE 20. $0 < \omega < a$. There are two fixed points of the flow on the circle. One is stable and the other is unstable.

the episode entitled 'Talking to Strangers'. About halfway through the program shows a really neat clip of thousands of Malaysian fireflies flashing in synchrony. A simple model of this was developed by Ermentrout and Rinzel in 1984. Suppose that $\theta(t)$ is the phase of a firefly's flashing rhythm, with $\theta(0)$ being the flash. Then in the absence of stimuli, the fireflies have a natural frequency, say ω. Now, suppose you point a flashlight at a firefly and turn it off and on repeatedly, at some different rate Ω. So we have that $\dot{\phi} = \Omega$, where $\phi(0)$ represents the flashlight turning on, and $\phi(t)$ is the phase of the stimulus's rhythm. If $\Omega > \omega$ then the firefly responds by speeding up its flashing in order to try and synchronise while if $\omega > \Omega$ then it slows down. A model that incorporates this is the following:

$$\dot{\theta} = \omega + A\sin(\phi - \theta).$$

The parameter A is called *the resetting strength*. It is a measure of the fireflies ability to modify its instantaneous flashing. If the firefly syncs up with the stimulus, it is called *entrainment*. The question now is, when does entrainment occur? To answer this, we look at the phase difference, $\delta = \phi - \theta$. We have then that

$$\dot{\delta} = \Omega - \omega - A\sin(\delta). \tag{4.2}$$

Finally, if we make the change of variables $\mu := \frac{\Omega-\omega}{A}$ and $\tau = At$ with $' := \frac{d}{d\tau}$ then eq. (4.2) becomes

$$\delta' = \mu - \sin\delta, \tag{4.3}$$

which is the same as the nonuniform oscillator discussed earlier. The parameter μ is a measure of the relative frequency difference versus the resetting strength. When μ is small, the frequencies are close together and we expect entrainment. If $\mu = 0$ all frequencies move towards the stable fixed point at $\delta_* = 0$. When $0 < \mu < 1$ we still have a stable fixed point $\delta_* > 0$, and the firefly eventually entrains here as well. In this case however it won't flash at the same time as the stimulus, but will be what is called *phase locked.* It will flash with the same relative difference to the stimuli's flash every period. In this case since $\delta_* > 0$ the stimulus always flashes ahead of the firefly. (If we considered $\Omega < \omega$ we could see the opposite effect.) We continue to increase μ past the bifurcation point $\mu = 1$. For $\mu > 1$ there is no fixed point, no phase locking, and what happens is called *phase drift.* The phase difference increases indefinitely (of course when $\delta = 2\pi$ you don't notice this and the firefly seems entrained for a while). The other thing to notice is that the phases don't separate at a uniform rate. When δ is close to $\frac{\pi}{2}$, the phase separation is faster.

Finally, we consider what happens if $\mu = 1 + \varepsilon$. Because we are close to the point where there is a bifurcation, we have that for θ near $\frac{\pi}{2}$, $\dot{\theta}$ is very small, meaning θ changes very slowly. This is called a *bottleneck* effect or a *ghost* effect, because the presence of the (stable) equilibrium is still 'felt' even after it is not there any more.

5. Glycolysis

In this example, we are going to put just about everything we've been doing for the last two months together and perform some qualitative analysis on a simplified model of a part of a biochemical process called *glycolysis*. Glycoysis is the process by which cells convert sugar into energy in the form of ATP (adenosine triphosphate). It is a complicated metabolic pathway, that involves numerous intermediary states. During the process, it has been observed that the concentrations of some of the intermediary biological chemicals *oscillate* in time. The oscillations have a period of about 30 seconds to sometimes as much as 20 minutes.

The model, due to Sel'kov (1968), is

$$\begin{aligned} \dot{x} &= -x + ay + x^2 y, \\ \dot{y} &= b - ay - x^2 y, \end{aligned} \tag{5.1}$$

where x is the concentration of a chemical called ADP (adenosine diphosphate), and y is the concentration of F6P (fructose-6-phosphate), both of which are intermediate chemicals occurring in the metabolic pathway. The parameters a, b are both positive. The parameter a represents the rate at which the F6P is converted into ADP (how fast the ADP 'eats' the F6P and makes more ADP). The parameter b, loosely speaking, represents the amount of sugar (glucose) available to the cell that can be used in the glycolysis.

The first step is to plot the nullclines. The equation for these are

$$\dot{x} = 0 \Leftrightarrow y = \frac{x}{a + x^2} \quad \text{, while} \quad \dot{y} = 0 \Leftrightarrow y = \frac{b}{a + x^2}.$$

In addition to plotting the nullclines, we can plot the general direction of the flow in the relevant regions of the first quadrant (see Figure 21). Further, we claim that the dashed lines together with the x- and y-axes together define a trapping region for the flow (that is a compact set that gets mapped to its interior under the flow). The only hard part is showing that the dashed line from $(b, b/a)$ with slope -1 down to the line where $\dot{x} = 0$ has arrows on it which always point to the interior of the region. To see this, consider the quantity, $\dot{x} + \dot{y}$. Substituting the values of the vector field in on this, we have

$$\dot{x} + \dot{y} = -x + ay + x^2 y + b - ay - x^2 y = b - x.$$

So in other words, if $b < x$ then we have that $\dot{y} + \dot{x} < 0$. This means that $\dot{y} < -\dot{x}$, or on the line with slope -1 when $x > b$ and up until the vertical nullcline, the arrow of the vector field is pointing downward, and hence into the region. So we can conclude that we have a trapping region.

Inside the trapping region, we see that there is a single critical point $(b, \dfrac{b}{a + b^2})$. Evaluating the linearisation of the flow at this point we have

$$DF\left(b, \frac{b}{a + b^2}\right) = \begin{pmatrix} -1 + \frac{2b^2}{a+b^2} & a + b^2 \\ -\frac{2b^2}{a+b^2} & -a - b^2 \end{pmatrix}.$$

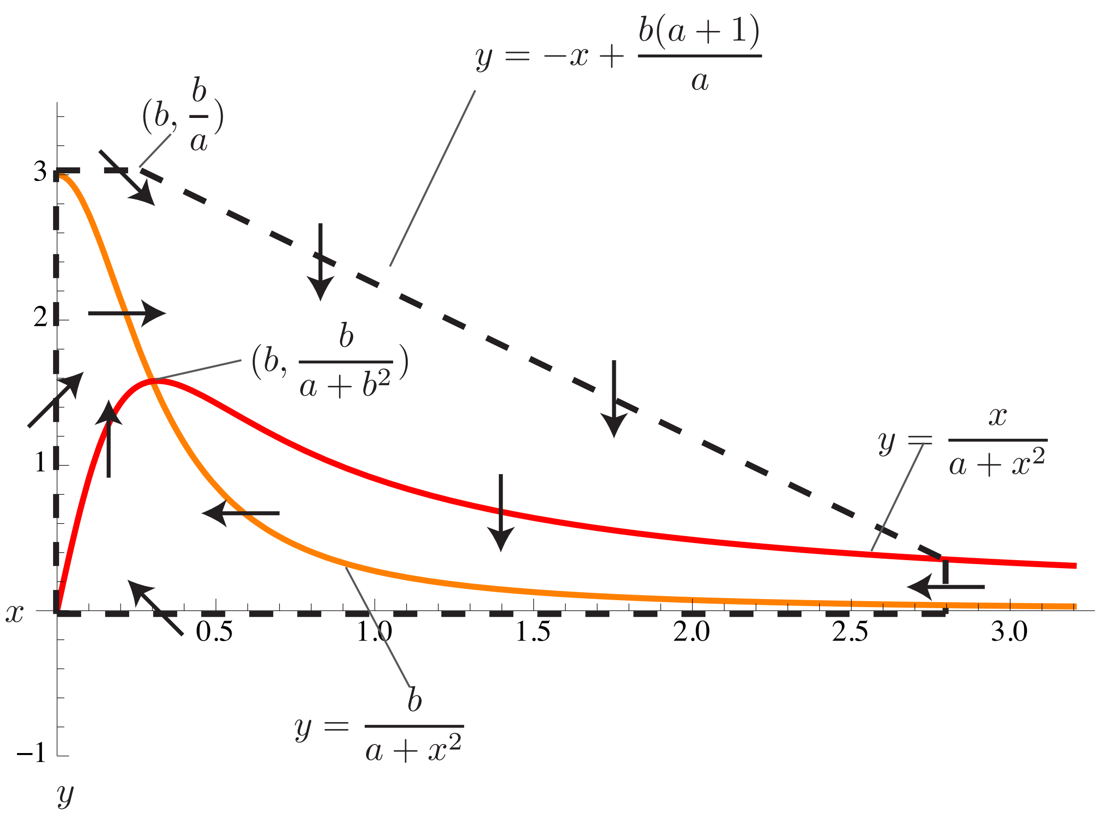


Figure 21. The nullclines, and direction of the flow of eq. (5.1) along them. The vertical nullcline is in red, while the horizontal nullcline is in orange. There is a critical point at $(b, \frac{b}{a+b^2}.)$ The black, dashed lines form a trapping region.

The eigenvalues of this guy are a mess, but we can see that the determinant $\delta = a+b^2 > 0$ and so we have a stable fixed point when the trace $\tau = -1 - a - b^2 + \frac{2b^2}{a+b^2} < 0$ and an unstable fixed point when $\tau > 0$. We can plot the line $\tau = 0$ in the (a, b) plane and we see that in the (a, b) plane this cuts out a closed region where a and b are both positive. We have that inside the region $\tau > 0$, while outside it $\tau < 0$. See Figure 22. Inside the shaded region, we have a trapping region with a single, repelling fixed point inside it. We can therefore cut a small open ball from the trapping region, and we have an annular neighborhood with no fixed points, and some points on the boundary which get mapped to the interior by the flow. We can thus conclude from the theorem following the Poincaré–Bendixson theorem that we have a limit cycle inside the region. If we fixed a value of a, and varied b, such that we were to intersect the shaded region (the red curve in Figure 22), we would have a stable fixed point, which would then destabilise and a stable limit cycle would emerge, and then disappear again as we continued to increase the parameter b. Thus we have that there would be two supercritical Hopf bifurcations (one at each end of the shaded region). A sketch of the bifurcation diagram is in Figure 23.

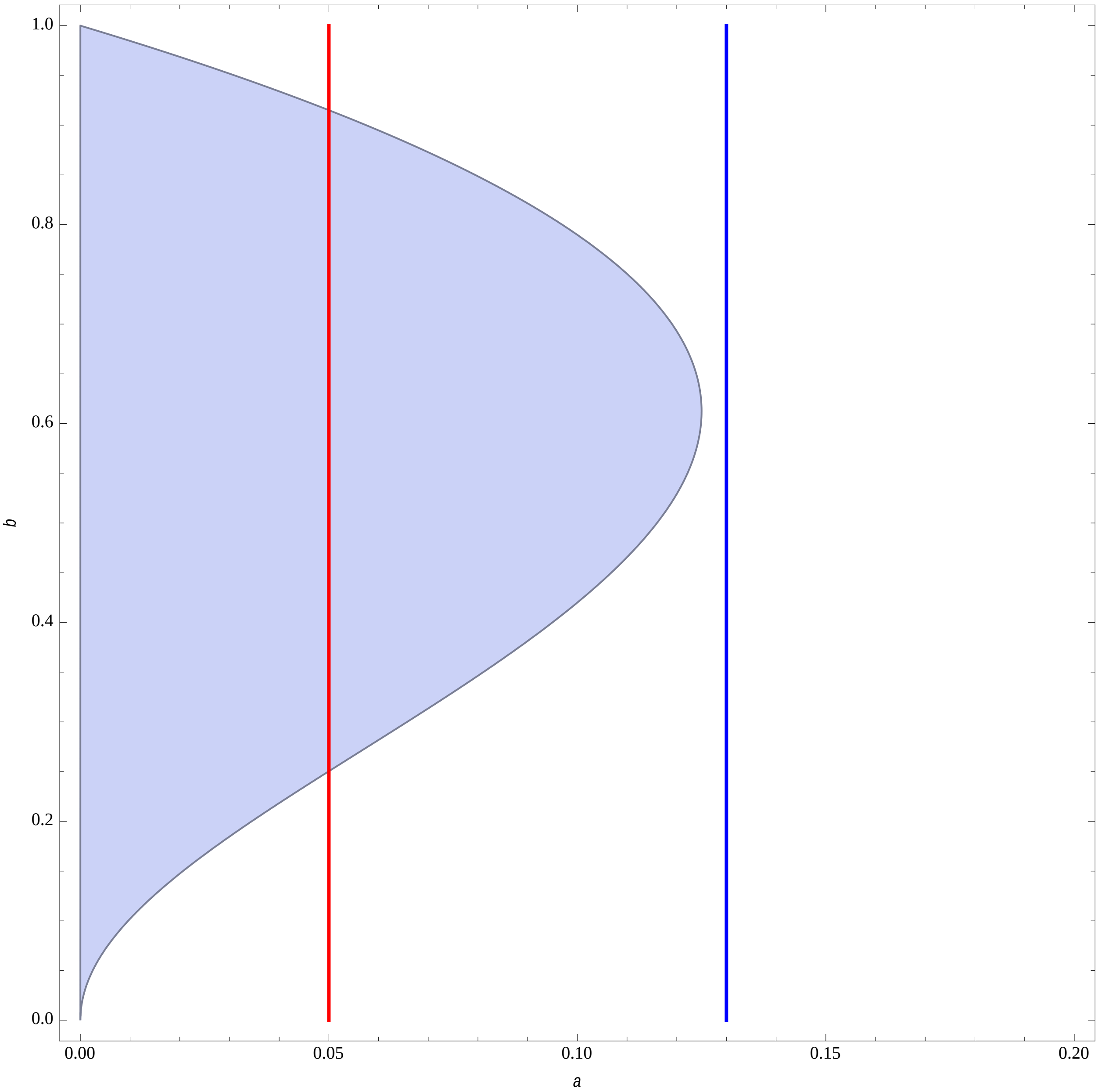


FIGURE 22. The black line is a plot of the curve $\tau = 0$, where τ is the trace of the Jacobian at the fixed point inside the trapping region. The shaded region is where $\tau > 0$ and the fixed point is unstable, while the white region is where the fixed point is stable.

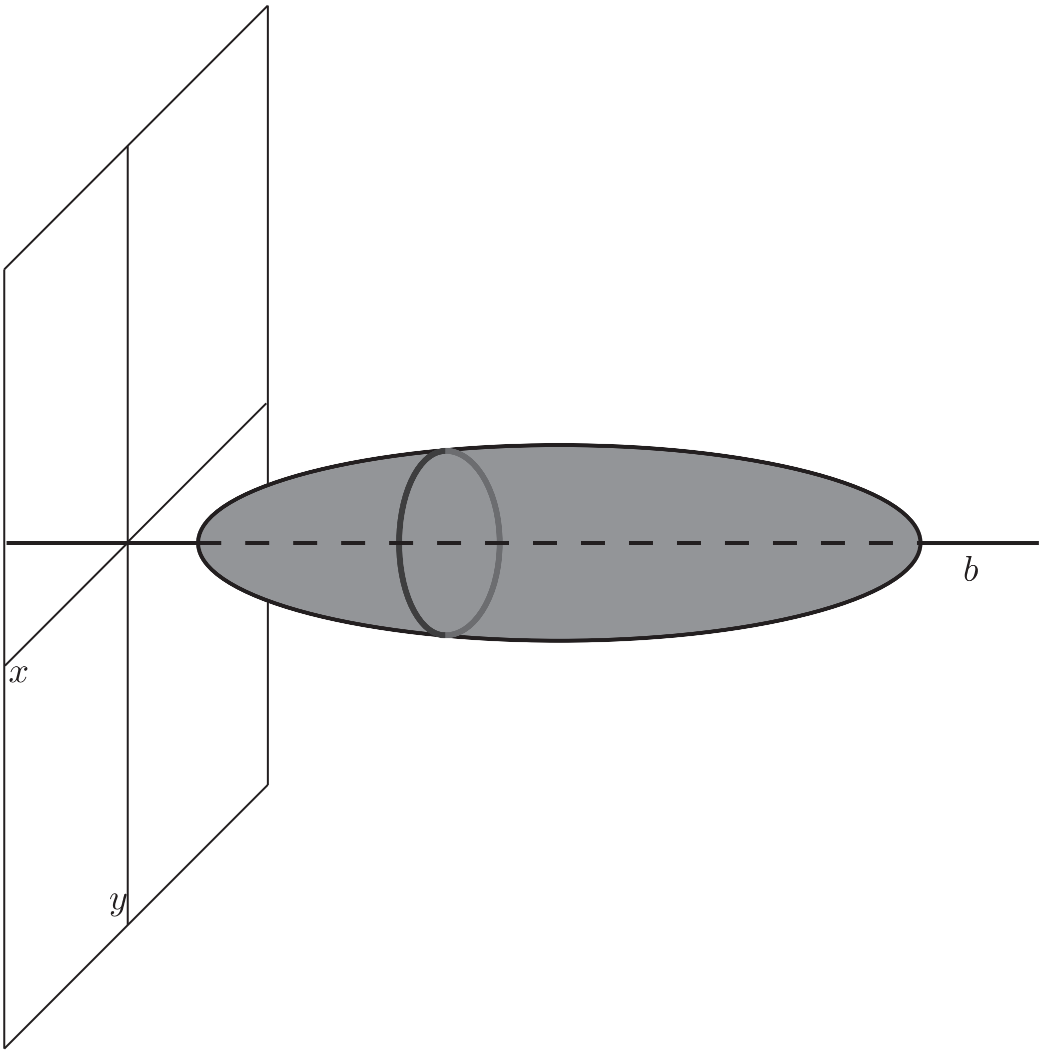


FIGURE 23. A qualitative sketch of the bifurcation diagram for the shaded region in Figure 22. Note that the stable limit cycle does not emerge from the origin, but at the value of b where the shaded region begins. The limit cycle then closes up again in another supercritical Hopf bifurcation as b is increased. Note that the limit cycle in this case is *not* actually an ellipse but is asymmetric. See Figures 26 to 28 for a quantitative picture of the limit cycle.

In terms of what is happening in metabolic process, increasing b (roughly) corresponds to increasing the amount of sugar (glucose) you are feeding the cells, so if you give it a little bit, the intermediary chemicals in the metabolic pathway each go straight to the fixed point (i.e. - the amount of ADP and F6P are driven directly to their fixed values in the system). Then as b is increased into the region where a limit cycle emerges, we have that the amount of ADP and F6P *oscillate* in the reaction. Then as the parameter is increased further, the system returns to one with a stable attracting fixed point.

We remark here, that if you were to fix a outside the region where varying b would take you through the shaded region in Figure 22 (as in the blue line), then you wouldn't have any bifurcations, and you would just have a stable fixed point. It is also worth noting that for a fixed value of a the curve that is the trace and determinant of the Jacobian in the (τ, δ) plane is parametrized by the value of b. It is worth having a look at these, so you can see for yourself that the trace of the Jacobian $DF(b, \frac{b}{a+b^2})$ goes from negative to positive and then back to negative. See Figure 24.

Lastly, you can (and should) examine eq. (5.1) into pplane. I have included a few phase portraits for varying b for a fixed a when you're in the parameter regime where a limit cycle occurs, and one figure showing the phase portrait for when you are outside of this regime.

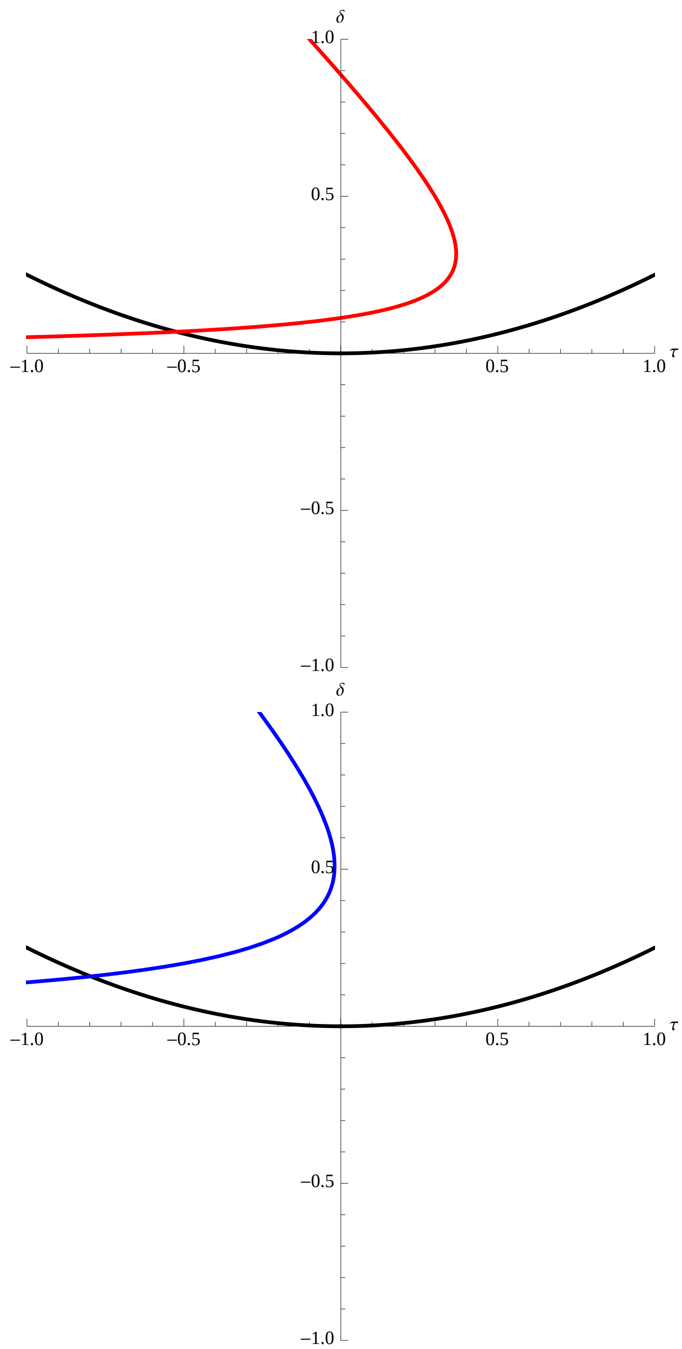


FIGURE 24. Curves of the trace and determinant of the Jacobian of eq. (5.1) at the fixed point considered as parametrized curves, parametrized by the value of b for a fixed value of a. The values correspond to the values in Figure 22. The red curve entering the right half plane means that the fixed point is destabilising. The black curve is the curve $\tau^2 - 4\delta = 0$.

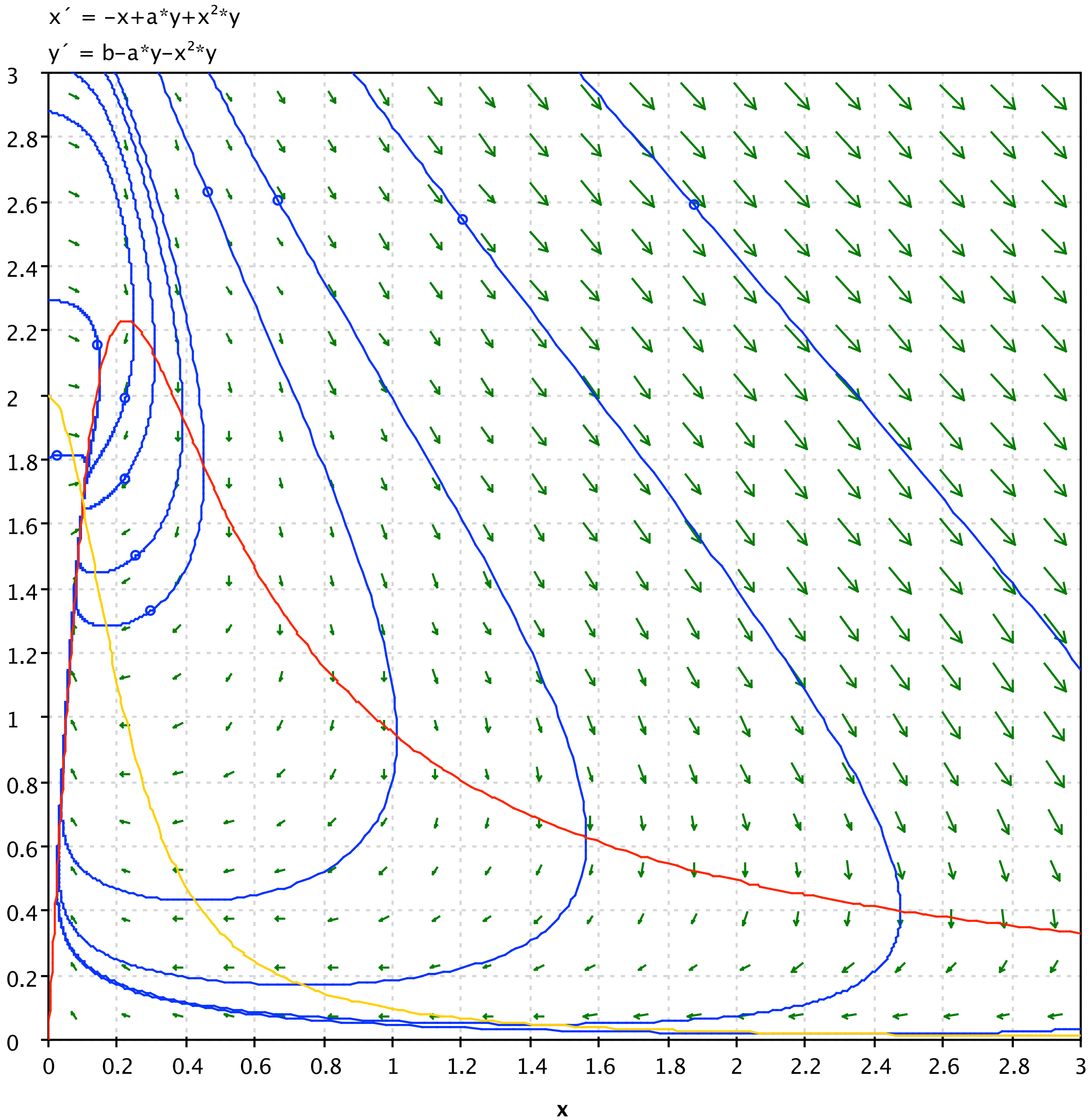


FIGURE 25. A phase portrait of eq. (5.1). The value of $a = 0.05$ and $b = 0.1$. There is no limit cycle observed, but instead is a stable, attracting fixed point.

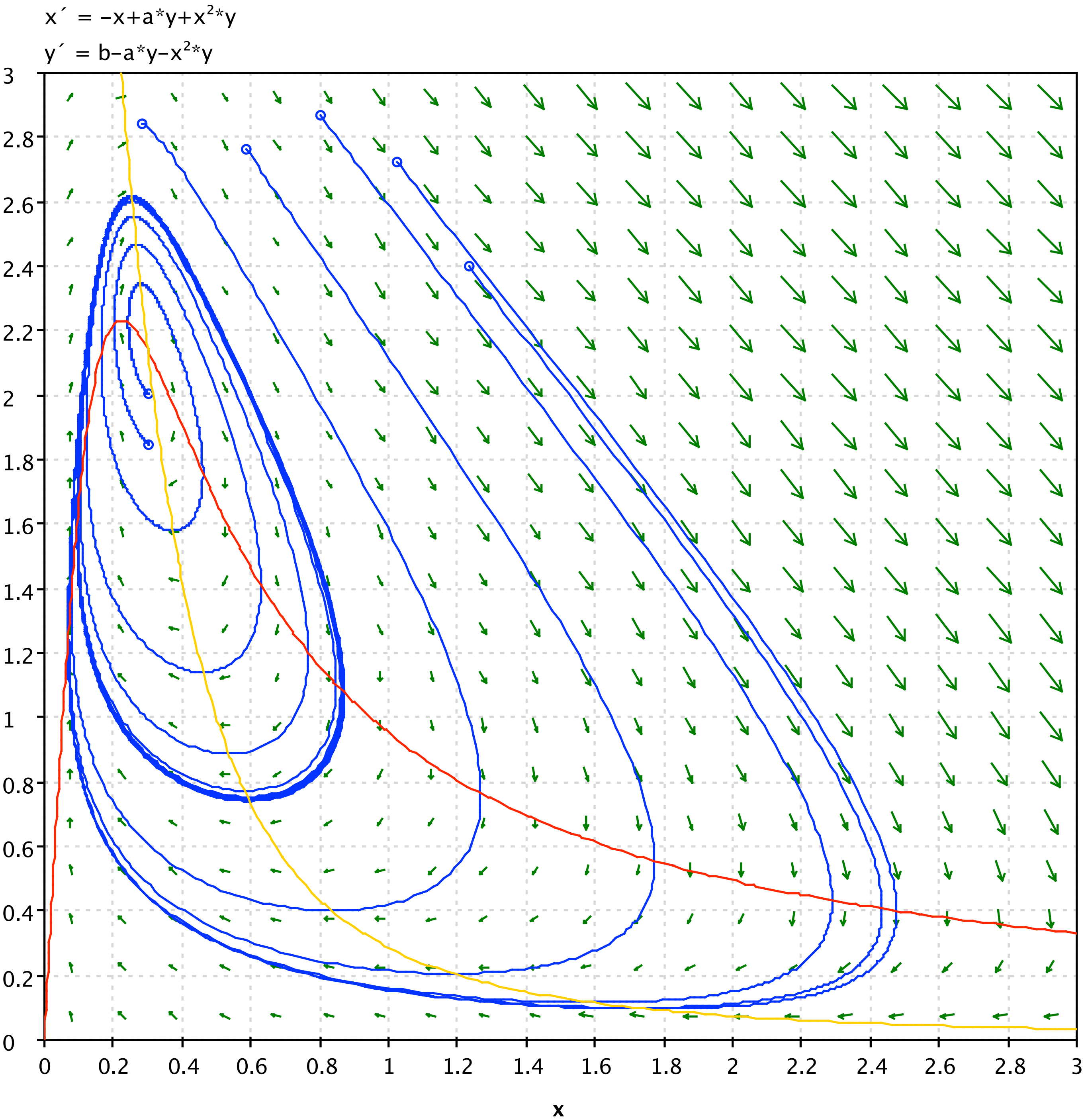


Figure 26. A phase portrait of eq. (5.1). The value of $a = 0.05$ and $b = 0.3$. The limit cycle has appeared.

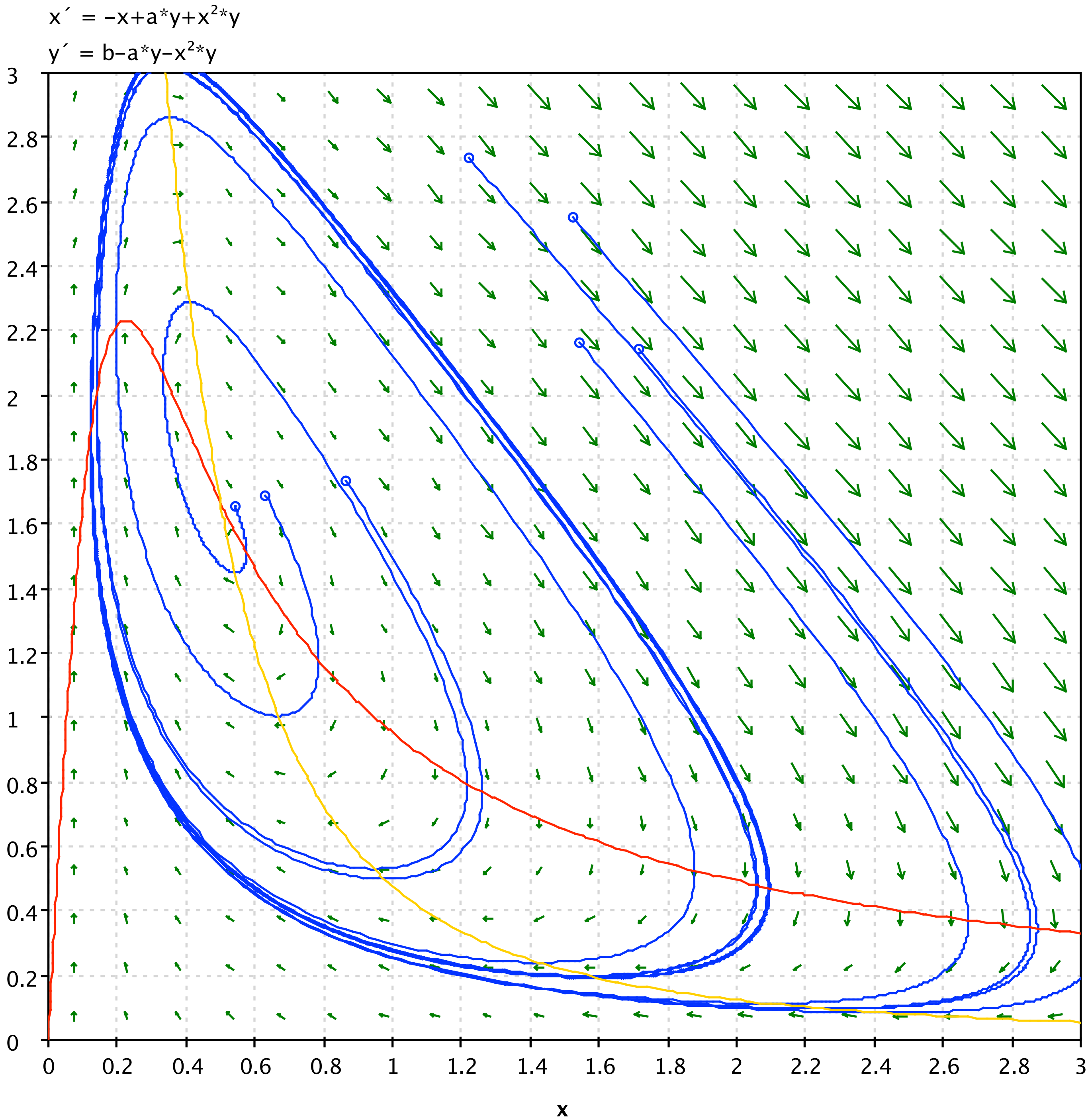


FIGURE 27. A phase portrait of eq. (5.1). The value of $a = 0.05$ and $b = 0.5$. The limit cycle is still present, and has grown in amplitude.

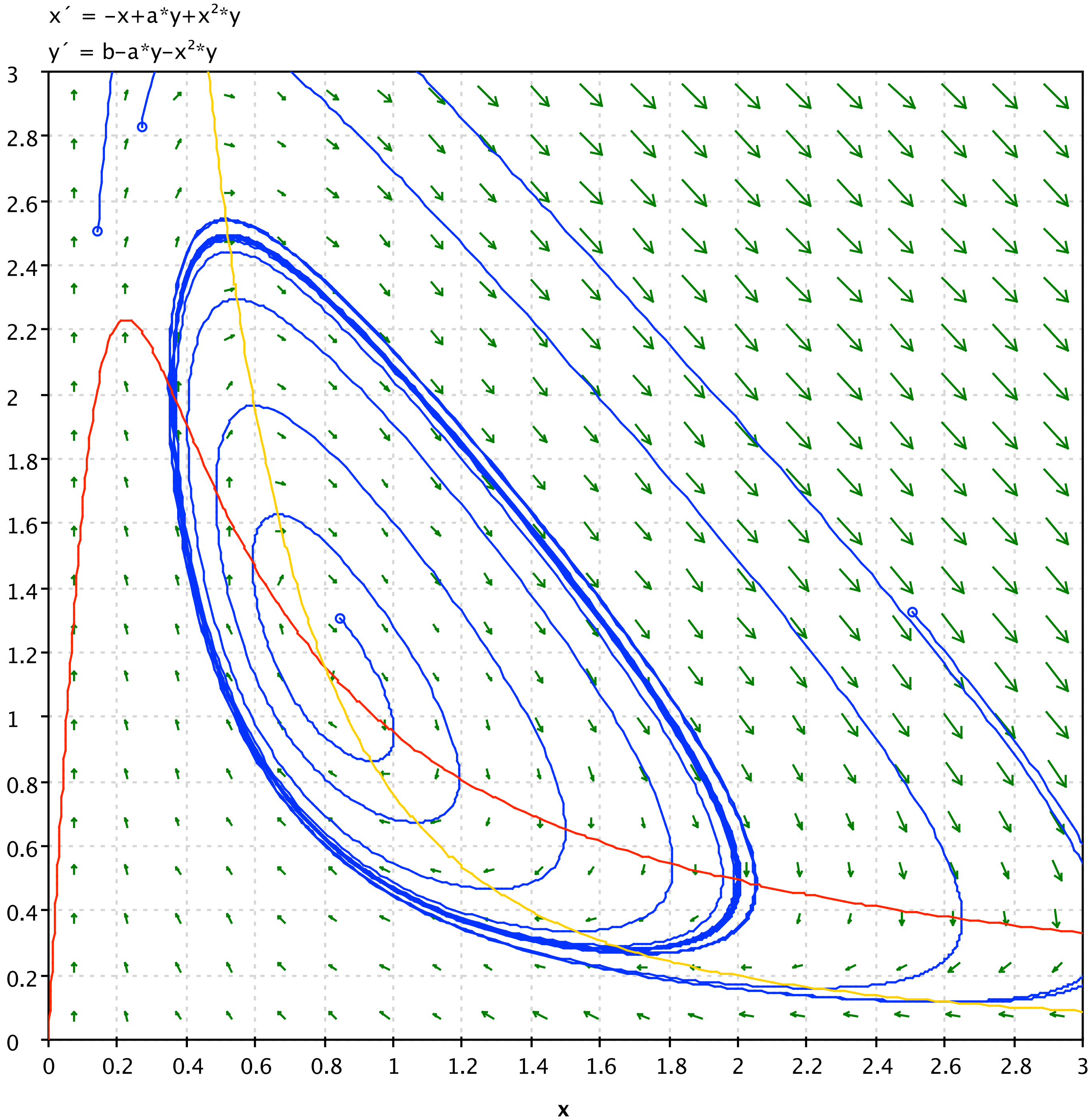


Figure 28. A phase portrait of eq. (5.1). The value of $a = 0.05$ and $b = 0.8$. The limit cycle is still present.

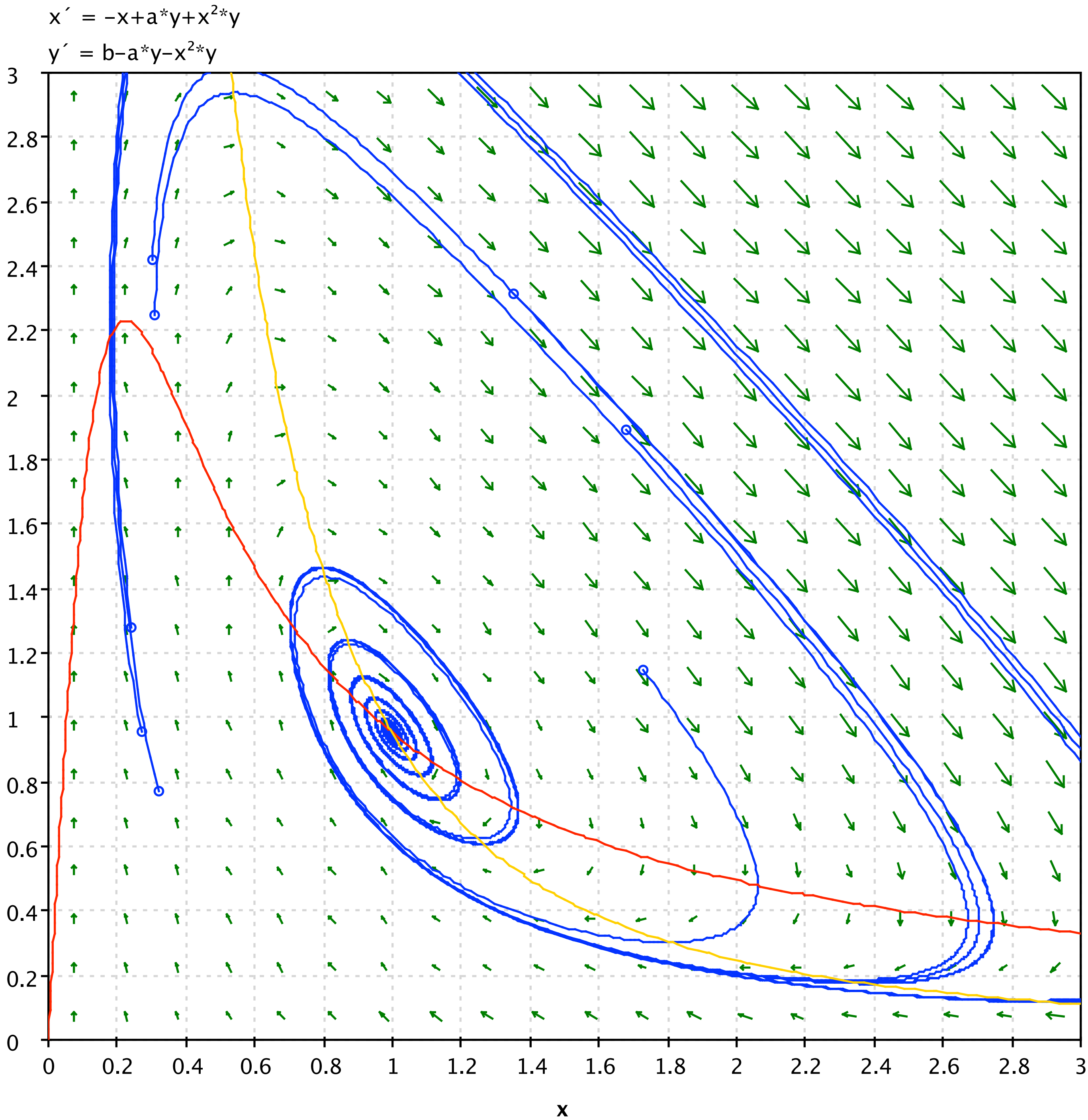


Figure 29. A phase portrait of eq. (5.1). The value of $a = 0.05$ and $b = 1$. The limit cycle has disappeared again, and the fixed point has returned to being an attractor.

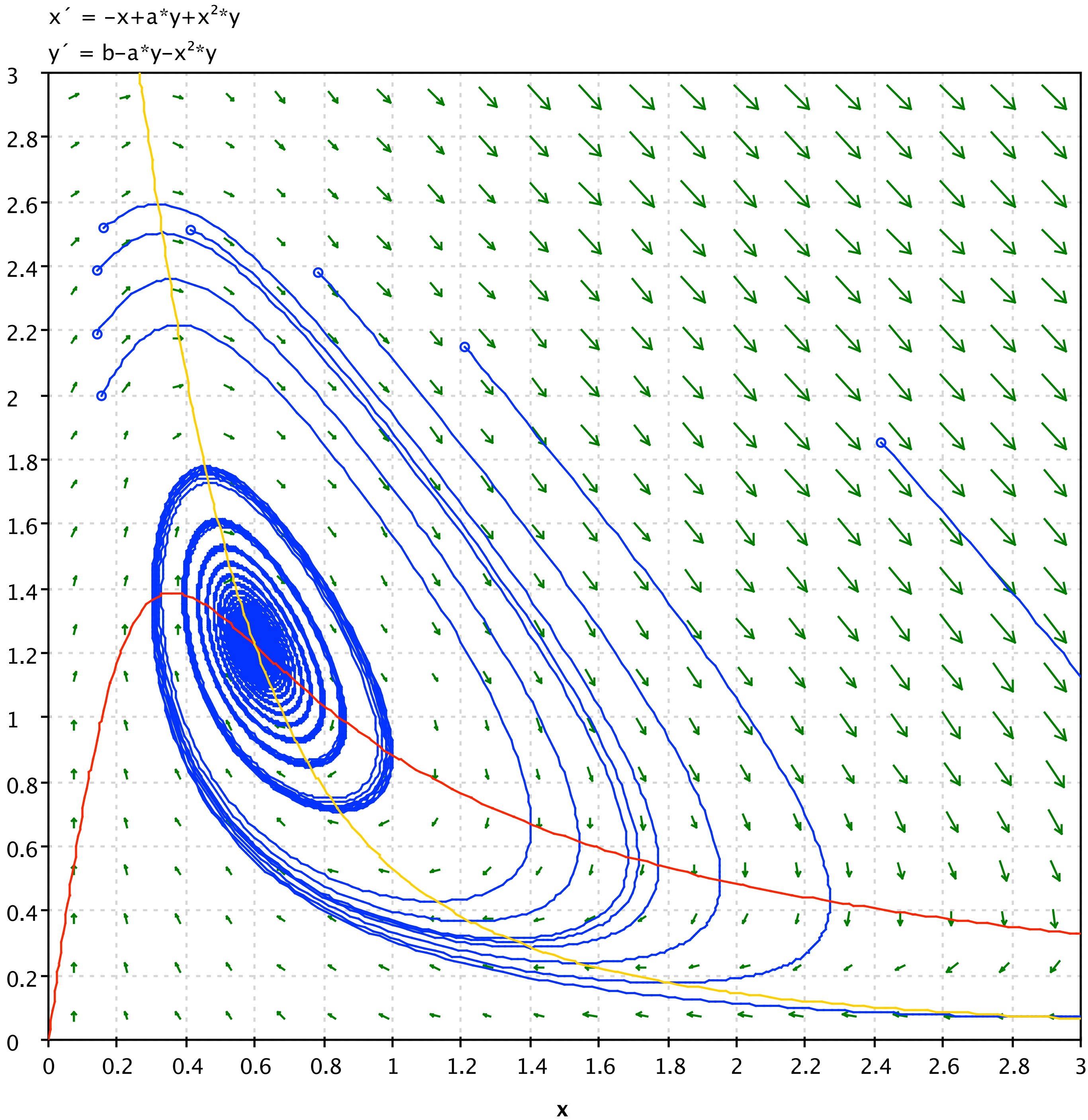


FIGURE 30. A phase portrait of eq. (5.1). The value of $a = 0.13$ and $b = 0.6$. There is no limit cylcle. The fixed point is an attractor (though a weak one).

6. A Catastrophe!

In this section we're going to investigate the beginning of a new field of mathematics which branches off (or bifurcates from) of bifurcation theory; catastrophe theory. We start with a supercritical pitchfork birfurcation

$$\dot{x} = -x^3 + ax \qquad a \in \mathbb{R} \tag{6.1}$$

with bifurcation diagram given in Figure 31.

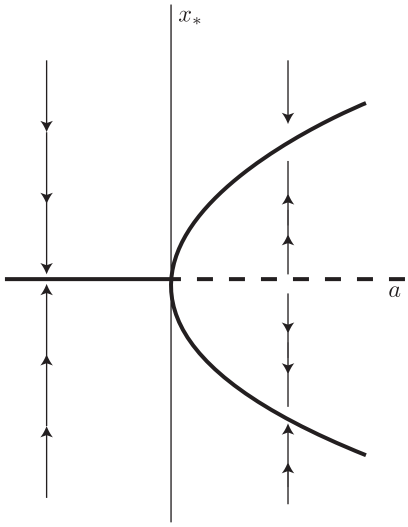


Figure 31. The bifurcation diagram of the supercritical pitchfork bifurcation

We recall that when $a \leq 0$, the right hand side, $f(x) = -x^3 + ax$ intersects the x axis once at $x = 0$, while when $a > 0$ there are three critical points, one at $x = 0$ and two more at $\pm\sqrt{a}$. See Figure 32.

Now the idea is to consider

$$\dot{x} = -x^3 + ax + b \qquad a, b \in \mathbb{R}. \tag{6.2}$$

What does adding the parameter b do? If $a \leq 0$, then not that much. It just raises or lowers the graph of $f(x)$, and the number of critical points and their stability data remain unchanged. See Figure 33. But if $a > 0$, then there is a critical value of $b = \pm b_*$ after which we will not have three critical points any more, but just one, and similarly, below which we will again see the disappearance of two critical points, from three to one. See Figure 34. We can see further, that we will have a pair of saddle-node bifurcations opposing each other in the (x_*, b) plane. The bifurcation diagram in this case is in Figure 35. One thing to remark is that for certain values of the parameter b (in fact $|b| < b_*$) there are two coexisting stable steady states simultaneously. This is sometimes referred to as *bistability*. Remember this. This will become important later on.

Okay, so that is what happens when we fix a and move b. What about if we fix b and move a. Suppose we fixed $b > 0$, and start with an $a < 0$. Then this just lifts the first graph in Figure 32 up. But, the key idea is that now the value $a = 0$ is no longer a bifurcation point. In fact, we have a new bifurcation value of our parameter $a = a_*$ where we go from having a single stable critical point to two stable critical points and one unstable one. Moreover, we no longer have a pitchfork bifurcation. This is because we always (provided $b \neq 0$) have a single stable critical point (which is the same sign as b), and so what we have instead is a saddle node bifurcation at the value $a = a_*$. See Figure 36. The bifurcation diagrams for $b > 0, b < 0$ and $b = 0$ (again, for completeness) are given in Figure 37.

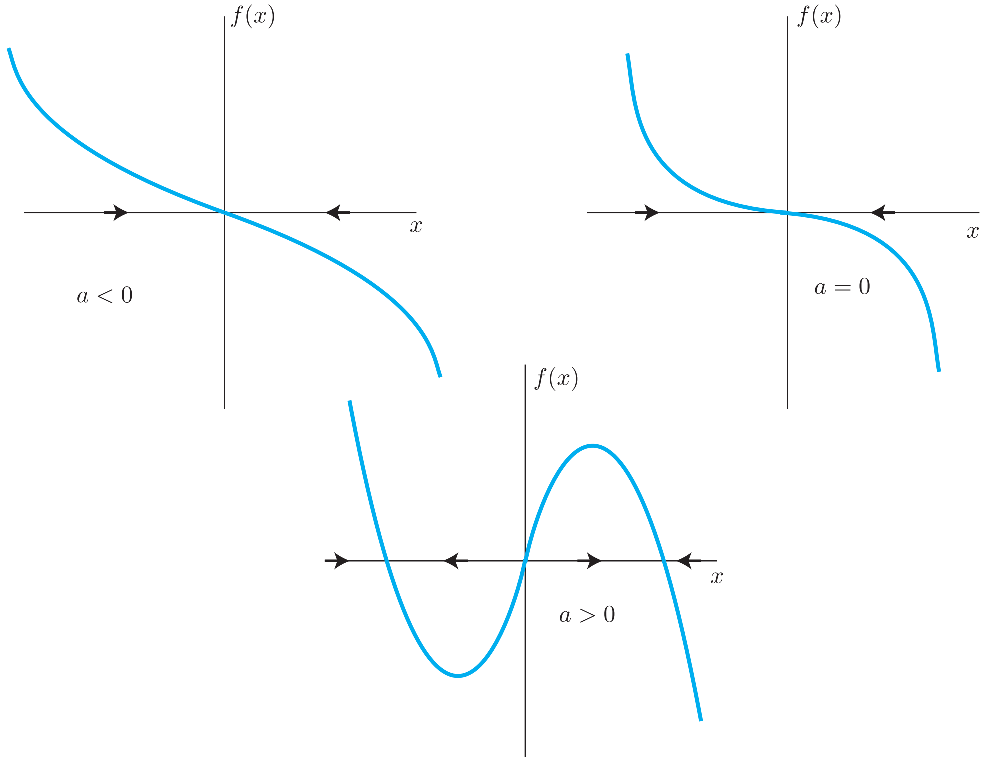


FIGURE 32. Plots of the right hand side of eq. (6.1), as well as the phase line on the x-axis for various values of the parameter a

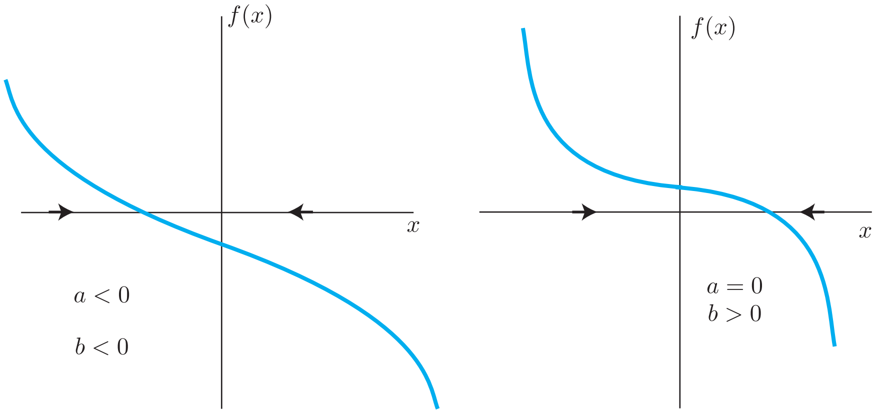


FIGURE 33. Plots of the right hand side of eq. (6.2), for various values of the parameter b when the parameter $a \leq 0$.

What we see is that the bifurcation diagrams are themselves bifurcating. That is we have a bifurcation of bifurcations. We have a saddle-node bifurcation which merges with another, stable critical point at $b = 0$ to become a pitchfork bifurcation, and then breaks up again to become a different saddle node bifurcation with a different stable critical point. Likewise if we were to return to Figure 35 and vary a, we would see that as we lowered a, the two saddle node bifurcations would merge towards each other, until they intersected at $a = b = 0$, and then we wouldn't have any bifurcation at all. Just a location of the single, stable critical point.

Now to draw the 'full' bifurcation diagram. We want a plot of the location of the fixed points x_* as we vary a and b. This is given in Figure 38. If we fix $a < 0$ there is a

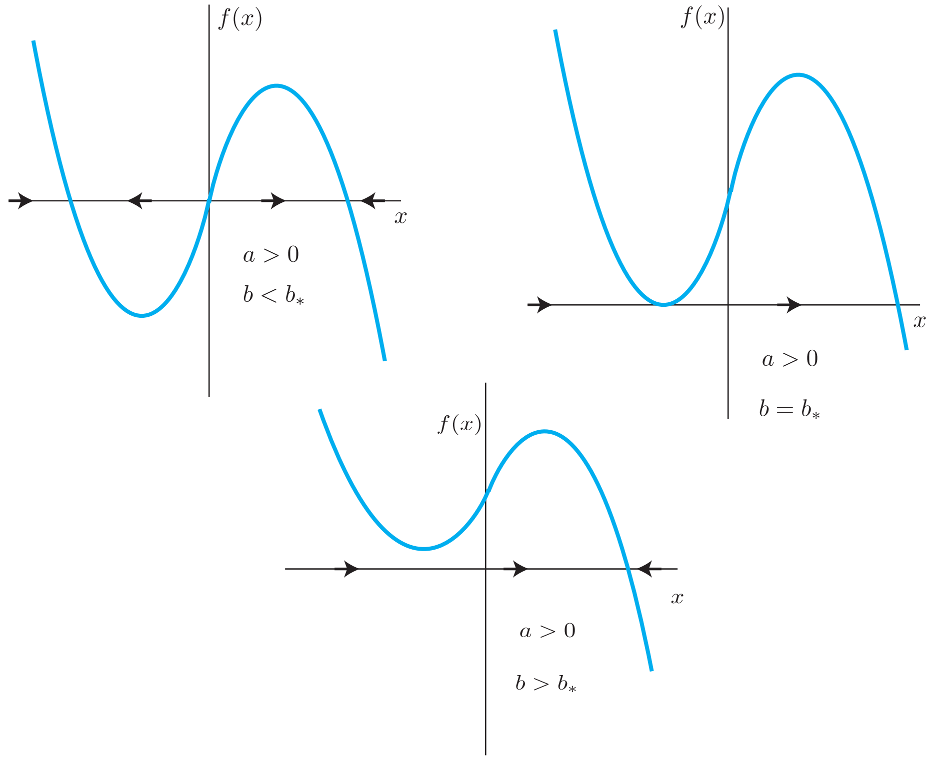


FIGURE 34. Plots of the right hand side of eq. (6.2), as well as the phase line on the x-axis for various values of the parameter b when $a > 0$.

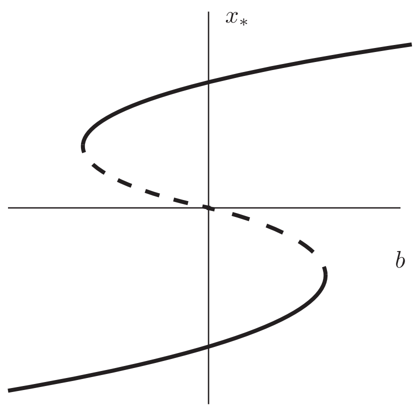


FIGURE 35. The bifurcation diagram for eq. (6.2) for a fixed $a > 0$. For certain values of b we have bistability, i.e. there are two coexisting steady states. Remember this for later.

single stable critical point for all values of b. Then as we increase a beyond 0, we see the bistability as we vary the parameter b. Likewise, if we fix $b = 0$ and vary a we see that a pitchfork bifurcation occurs, while for a fixed $b \neq 0$ and varying a we can see that at some point the fold (the location of the saddle-node bifurcation in the a parameter) will move out so that b is then in the region of bistability for that a. The saddle-node bifurcation is sometimes referred to as a *fold* bifurcation.

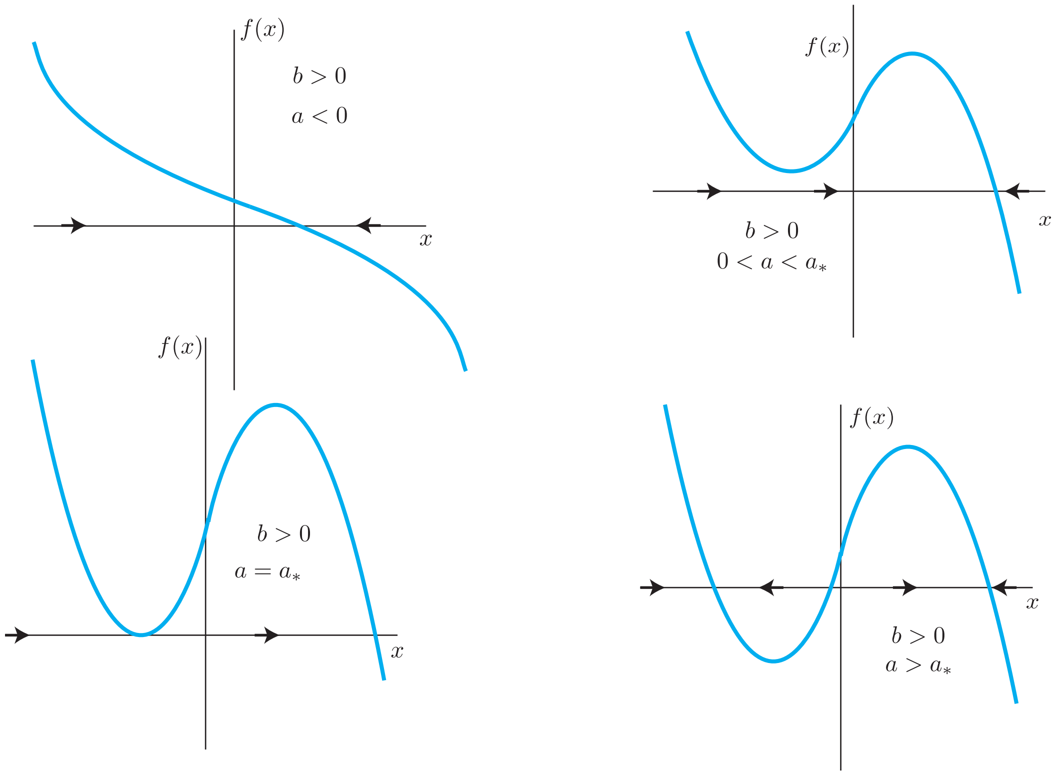


FIGURE 36. Plots of the right hand side of eq. (6.2), as well as the phase line on the x-axis for various values of the parameter a when $b > 0$.

Now what about the values a_* and b_*. They occur when the 'knee' of the cubic $f(x) = -x^3 + ax + b$ touches the x axis. That is, when the equations $-x^3 + ax + b = 0$ and $f'(x) = -3x^2 + a = 0$ are simultaneously satisfied. Solving for $f'(x_*) = 0$ gives $x_* = \pm\sqrt{\frac{a}{3}}$. then plugging this value in for x_* and using $f(x_*) = 0$ and rearranging gives a relationship between a and b

$$\left(\pm\sqrt{\frac{a}{3}}\right)\left(-\frac{2}{3}a\right) = b \quad \text{or,} \quad 27b^2 = 4a^3.$$

Now what is this? The equation $4a^3 = 27b^2$ (which some of you may recognise as the so-called *cubic discriminant* of $f(x) = -x^3 + ax + b$) is a curve of points in the (a, b) plane where the bifurcations in each parameter occur. The bifurcation curve if you will. If we were to project the bifurcation diagram in Figure 38 onto the (a, b) plane, and keep track of where the bifurcations occur, we would see this curve. See Figure 39.

To the 'right' of the curve (i.e those values of (a, b) where $27b^2 - 4a^3 < 0$), we have that the system will have three critical points, one unstable and two stable. While to the 'left" of the curve (i.e. where $27b^2 - 4a^3 > 0$), we have that the system will have one stable critical point. Moreover, the curve itself represents two sets of saddle-node bifurcations which meet at a pitchfork bifurcation. This point where the curve of saddle-node (or fold) bifurcations meet is called a 'cusp point' from geometry, and it is often said that in Figure 39 there are two curves of folds meeting at a cusp. This - families of bifurcations meeting at another, different bifurctaion, was what was first called a catastrophe. Now, actually, in order to 'classify' catastrophes, people also call the fold bifurcation a catastrophe as well.

We remark that along each of the fold curves we have that $f'(x_*) = 0$, the usual loss of hyperbolicity associated with a bifurcation. However, we also note that $f''(x_*) =$

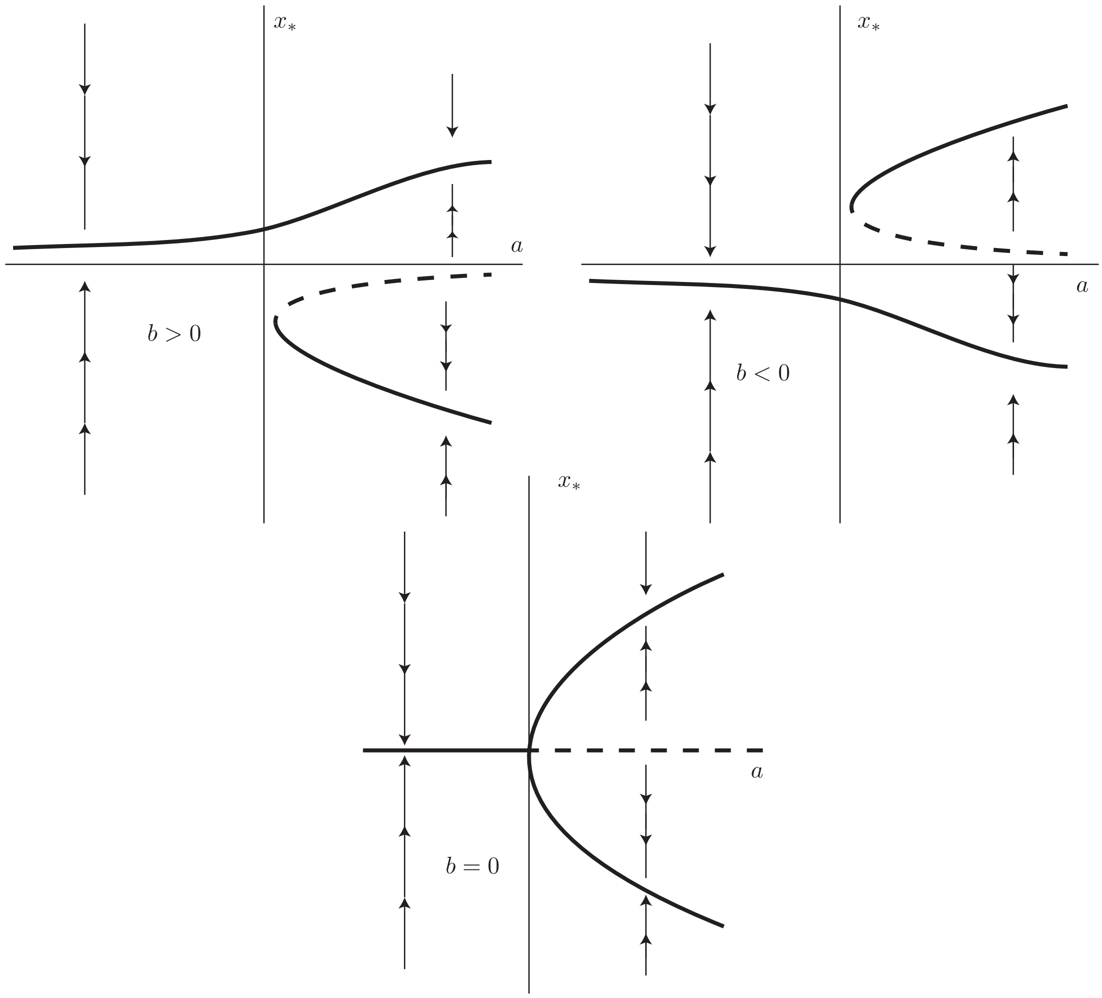


FIGURE 37. A bifurcation of bifurcations. The bifurcation diagrams of x_* in terms of a for various values of the parameter b. When $b > 0$ there is a saddle node bifurcation above the a axis which then turns into a pitchfork bifurcation for $b = 0$ and then becomes a saddle node bifurcation when $b < 0$.

$-6\left(\pm\sqrt{\dfrac{a}{3}}\right) \neq 0$ if we are away from the cusp point, but $f''(x_*) = 0$ at the cusp point (because $a = 0$ there). So we have a sort of 'double' loss of hyperbolicity. We have that $f'(x_*) = f''(x_*) = 0$. We remark that $f'''(x_*) = -6 \neq 0$. This gives us a sort of hierarchical structure naturally imbued in catastrophe theory. A sort of 'dimension' (actually it is a 'codimension', but no matter) for each catastrophe exists and is determined by how many derivatives of the right hand side of the system vanish at said catastrophe (or bifurcation).

So for example if we were to consider the system

$$\dot{x} = x^4 + ax^2 + bx + c. \tag{6.3}$$

We have three parameters. And moreover solving $f(x; a, b, c) = 0$ and $f'(x; a, b, c) = 0$ gives a relationship describing a *surface* in parameter space where a fold bifurcation is occuring, while if we mandate $f''(x; a, b, c) = 0$ as well, we will have *lines* of cusp bifurcations occurring (at the intersection of the surfaces of folds). These cusps would then intersect at a single point, which would give us a new catastrophe. This one is called *the swallowtail catastrophe*. For a picture of the surfaces, lines and swallowtail point, see Figure 40.

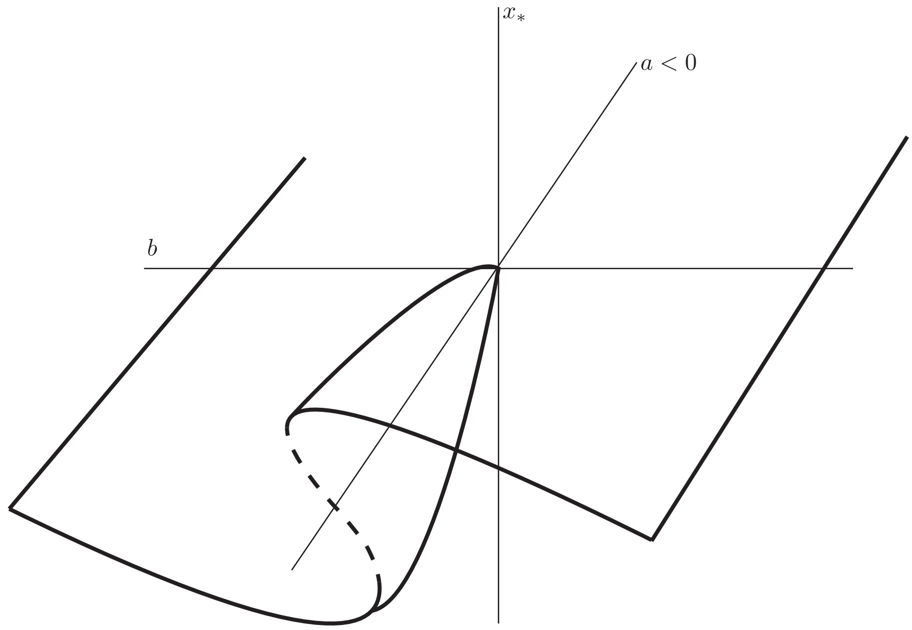


FIGURE 38. The full bifurcation diagram for eq. (6.2). You can visualise this as a folded piece of paper with a pinch point at the origin.

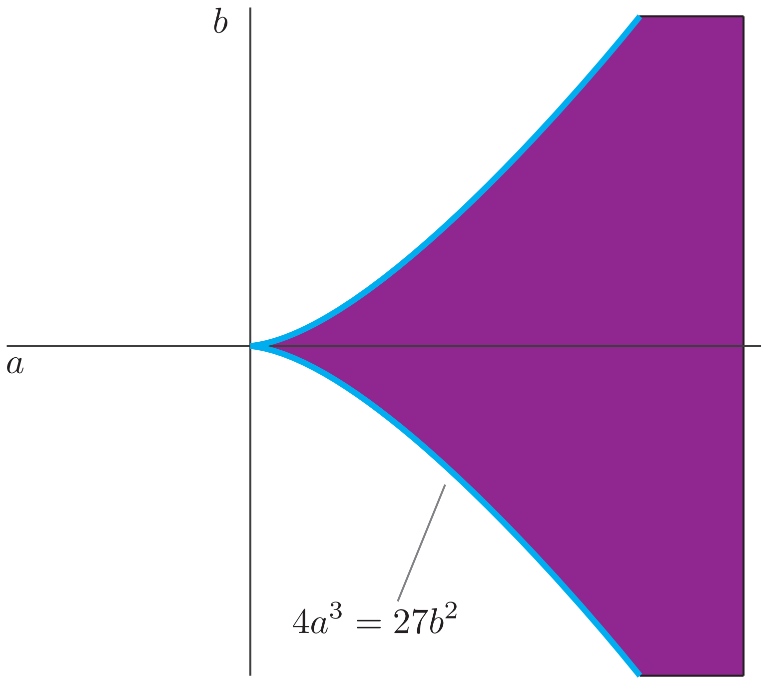


FIGURE 39. The curve $4a^3 = 27b^2$. The shaded region is where $27b^2 - 4a^3 < 0$ and there are three critical points of eq. (6.2) two stable and one unstable. The white region is where $27b^2 - 4a^3 > 0$ and there is only one stable critical point. The curve $27b^2 - 4a^3$ itself represents two curves of saddle node bifurcations meeting at a cusp point, where a pitchfork bifurcation is occurring.

There are a few more catastrophes that have names, *the butterfly catastrophe:*

$$x^5 + ax^3 + bx^2 + cx + d,$$

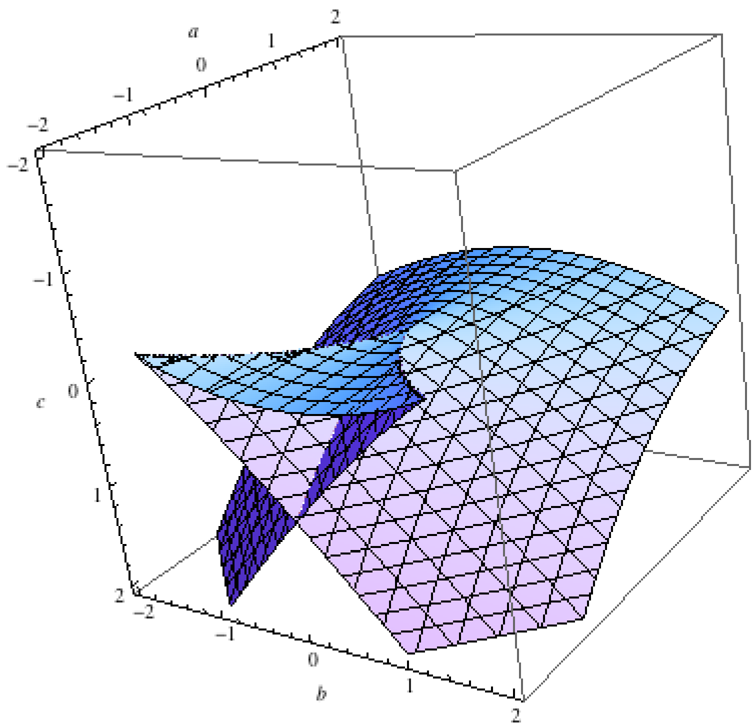


FIGURE 40. A plot of the swallowtail catastrophe in parameter space. There are three surfaces of folds, each intersecting in a line of cusp singularities, which all three meet in a swallowtail point.

which I am not going to draw for you (since I don't yet know how to draw in 4 dimensions). Then there are some catastrophes that you need two dimensions in your phase portrait to find. The ones with names require four or fewer parameters. They are (for completeness), the *parabolic umbilic, elliptic umbilic*, and *hyperbolic umbilic catastrophes.*

7. Hysteresis and the van der Pol oscillator

In the discussion about the cusp catastrophe, we saw that there was a region of so-called bistability. Let's return to this set up. Recall the bifurcation diagram of eq. (6.2) for a fixed $a > 0$ as b varies through $\mathbb{R}$, this time along with some of the phase lines in Figure 41.

The idea now is to think about the system, but to allow the parameter b to 'drift' slowly. You can think of this as follows: if the parameter b in the system were dependent on the asymptotic end state of the system given in eq. (6.2). Then we have that solutions would tend towards the steady state $x_*(a, b)$, but as they got close and didn't move for a while, the parameter's dependence kicks in and b changes ever so slightly. This will slightly change our system, and we will have an initial condition that is close to the previous value of b's asymptotic end state, which will then tend towards the stable fixed point in the system. The trouble comes when the parameter 'drifts' beyond the 'knee' of the stable branch in the bifurcation diagram in Figure 41.

For example, suppose that $\dot{b} = -\varepsilon x$ where ε is a small number. Then we would be considering the system

$$\begin{aligned} \dot{x} &= -x^3 + ax + b \\ \dot{b} &= -\varepsilon x. \end{aligned} \tag{7.1}$$

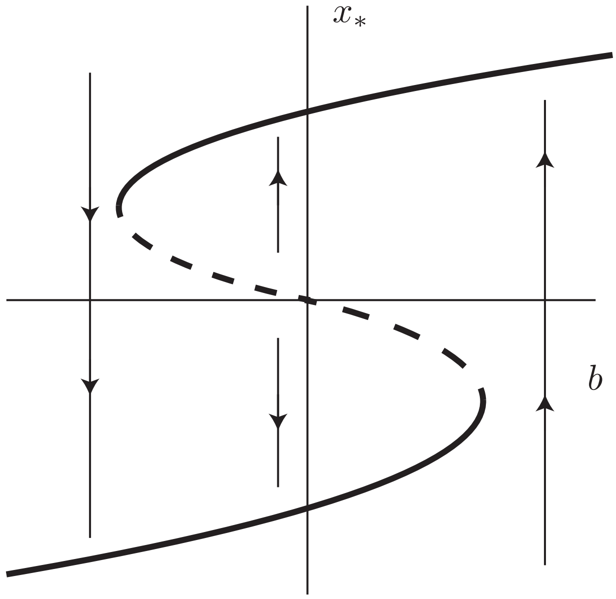


FIGURE 41. The bifurcation diagram for eq. (6.2) for a fixed $a > 0$. For certain values of b there is bistability, i.e. there are two coexisting steady states.

The sign on $\dot{b}$ means that if we were to start on the lower branch of the stable equilibrium in the bifurcation diagram, we would see that our end-states would 'drift' right. All the way to the edge of the knee, to the saddle node bifurcation in the parameter b. We would then move off the knee and out of the region of bistability, and then we would 'zip' straight to the other equilibrium point, the one on the upper branch. We would then stay near there, and b would begin to get smaller again. What you should first notice here is that the parameter value would return to below where it was when it jumped branches, but would not return to the original stable fixed point. This phenomena (in dynamical systems at least) is called *hysteresis*. You should compare this to what happens with a pitchfork bifurcation à la Figure 31. When you increase a beyond the bifurcation point, you jump to one of the stable branches, but when you lower a again, you return to your original stable fixed point. This is not what happens here. You need to lower b down considerably all the way down to the other knee, and then raise it again, in order to return to your original stable fixed point. In fact, with system eq. (7.1) this is exactly what happens, and you can see that we have a periodic orbit. See Figure 42. Such systems as in eq. (7.1) are called *slow-fast* systems, and the periodic orbit that we found is called a *relaxation oscillation*. This is because it spends a long time passing through the b parameter regime, and then when it hits the 'knee', it jumps to the other stable equilibrium. Then it slowly drifts back. So we have an oscillation of sorts, it is just occurring on two different time scales. For a plot of x vs t see Figure 43.

Example 7.1 (The van der Pol oscillator)**.** The quintessential example of a relaxation oscillation is due to Balthazar van der Pol, a researcher at Phillips who wrote down the equation that bears his name in 1927 when he was trying to find an electric circuitry model of the human heartbeat. The van der Pol equation has since become a prototypical model for relaxation oscillations in a wide variety of fields, from physics to chemistry, and of course biology. A nice, biological example that you might want to look into is due to

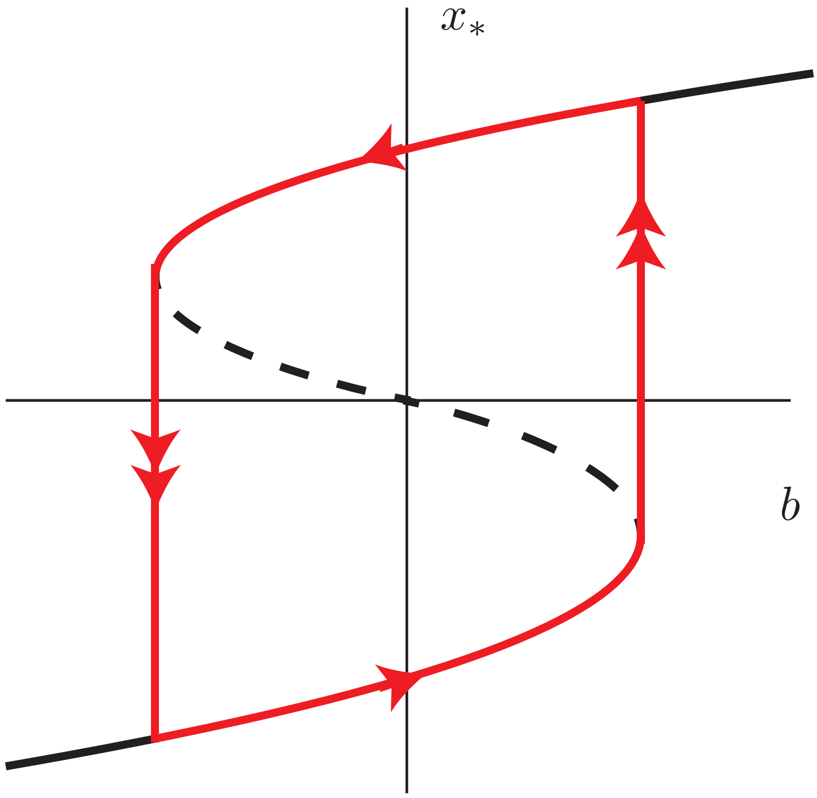


FIGURE 42. The relaxation oscillation (red, periodic orbit) that occurs in system eq. (7.1) as we allow the parameter b to vary slowly in the prescribed way. The double arrows indicates that travel on these lines is considerably faster than along the stable branch of the bifurcation diagram.

Fitzhugh and Nagumo who used what is effectively the van der Pol oscillator to describe spiking in giant squid axons. In any case, without further ado, the van der Pol equation is

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0. \tag{7.2}$$

Or equivalently as a system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\mu(x^2 - 1)y - x. \end{aligned} \tag{7.3}$$

You can see right away that $(0, 0)$ is a fixed point of the equation. We see that the Jacobian of the right hand side at $(0, 0)$ is given as $DF(0, 0) = \begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix}$, which has determinant equal to $1 > 0$ for all μ, and trace equal to μ. So we have that the origin is stable when $\mu < 0$ and unstable when $\mu > 0$. Further the eigenvalues of the Jacobian at the origin are given as $\lambda_{\pm} = \dfrac{\mu \pm \sqrt{\mu^2 - 4}}{2}$, so as μ crosses through the origin we have a pair of eigenvalues crossing the imaginary axis. We don't have a regular Hopf bifurcation though, as no limit cycle emerges out of the origin. The fastest way to see this is to use Bendixson's negative criterion $f_x + g_y = -\mu(x^2 - 1)$ which if $\mu > 0$, is of one sign for all

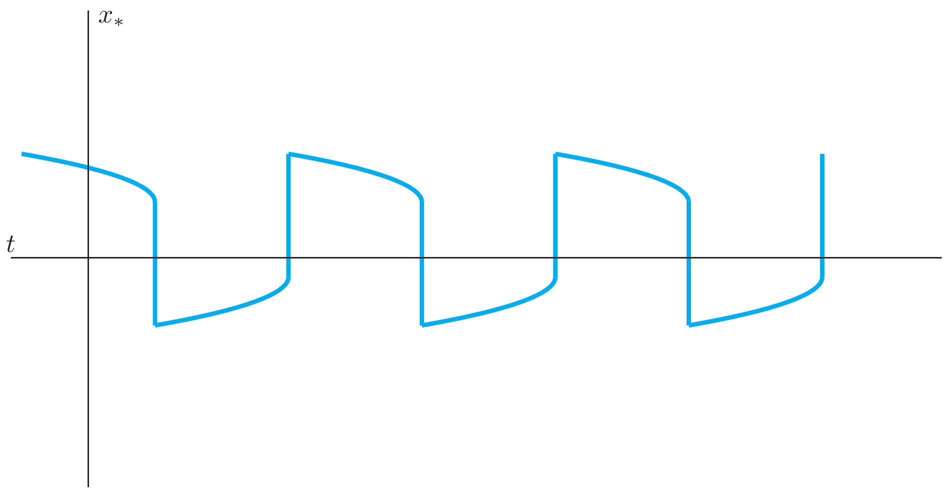


FIGURE 43. A sketch of the plot of the value x_* in the relaxation oscillation versus t. The equation takes a while to increase or decrease beyond a critical x value the location of the saddle node bifurcation in the (b, x) plane, and then 'zips' to the other stable fixed point. It then repeats in the other direction resulting in an odd periodic orbit.

$|x| < 1$, so we don't have a periodic orbit in this region. You can also heuristically see this by observing that when $\mu = 0$, eq. (7.3) becomes the *linear* equation for a centre

$$\dot{x} = y$$
$$\dot{y} = -x$$

which has an infinite number of periodic orbits. So the origin is a degenerate Hopf bifurcation.

There is a limit cycle in the system however. The system eq. (7.3) is reminiscent of the system we discussed right before stating Liénard's first theorem. The only problem is that the first condition of Liénard's first theorem doesn't hold. However, there is another theorem, also due to Liénard which will give us what we want. We have the following:

Theorem 7.1 (Liénard's theorem II). *The equation $\ddot{x} + f(x)\dot{x} + g(x) = 0$ has a unique stable limit cycle (a unique, attracting periodic orbit) if f and g are C^1 and*

(1) The function $F(x) := \int_0^x f(u)du$ is an odd function for all x,
(2) $F(x)$ is zero only at $x = 0$ and at $x = \pm a$ for some $a \in \mathbb{R}$.
(3) $F(x) \to \infty$ as $x \to \infty$ monotonically for $x > a$.
(4) $g(x)$ is an odd function, and $g(x) > 0$ for $x > 0$.

You can see for yourself that the van der Pol equation satisfies these criteria for $\mu > 0$, and so we can conclude the existence of unique *stable* limit cycle for all $\mu > 0$. By setting $t = -t$, and appealing to Liénard's theorem II in this case, you can see the presence of a unique *unstable* limit cycle (repelling periodic orbit) when $\mu < 0$. What you can show (though we won't do this) is that the limit cycles tend towards the circle of radius 2 as $\mu \to 0$, stable for $\mu > 0$ and unstable for $\mu < 0$. We thus have that the bifurcation diagram for μ near 0 looks as in Figure 44.

So that is the van der Pol equation for μ near zero. What about when $\mu \gg 1$? Well, we still have that Liénard's theorem II says that there is a unique stable limit cycle, so what happens to it? In order to answer the question, we need to return to our original ODE, the one in eq. (7.2)

$$\ddot{x} + \mu\dot{x}(x^2 - 1) + x = 0.$$

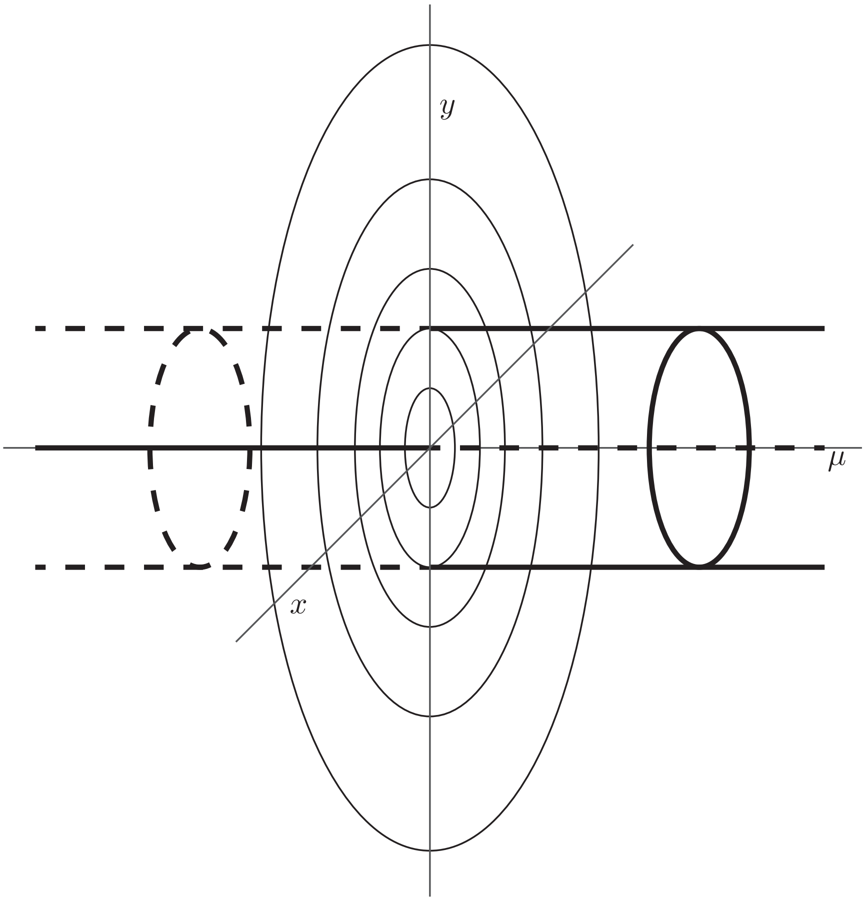


FIGURE 44. A sketch of the bifurcation diagram for the van der Pol equation, eq. (7.3). There is a degenerate Hopf bifurcation at $(\mu, x_*, y_*) = (0, 0, 0)$ and the limit cycle (which exists because of Liènard's theorem II) switches stability as μ passes through zero as well.

The idea now is to introduce new variables instead of the usual $\dot{x} = y$ ones from eq. (7.3). If we let

$$F(x) := \frac{x^3}{3} - x \quad \text{and,} \quad w = \dot{x} + \mu F(x),$$

then,

$$\dot{w} = \ddot{x} + \mu F'(x)\dot{x} = -x.$$

So our original equation from (7.2) is equivalent to

$$\begin{aligned} \dot{x} &= w - \mu F(x) \\ \dot{w} &= -x. \end{aligned} \tag{7.4}$$

We then let $b = \dfrac{w}{\mu}$ and eq. (7.4) is equivalent to

$$\begin{aligned} \dot{x} &= \mu(b - F(x)) \\ \dot{b} &= -\frac{x}{\mu}. \end{aligned} \tag{7.5}$$

Finally, we make one more change of variables, this time of the independent variable. We set $\tau = \mu t$ so $\frac{dt}{d\tau} = \frac{1}{\mu}$. Then if we denote $' := \frac{d}{d\tau}$, we have that eq. (7.5) is equivalent to

$$\begin{aligned} x' &= b - F(x) \\ b' &= -\frac{x}{\mu^2}. \end{aligned} \tag{7.6}$$

(Incidentally, these coordinates, and really any that aren't the standard $\dot{x} = y \ldots$ are called *Liénard coordinates.*)

We see that eq. (7.6) is the same as eq. (7.1) that we wrote down when we were studying the bifurcation diagram of the bistable part of the cusp catastrophe. So we know (in these coordinates) exactly what the phase portrait should look like, and what the stable limit cycle should look like. It should be a relaxation oscillation in a fast-slow system. In this case, the phase portrait will be rotated about the $x_* = b$ axis, and in the plot the curve of the x_* vs b will be the vertical nullcline of the system in eq. (7.6).

Even though we know what the phase portrait, and the relaxation oscillation must look like, I have included some pplane diagrams of it at the end for completeness. Also included is a plot of the phase portrait of the limit cycle and a plot of x vs t, and b vs t in the Liénard and x vs t and y vs t in the original coordinates as well.

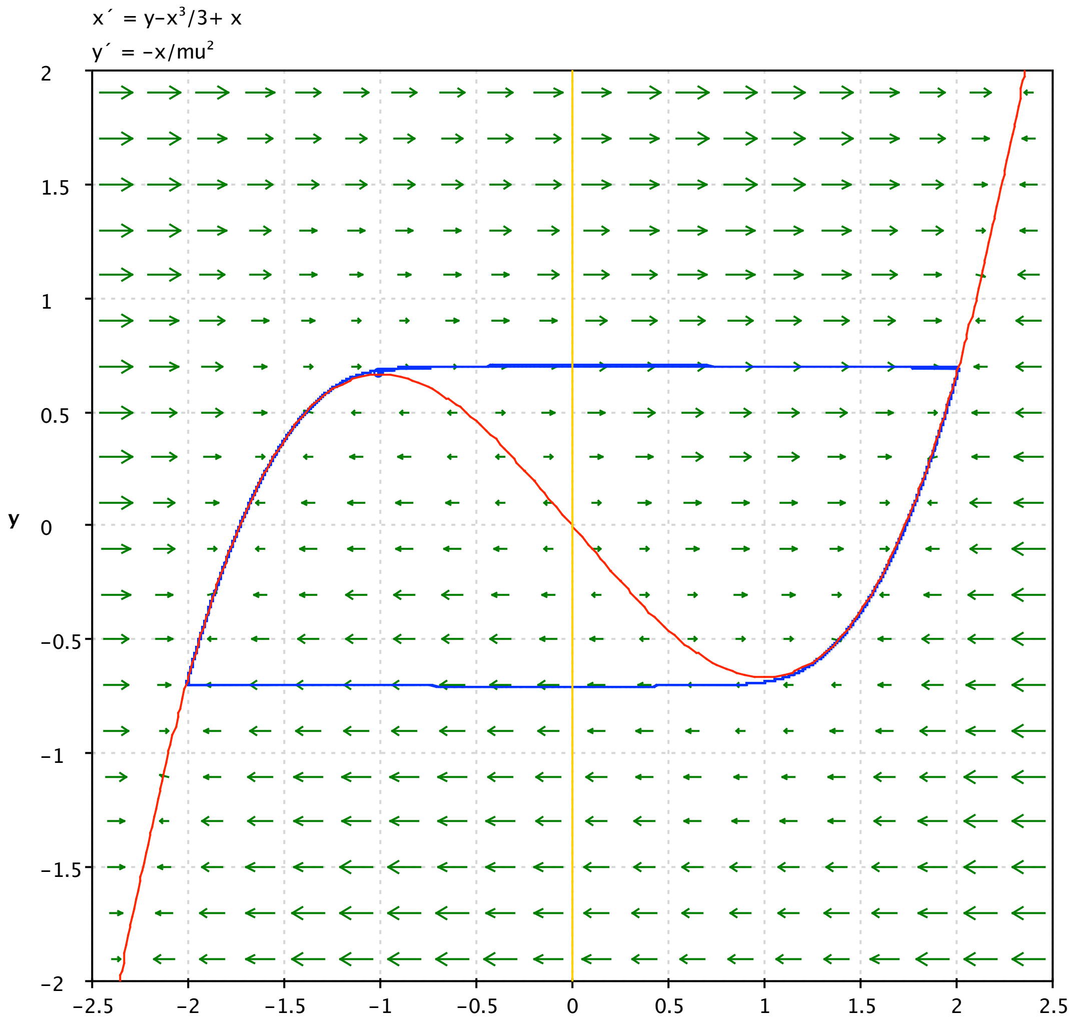


FIGURE 45. The phase portrait of eq. (7.6) as well as the nullclines and the stable limit cycle (blue). The vertical nullcline in this picture is the red cubic curve. Looking at the vector field, you can see that off the vertical nullcline, the flow is predominantly horizontal. So we have a slow creep along the vertical nullcline until the knee, and then a quick 'zip' to the other end of it (a quick 'zip' to the other stable fixed point in x if we view b not as a dynamic dependent variable, but rather as a 'drifting' parameter.)

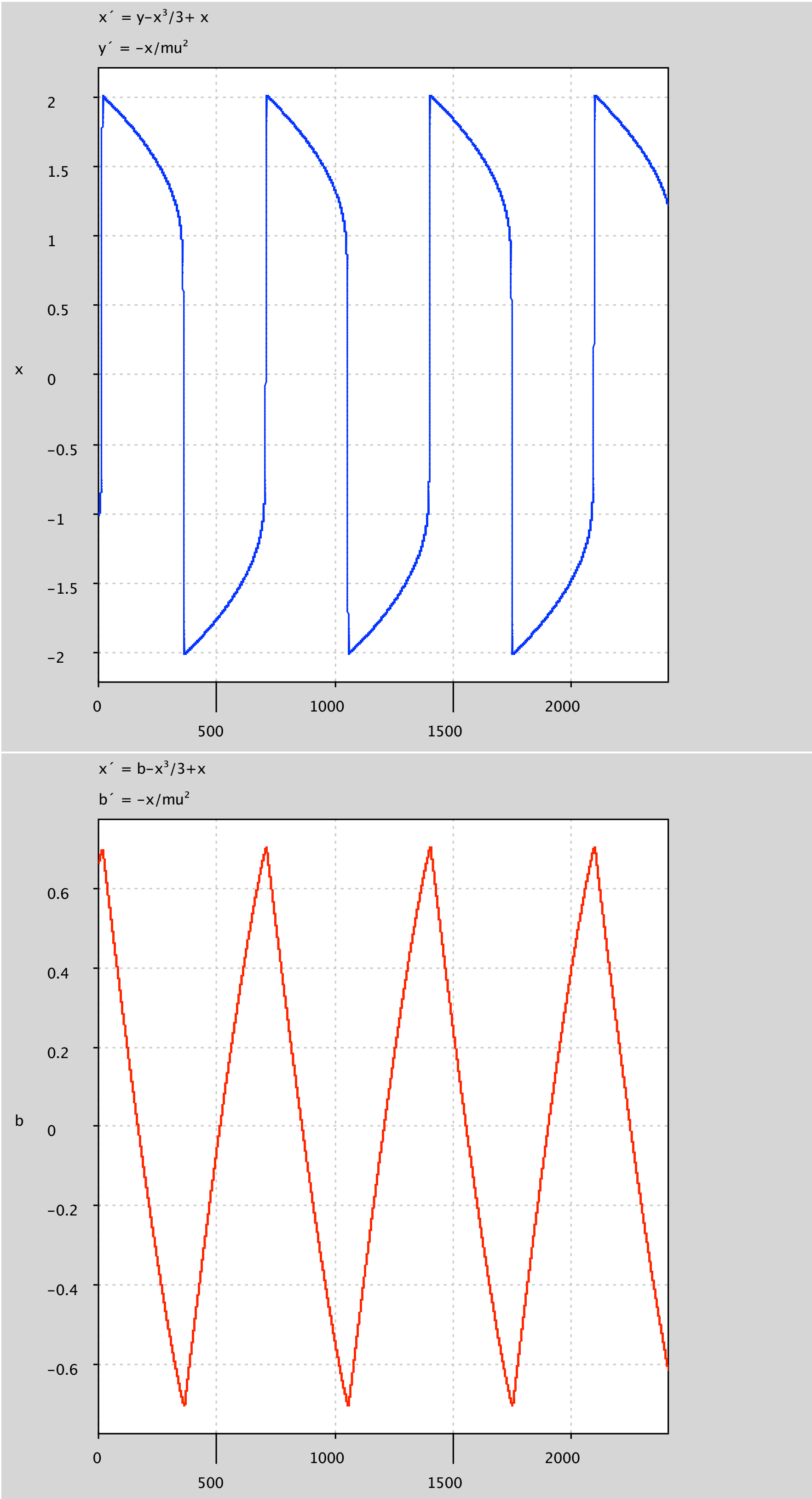


FIGURE 46. A plot of x vs t (top, blue online) and b vs t (bottom, red online). The x vs t curve agrees qualitatively with Figure 43.You can see that the fast transition in the x variable is happening at the peaks and troughs for the b variable.

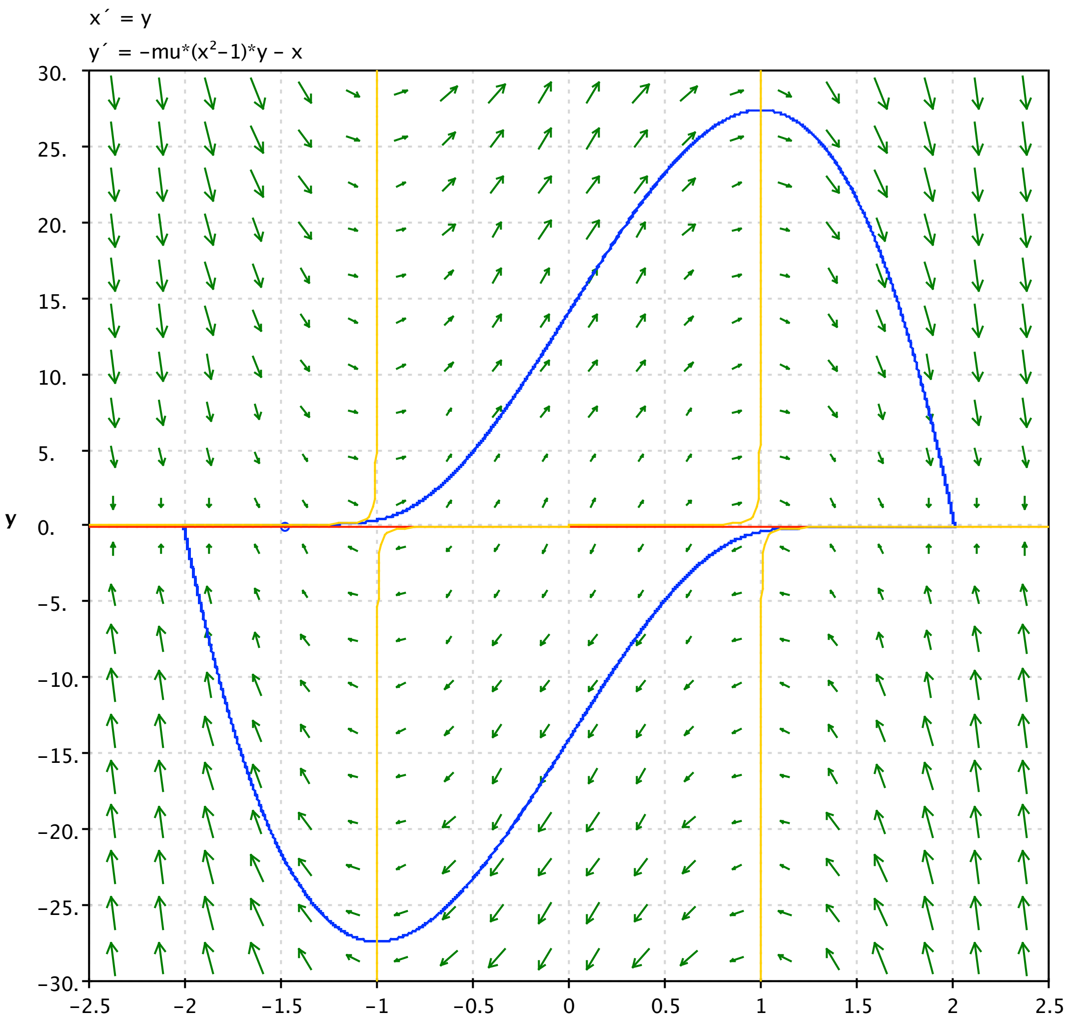
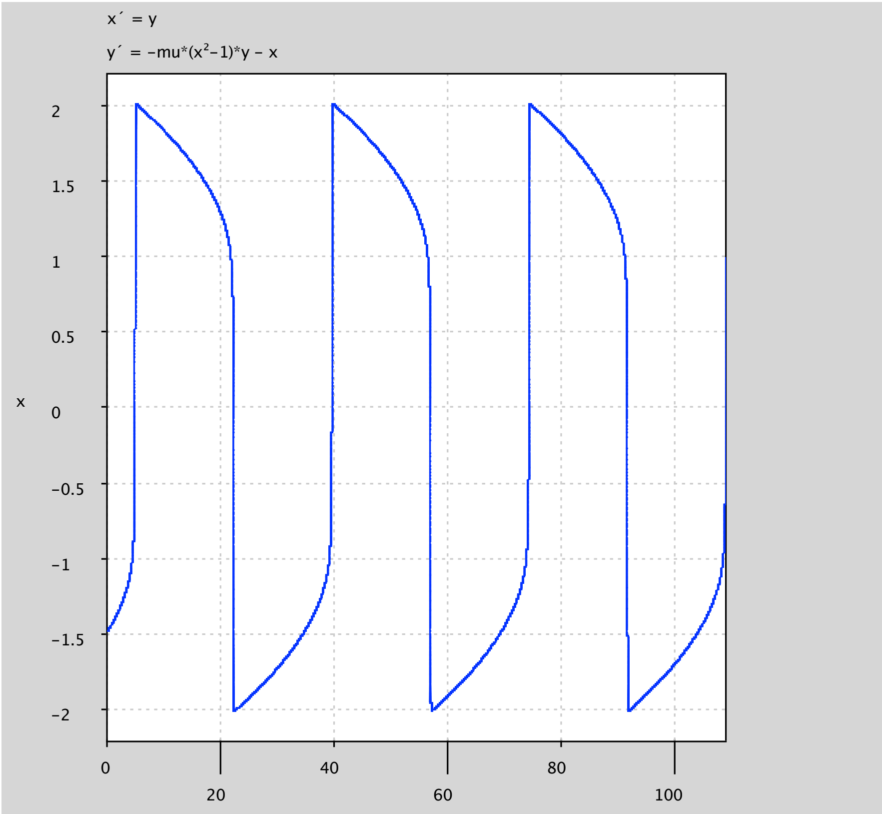


FIGURE 47. The phase portrait of eq. (7.3) as well as the nullclines and the stable limit cycle (blue). The solution spends a long time near the x-axis from $y = -2$ to $y = -1$ before quickly jumping and shooting very high in the y variable to the other 'stable' state at $x = 0$, and $y \in (2, 1)$. The limit cycle in this figure is much larger than the one in the Liénard coordinates.



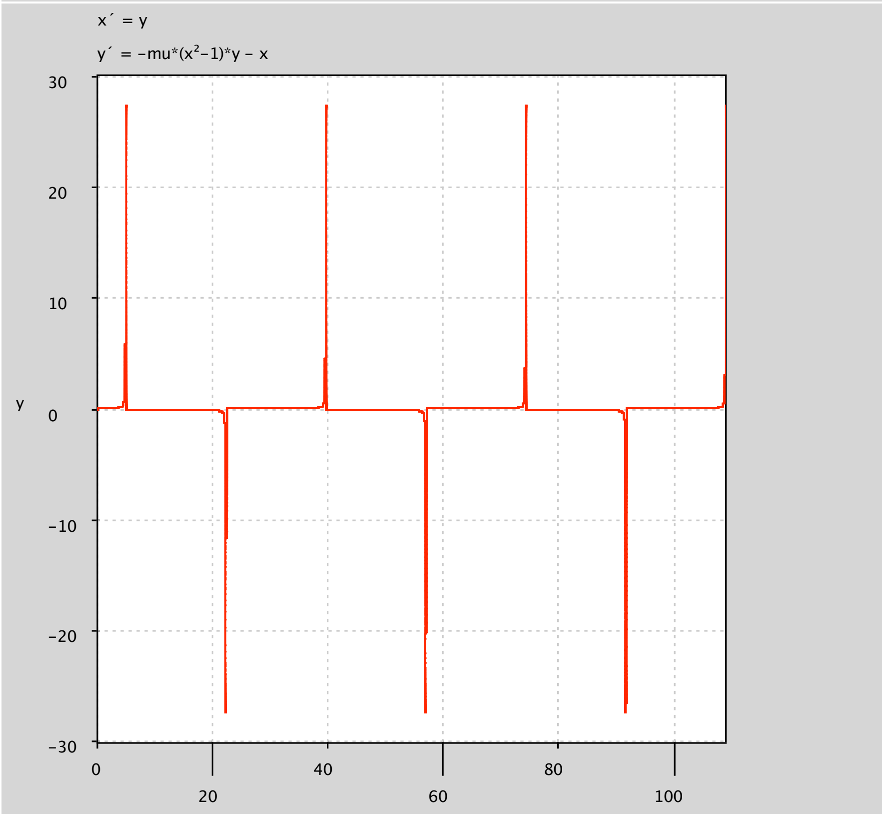


FIGURE 48. A plot of x vs t (top, blue online) and y vs t (bottom, red online). The x vs t curve again agrees qualitatively with Figure 43. This time though the y variables have very different pictures. In this figure one can see why eq. (7.2) might be a model for nerve axon firings.