DETECTING EIGENVALUES IN A FOURTH-ORDER NONLINEAR SCHRÖDINGER EQUATION WITH A NON-REGULAR MASLOV BOX

M. CURRAN, R. MARANGELL

Abstract. We use the Maslov index to study the eigenvalue problem arising from the linearisation about a standing wave in the fourth-order cubic nonlinear Schrödinger equation (NLSE). Our analysis is motivated by a recent work by Bandara et. al., in which the fourth-order cubic NLSE was shown to support infinite families of multipulse soliton solutions with various symmetry properties. By using a homotopy argument, we prove a lower bound for the number of real unstable eigenvalues. We also give a Vakhitov-Kolokolov type stability criterion, which is sufficient to determine the spectral stability or instability for certain classes of standing waves. The interesting aspects of this problem as an application of the Maslov index are the instances of non-regular crossings; in particular, we encounter crossing forms with degeneracies of both zero and nonzero rank. We handle such degeneracies by analysing the partial signatures of higher order crossing forms, using a definition of the Maslov index developed by Piccione and Tausk.

Contents

1. Introduction

The fourth-order cubic nonlinear Schrödinger (NLS) equation

$$i\Psi_t = -\frac{\beta_4}{24}\Psi_{xxxx} + \frac{\beta_2}{2}\Psi_{xx} - \gamma|\Psi|^2\Psi, \tag{1.1}$$

Date: November 25, 2024.

models the propagation of pulses in media with Kerr nonlinearity that are subject to both quartic and quadratic dispersion [KH94, ABK94, BGBK21, TABRdS19]. Here Ψ is the slowly varying complex envelope of the pulse, and β_2, β_4, and γ are real coefficients.

Solutions to (1.1) of the form $\Psi(x,t) = e^{i\beta t}\phi(x)$, $\beta \in \mathbb{R}$, are called *standing wave* solutions. Following the convention of [BGBK21], when the wave profile ϕ is a homoclinic orbit of the associated standing wave equation (given in (1.5)), we will call Ψ a *soliton solution* of (1.1). Karlsson and Höök [KH94] discovered an exact analytic family of soliton solutions to (1.1) with a squared hyperbolic secant profile. Akhmediev, Buryak and Karlsson [ABK94] observed oscillatory behaviour in the tails of solitons for certain values of β. Akhemdiev and Buryak [BA95] showed the existence of bound states of two-solitons (i.e. double-hump pulses ϕ) in the same parameter regime, and derived a stability criterion by analysing the dependence of the associated Hamiltonian on the energy. Karpman and Shagalov [Kar96, KS97, KS00] considered the extension of (1.1) to higher-order nonlinearities and multiple space dimensions. All of these works considered the case $\beta_4 < 0$ and $\beta_2 < 0$.

More recently, (1.1) has been the focus of a number of studies following the experimental discovery of *pure quartic solitons* (PQSs) in silicon photonic crystal waveguides [BRdSS+16]. These solitons exist through a balance of negative quartic dispersion and the nonlinear Kerr effect, for which $\beta_2 = 0$ and $\beta_4 < 0$. They have attracted much attention for their potential applicability to ultrafast lasers due to their favourable energy-width scaling [BRdSHE17, TABRdS19]. Following the discovery of PQSs, Tam *et al.* [TABRdS19] numerically investigated their existence and spectral stability. They also showed [TABRdS18, TABRdS20] that PQSs and solitons of the classical second-order NLS equation, for which $\beta_4 = 0$, are in fact part of a broader continuous family of soliton solutions to (1.1) for nonpositive dispersion coefficients β_4 and β_2.

Extending the work of Tam *et al.*, Bandara *et al.* [BGBK21] used a dynamical systems approach to find infinite families of multi-hump soliton solutions to (1.1) for $\beta_4 \neq 0$ and $\beta_2 \neq 0$. To do so, they identified solitons of (1.1) as orbits of the standing wave equation satisfied by the wave profile that are homoclinic to the origin. As a consequence of the standing wave equation being Hamiltonian, fourth-order and having two reversibility symmetries, infinitely many homoclinic solutions are created when the origin transitions from a real saddle (having only purely real eigenvalues) to a saddle focus (having complex conjugate eigenvalues) as a parameter is changed. This holds provided there exists a symmetric homoclinic orbit at the point of transition [CT93]. In parameter regimes where this spectral behaviour occurs, they use continuation techniques to numerically compute homoclinic orbits characterised as heteroclinic cycles between the origin and periodic orbits in the zero energy level (zero set of the Hamiltonian). The orbits are organised into infinite families according to the symmetry properties of the periodic orbits and the types of connections from the origin to them. They also use numerical simulations to investigate the stability of the waves computed. They found that while many of the multi-pulse solutions were unstable, some were only weakly unstable, and therefore possibly observable in experiments over a number of dispersion lengths.

More rigorous stability analyses were undertaken in [NP15] and [PA21]. Natali and Pastor [NP15] proved the orbital stability of the family of solutions found by Karlsson and Höök (realised as a single exact solution to the nondimensionalised version of (1.1), see (1.3)). As observed in [NP15] (see also [TABRdS20, §II]), this exact solution exists only for a fixed value of the frequency parameter β, and is *not* part of a continuous family of solutions in that parameter. The failure of the existence of such a family renders the classical results of Grillakis, Shatah and Strauss [GSS87, GSS90] inadmissible since [GSS87, Assumption 2] does not hold in this instance. Outside of the Karlsson and Höök solution, Parker and

Aceves [PA21] proved the existence of a primary single-hump soliton in certain parameter regimes of (1.1), along with an associated discrete family of multi-hump solitons. Under certain assumptions, they proved orbital stability of the primary pulse and spectral instability of the associated multi-pulses.

In this paper, we further develop the spectral stability theory for *arbitrary* single and multi-hump soliton solutions to (1.1). In particular, we seek to determine the existence of unstable eigenvalues of the associated linear operator

$$N = \begin{pmatrix} 0 & -L_- \\ L_+ & 0 \end{pmatrix}, \tag{1.2}$$

where L_+ and L_- are selfadjoint fourth order operators acting in $L^2(\mathbb{R})$. Our results are not confined to solitons satisfying Hypothesis 2, the first part of Hypothesis 3 and Hypothesis 4 in [PA21]. The main tool in our arsenal is a topological invariant known as the *Maslov index*, a signed count of the nontrivial intersections, or *crossings*, of a path of Lagrangian subspaces of $\mathbb{R}^{2n}$ with a fixed reference plane. Since the Maslov index is unable to detect complex eigenvalues, we will restrict our search to purely real unstable eigenvalues.

Our main results are as follows. In Theorem 1.2, we provide a lower bound for the number of positive real eigenvalues of (1.2). The bound is given in terms of the *Morse indices* (the number of positive eigenvalues) of L_+ and L_-, denoted $P := n_+(L_+)$ and $Q := n_+(L_-)$, as well as a correction term which represents the contribution to the Maslov index from the crossing corresponding to the zero eigenvalue of N. An immediate consequence of Theorem 1.2 is Corollary 1.5, a *Jones-Grillakis* type instability theorem [Jon88,Gri88,KP13] which gives a sufficient condition on P and Q for spectral instability of the underlying soliton. We also give a *Vakhitov-Kolokolov* (VK) type stability criterion [VK73, Pel11] in Theorem 1.6, where the spectral stability of solitons for which $P = 1$ and $Q = 0$ is determined by the sign of a certain integral.

Along the way, we prove Theorem 4.1, which equates P and Q to counts of the *conjugate points* (see Definition 3.10) of the operators L_+ and L_-, respectively. Thus, all of the required data for the lower bound in Theorem 1.2 occurs when the spectral parameter is zero. As highlighted in [BJ22], this is a convenient feature for numerical computations that is not afforded by a calculation using the Evans function [AGJ90]. In light of this, an alternate form of (1.19), which may be more useful for numerics, is given in Remark 5.5.

The key feature of the eigenvalue problem for N that makes it amenable to the Maslov index is the Hamiltonian structure of the eigenvalue equations when written as a spatially dynamic first order system in $\mathbb{R}^8$. This system therefore preserves Lagrangian planes. Moreover, in parameter regimes which guarantee the essential spectrum of N is confined to the imaginary axis, the first order system has an exponential dichotomy on the positive and negative half lines. This gives rise to two-parameter families of Lagrangian planes (in x and λ, the spatial and spectral parameters), known as the *unstable* and *stable bundles*, comprising the solutions that decay to zero exponentially as $x \to -\infty$ and $x \to +\infty$, respectively. These bundles are the central objects of our analysis; their nontrivial intersections at common values $x \in \mathbb{R}$ and $\lambda \in \mathbb{R}$ encode the positive real eigenvalues of interest. By exploiting homotopy invariance of the Maslov index, we will detect these eigenvalues by instead counting conjugate points, i.e. the nontrivial intersections of the unstable bundle, when the spectral parameter is zero, with the stable subspace of the asymptotic system.

Operators of the form of N are said [KKS04] to have the *canonical symplectic structure*. While the abstract spectral theory of such operators is well studied (see, for example, [KP13, §7] and the references therein), the soliton solutions of interest here are not in general part of a continuous family in the frequency parameter β (as, for example, the Karlsson and Höök solution is not). For the same reason that [GSS90] is not applicable in this instance, this places the current problem outside the scope of the spectral theory given in [KP13, §7], since [KP13, Hypothesis 5.2.5(f)] fails in this case. We emphasise that in this work, we make no comment on the orbital stability of the soliton solutions of interest.

The Maslov index has been used to study the spectrum of homoclinic orbits in a number of works [Jon88, BJ95, CJ18, Cor19, CH14, HLS18, BCJ+18, How23, How21], namely as a tool to detect real unstable eigenvalues. If monotonicity in the spectral parameter holds, as is often the case in selfadjoint problems [HLS18, BCJ+18, How23], then it is possible to give an exact count of such eigenvalues in terms of the Maslov index of a Lagrangian path in the spatial parameter. Howard, Latushkin and Sukhtayev [HLS18] proved the equality of the Morse and Maslov indices for matrix-valued Schrödinger operators with asymptotically constant symmetric potentials on the line, applying their results to determine the stability of nonlinear waves in various reaction-diffusion systems. Beck *et al.* [BCJ+18] proved the instability of pulses in gradient reaction-diffusion systems, generalising the instability result for pulses in scalar reaction-diffusion equations (see [KP13, §2.3.3]). Jones [Jon88] gave a stability criterion for soliton solutions of the cubic NLS equation with a spatially dependent nonlinearity. Bose and Jones [BJ95] proved the stability of a travelling wave in a coupled two-variable reaction diffusion system with diffusion in one variable, in which they used the Maslov index to locate eigenvalues in gradient systems. Chen and Hu [CH14] proved stability and instability criteria for standing pulses in a doubly-diffusive FitzHugh-Nagumo equation. Chardard, Dias and Bridges [CDB09b, CDB09a, CDB11] developed numerical tools to compute the Maslov index and study the stability of homoclinic orbits in Hamiltonian systems. Cornwell and Jones [Cor19, CJ18] analysed travelling waves in skew-gradient systems. In their work, despite not having a Hamiltonian structure the eigenvalue equations preserve Lagrangian planes for a nonstandard symplectic form. By showing monotonicity in the spectral parameter [Cor19] and computing the Maslov index at all conjugate points in the travelling wave co-ordinate, they proved [CJ18] the stability of a travelling pulse in a doubly diffusive FitzHugh-Nagumo system.

The current problem is distinguished from previous works by the presence of *non-regular* crossings. In their work [RS93], Robbin and Salamon require that all crossings be *regular* i.e. that the crossing form is nondegenerate. This is then topologically extended to all continuous Lagrangian paths (i.e. to those with non-regular crossings) via homotopy invariance. As pointed out in [GPP04b, GPP04a], one issue with extending the definition in this way is the destruction of potentially important information encoded in any non-regular crossings. For example, in the present paper, it turns out that the contribution of the crossing corresponding to the zero eigenvalue of N, which is non-regular with respect to the spectral parameter, is determined by the kernel and generalised kernel of N. It is not clear how a homotopy argument would capture the same information. Additionally, the technical details of perturbing the path typically requires breaking the structures that generate the path in the first place, making analytical calculations of such perturbations difficult.

We therefore use the approach of [PT08] to locally compute the Maslov index directly through the use of *higher-order crossing forms*. These generalise the (first-order) crossing form defined in [RS93], and allow us to calculate the contribution to the Maslov index from non-regular crossings without perturbative arguments. In their work, Deng and Jones [DJ11] derived a second order crossing form in the case where the crossing form was identically zero. This was recently extended in [BJP24] to the cases when *all* lower-order forms are identically zero. Our formulas, adapted from [PT08, GPP04a], do not require this assumption. Moreover, we do not write the Lagrangian path locally as the graph of an infinitesimally symplectic matrix, instead relying solely on the construction of *root functions* and *degeneracy spaces* (see Section 3.1).

Specific instances of non-regular crossings in the present paper include the conjugate point at the top left corner of the *Maslov box*, which corresponds to the zero eigenvalue of N, as well as crossings in the spatial parameter. For the former, we find that the crossing form in the spectral parameter is identically zero; this is a feature of eigenvalue problems for operators with the canonical symplectic structure [CCLM23]. For crossings in the spatial parameter, the crossing form is degenerate, but in general not identically zero, and so the approaches of [DJ11, CCLM23, BJP24] do not apply. This phenomenon appears to be the result of the eigenvalue equations being fourth order, and has been encountered in [How21, How23]. There, the authors circumvented the issue via a semi-definiteness argument; in the present setting, we elect to use higher order crossing forms in an effort to generalise previous efforts with degenerate crossing forms.

In [CCLM23], a similar lower bound to that in Theorem 1.2 was derived for an eigenvalue problem of the form of (1.14) on a compact interval, where $L_\pm$ are Schrödinger operators. In that work, the correction term was computed via an analysis of the *eigenvalue curves*, which represent the evolution of the eigenvalues as the spatial domain is shrunk or expanded. That the spatial domain is the whole real line in the present setting precludes the approach taken in [CCLM23], where the technical details required the spatial domain to be compact.

1.1. **Statement of main results.** Following [BGBK21, TABRdS20, TABRdS19], we will treat the case of negative quartic dispersion, $\beta_4 < 0$, nonzero quadratic dispersion $\beta_2 \neq 0$ and positive Kerr nonlinearity $\gamma > 0$, giving rise to the following nondimensionalised version of (1.1)

$$i\psi_t = \psi_{xxxx} + \sigma_2\psi_{xx} - |\psi|^2\psi, \tag{1.3}$$

where $\psi : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ and $\sigma_2 = \operatorname{sign} \beta_2$. (For the transformations used to obtain (1.3) from (1.1) for $\beta_4 < 0, \beta_2 \neq 0$, we refer the reader to [BGBK21, Table 1].) The modifications needed to treat pure quartic solitons for which $\beta_2 = 0$ will be given in Section 7.

Our focus will be to determine the spectral stability of standing wave solutions

$$\psi(x,t) = e^{i\beta t}\phi(x), \qquad \phi(x) \in \mathbb{R}, \tag{1.4}$$

to (1.3), subject to perturbations in $L^2(\mathbb{R};\mathbb{C})$. Substituting (1.4) into (1.3) shows that the wave profile ϕ satisfies the *standing wave equation*

$$\phi'''' + \sigma_2\phi'' + \beta\phi - \phi^3 = 0. \tag{1.5}$$

Using the change of variables

$$\phi_1 = \phi'' + \sigma_2\phi, \qquad \phi_2 = \phi, \qquad \phi_3 = \phi', \qquad \phi_4 = \phi''', \tag{1.6}$$

we may write (1.5) as the first order Hamiltonian system

$$\begin{pmatrix}\phi_1'\\ \phi_2'\\ \phi_3'\\ \phi_4'\end{pmatrix} = \begin{pmatrix}\phi_4 + \sigma_2\phi_3\\ \phi_3\\ \phi_1 - \sigma_2\phi_2\\ -\sigma_2\phi_1 + \phi_2 - \beta\phi_2 + \phi_2^3\end{pmatrix}. \tag{1.7}$$

Motivated by the families of homoclinic orbits discovered in [BGBK21], we consider orbits of (1.7) that are homoclinic to the origin, which correspond to soliton solutions of (1.3). An example is given by the exact solution found by Karlsson and Höök [KH94],

$$\phi_{\mathrm{KH}}(x) = \sqrt{\frac{3}{10}}\operatorname{sech}^2\left(\frac{x}{2\sqrt{5}}\right), \tag{1.8}$$

which solves (1.5) for the specific values $\beta = 4/25$ and $\sigma_2 = -1$.

We will assume that the origin in (1.7) is hyperbolic. Noting that the eigenvalues μ of the linearisation about the origin solve

$$\mu^2 = \frac{1}{2}\left(-\sigma_2 \pm \sqrt{1-4\beta}\right) \tag{1.9}$$

(where we used that $\sigma_2^2 = 1$), hyperbolicity holds provided

$$\begin{cases} \beta > 0 & \text{if} \quad \sigma_2 = -1 \\ \beta > \frac{1}{4} & \text{if} \quad \sigma_2 = 1. \end{cases} \tag{1.10}$$

For reasons soon to be discussed, in the first part of (1.10) we additionally require

$$\beta \neq \frac{1}{4} \quad \text{if} \quad \sigma_2 = -1. \tag{1.11}$$

Linearising (1.3) by substituting the complex-valued perturbation

$$\psi(x,t) = \left[\phi(x) + \varepsilon\left(u(x) + iv(x)\right)e^{\lambda t}\right]e^{i\beta x} \tag{1.12}$$

for $u, v \in L^2(\mathbb{R};\mathbb{R})$ into (1.3), collecting $O(\varepsilon)$ terms and separating into real and imaginary parts leads to the following linearised dynamics in u and v:

$$\begin{aligned} -u'''' - \sigma_2 u'' - \beta u + 3\phi^2 u &= \lambda v \\ -v'''' - \sigma_2 v'' - \beta v + \phi^2 v &= -\lambda u. \end{aligned} \tag{1.13}$$

We can write (1.13) as the spectral problem

$$N\begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \end{pmatrix}, \tag{1.14}$$

where N is the unbounded and densely defined linear operator

$$N = \begin{pmatrix} 0 & -L_- \\ L_+ & 0 \end{pmatrix}, \qquad \begin{cases} L_- = -\partial_x^4 - \sigma_2\partial_x^2 - \beta + \phi^2, \\ L_+ = -\partial_x^4 - \sigma_2\partial_x^2 - \beta + 3\phi^2, \end{cases} \tag{1.15}$$

with

$$\text{dom}(N) = H^4(\mathbb{R}) \times H^4(\mathbb{R}), \qquad \text{dom}(L_\pm) = H^4(\mathbb{R}). \tag{1.16}$$

Our goal is to determine whether the spectrum of N intersects the open right half plane. Because N is Hamiltonian, its spectrum has four-fold symmetry in $\mathbb{C}$, and instability follows from any part of the spectrum lying off the imaginary axis. We will see in Section 2 that the essential spectrum of N is confined to the imaginary axis and bounded away from the origin under (1.10). Since our symplectic analysis requires that the eigenvalue parameter be real, our task, then, will be to detect positive real point spectrum of N. Our main result is a lower bound for this spectral index in terms of the Morse indices of the operators $L_\pm$, which are selfadjoint with the domain (1.16) (see, for example, [Wei87]). The Morse indices of $L_\pm$ are only well-defined if their essential spectra are confined to the negative half line, and we show in Section 2 that this is indeed the case provided *both* (1.10) and (1.11) hold.

We point out that the equation $L_-\phi = 0$ is just (1.5), and, differentiating (1.5) with respect to x, we have $L_+\phi_x = 0$. Thus

$$0 \in \text{Spec}(L_-) \cap \text{Spec}(L_+) \tag{1.17}$$

with $\phi \in \ker(L_-)$ and $\phi_x \in \ker(L_+)$. We make the following simplicity assumption.

Hypothesis 1.1. $\ker(L_-) = \text{span}\{\phi\}$ and $\ker(L_+) = \text{span}\{\phi_x\}$.

Notice that when $\lambda = 0$, the eigenvalue equations (1.14) decouple into two independent equations, $L_- v = 0$ and $L_+ u = 0$, so that $\ker(N) = \ker(L_+) \oplus \ker(L_-)$. Hypothesis 1.1 therefore implies that $\ker(N) = \mathrm{span}\{(\phi_x, 0)^\top, (0, \phi)^\top\}$.

Let us denote

$$P := \#\{\text{positive eigenvalues of } L_+\},$$
$$Q := \#\{\text{positive eigenvalues of } L_-\},$$
$$n_+(N) := \#\{\text{positive real eigenvalues of } N\},$$

and define the quantities

$$\mathcal{I}_1 := \int_{-\infty}^{\infty} \phi_x \, \widehat{v} \, dx, \qquad \mathcal{I}_2 := \int_{-\infty}^{\infty} \phi \, \widehat{u} \, dx, \tag{1.18}$$

where $\widehat{v}$ is any solution in $H^4(\mathbb{R})$ to $-L_- v = \phi_x$ and $\widehat{u}$ is any solution in $H^4(\mathbb{R})$ to $L_+ u = \phi$. Under Hypothesis 1.1 and the conditions (1.10)–(1.11), our main result is the following.

Theorem 1.2. *Suppose $\mathcal{I}_1, \mathcal{I}_2 \neq 0$. The number of positive, real eigenvalues of the operator N satisfies*

$$n_+(N) \geq |P - Q - \mathfrak{c}|, \tag{1.19}$$

where

$$\mathfrak{c} = \begin{cases} 1 & \mathcal{I}_1 > 0, \, \mathcal{I}_2 < 0, \\ 0 & \mathcal{I}_1 \mathcal{I}_2 > 0, \\ -1 & \mathcal{I}_1 < 0, \, \mathcal{I}_2 > 0. \end{cases} \tag{1.20}$$

Remark 1.3. The equations $-L_- v = \phi'$ and $L_+ u = \phi$ each satisfy a solvability condition that guarantees the existence of solutions $\widehat{u}$ and $\widehat{v}$. In the case that either $\mathcal{I}_1$ or $\mathcal{I}_2$ vanishes, an extra calculation is needed to compute the correction term $\mathfrak{c}$ (the definition of which is given in (3.46)); for details, see Section 7). Finally, our theorem will also hold in the case of any integer power-law nonlinearity in (1.3) as studied in [KS97, KS00], i.e. in the case of standing wave solutions to

$$i\psi_t = \psi_{xxxx} + \sigma_2 \psi_{xx} - |\psi|^{2p} \psi, \qquad p \in \mathbb{Z}^+. \tag{1.21}$$

(See Remark 4.5.) However, with the standing wave solutions of [BGBK21] in mind, we have stated our results for the cubic case.

Remark 1.4. In this work we make no comment on the existence of soliton solutions to (1.3) (i.e. orbits homoclinic to the origin in (1.7)). Rather, we prove that if such a solution exists, then its associated linearised operator N satisfies Theorem 1.2.

The following Jones-Grillakis instability theorem [Jon88, Gri88, KP13] is an immediate consequence of Theorem 1.2.

Corollary 1.5. *Standing waves for which $P - Q \neq -1, 0, 1$ are spectrally unstable.*

We also have the following Vakhitov-Kolokolov type criterion [VK73, Pel11].

Theorem 1.6. *Suppose $P = 1$ and $Q = 0$. The standing wave $\widehat{\psi}$ is spectrally unstable if $\mathcal{I}_2 > 0$ and is spectrally stable if $\mathcal{I}_2 < 0$.*

Remark 1.7. If there exists a C^1 family of solutions $\beta \to \partial_\beta \phi(x; \beta) \in H^4(\mathbb{R})$ to the standing wave equation, then $\widehat{u} = \partial_\beta \phi(x; \beta)$ and the integral $\mathcal{I}_2$ is precisely that appearing in the Vakhitov-Kolokolov criterion for standing waves in the classical (second-order) NLS equation (see [Pel11, §4.2]), i.e.

$$\mathcal{I}_2 = \frac{1}{2} \frac{\partial}{\partial \beta} \int_{-\infty}^{\infty} \phi^2 dx. \tag{1.22}$$

We note, however, that in general this is not the case; for example, the Karlsson and Höök solution (1.8) is not a member of a smooth family of solutions $\beta \mapsto \phi(\cdot;\beta)$, since there is no solution of the form of (1.8) to (1.5) for $\beta \neq 4/25$, as observed in [NP15].

The paper is organised as follows. In Section 2 we write down the first order system associated with (1.14), and compute the essential spectra of the operators L_-, L_+ and N. We also define the stable and unstable bundles, the main objects of our analysis. In Section 3, we provide some background material on the Maslov index which includes the definition of higher order crossing forms due to Piccione and Tausk [PT08], before setting up the homotopy argument that will lead to the proof of the lower bound in Theorem 1.2. In Section 4 we use the Maslov index to prove that the Morse index of each of the operators L_- and L_+ is equal to the associated number of conjugate points (defined in Section 3). In Section 5, we prove Theorems 1.2 and 1.6. In Section 6 we apply our theory to confirm the spectral stability of the Karlsson and Höök solution (1.8), which will involve numerically computing the number of conjugate points of the associated operators L_- and L_+. In Section 7 we give some concluding remarks on our analysis, and in Appendix A we complete the proof of Theorem 4.1 via the removal of a certain hypothesis used in the proof in Section 4.

2. Set-up

We first compute the essential spectra of the operators $L_\pm, N$. Using the change of variables

$$\begin{aligned} u_1 &= u'' + \sigma_2 u, & u_2 &= u, & u_3 &= u', & u_4 &= u''', \\ v_1 &= v'' + \sigma_2 v, & v_2 &= -v, & v_3 &= -v', & v_4 &= v''', \end{aligned} \tag{2.1}$$

we convert (1.13) to the (infinitesimally symplectic) first order system

$$\begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix}' = \left(\begin{array}{cccc|cccc} & & & & \sigma_2 & 0 & 1 & 0 \\ & 0 & & & 0 & -\sigma_2 & 0 & 1 \\ & & & & 1 & 0 & 0 & 0 \\ & & & & 0 & 1 & 0 & 0 \\ \hline 1 & 0 & -\sigma_2 & 0 & & & & \\ 0 & -1 & 0 & -\sigma_2 & & 0 & & \\ -\sigma_2 & 0 & \alpha(x) & \lambda & & & & \\ 0 & -\sigma_2 & \lambda & \eta(x) & & & & \end{array}\right) \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix}, \tag{2.2}$$

where

$$\alpha(x) := 3\phi(x)^2 - \beta + 1, \qquad \eta(x) := -\phi(x)^2 + \beta - 1.$$

Setting

$$B = \begin{pmatrix} \sigma_2 & 0 & 1 & 0 \\ 0 & -\sigma_2 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad C(x;\lambda) = \begin{pmatrix} 1 & 0 & -\sigma_2 & 0 \\ 0 & -1 & 0 & -\sigma_2 \\ -\sigma_2 & 0 & \alpha(x) & \lambda \\ 0 & -\sigma_2 & \lambda & \eta(x) \end{pmatrix},$$

we can write (2.2) as

$$\mathbf{w}_x = A(x;\lambda)\mathbf{w}, \tag{2.3}$$

where

$$\mathbf{w} = (u_1, v_1, u_2, v_2, u_3, v_3, u_4, v_4)^\top, \qquad A(x;\lambda) = \begin{pmatrix} 0 & B \\ C(x;\lambda) & 0 \end{pmatrix}. \tag{2.4}$$

The asymptotic system for (2.2) is given by

$$\mathbf{w}_x = A_\infty(\lambda)\mathbf{w}, \tag{2.5}$$

where

$$A_\infty(\lambda) := \lim_{x\to\pm\infty} A(x,\lambda).$$

(The endstates as $x \to \pm\infty$ are the same because ϕ is homoclinic to the origin.) It now follows from [KP13, Theorem 3.1.11] that the essential spectrum of N is given by the set of $\lambda \in \mathbb{C}$ for which the matrix $A_\infty(\lambda)$ has a purely imaginary eigenvalue. A short calculation shows that

$$\mathrm{Spec}_{\mathrm{ess}}(N) = \{\lambda \in \mathbb{C} : \lambda^2 = -\left(-k^4 + \sigma_2 k^2 - \beta\right)^2 \text{ for some } k \in \mathbb{R}\}. \tag{2.6}$$

Under (1.10), it follows that $\mathrm{Spec}_{\mathrm{ess}}(N) \subset i\mathbb{R}\backslash\{0\}$.

The essential spectra of the operators $L_\pm$ is computed similarly. The first order systems associated with the eigenvalue equations for each of the operators L_+ and L_- will be given in Section 4 (see (4.3) and (4.6)). It follows from a similar calculation on the asymptotic matrices associated with those systems that

$$\mathrm{Spec}_{\mathrm{ess}}(L_\pm) = \{\lambda \in \mathbb{R} : \lambda = -k^4 + \sigma_2 k^2 - \beta \text{ for some } k \in \mathbb{R}\}. \tag{2.7}$$

Given its biquadratic structure, if the equation in (2.7) has no real roots for k then the essential spectra of L_+ and L_- will be confined to the negative half line. The equation in (2.7) has no real roots for k if and only if the associated discriminant is positive, i.e.

$$16\beta^3 - 8\beta^2 + \beta = \beta(4\beta - 1)^2 > 0, \tag{2.8}$$

and, in addition, we have either

$$-8\sigma_2 > 0 \qquad \text{or} \qquad 4\beta - 1 > 0. \tag{2.9}$$

(See [Ree22], and note we have used that $\sigma_2^2 = 1$). Both (2.8) and (2.9) are satisfied for the values of β given in (1.10), (1.11). For these values of β we therefore have

$$\mathrm{Spec}_{\mathrm{ess}}(L_\pm) = \begin{cases} (-\infty, -\beta) & \sigma_2 = -1, \\ (-\infty, -\beta + \frac{1}{4}] & \sigma_2 = 1, \end{cases} \tag{2.10}$$

so that $\mathrm{Spec}_{\mathrm{ess}}(L_\pm) \subset \mathbb{R}^-$. In addition to hyperbolicity of the asymptotic matrices for the L_+ and L_- eigenvalue problems, the values of β given in (1.10), (1.11) will actually guarantee that, for all λ lying to the right of the essential spectrum in (2.10), those asymptotic matrices have an equal number of eigenvalues with positive and negative real part.

Note that the assumptions (1.10) actually guarantee that the matrix $A_\infty(\lambda)$ is hyperbolic, with an equal number of eigenvalues with positive and negative real part. Precisely, the eight eigenvalues are

$$\pm\frac{\sqrt{-\sigma_2 \pm \sqrt{1 - 4\beta \pm 4\lambda i}}}{\sqrt{2}}. \tag{2.11}$$

We denote the corresponding four-dimensional stable and unstable subspaces by $\mathbb{S}(\lambda)$ and $\mathbb{U}(\lambda)$ respectively.

Next, since $\mathrm{Spec}_{\mathrm{ess}}(N) \subset i\mathbb{R}\backslash\{0\}$, the operator $N - \lambda I$ of (1.14)–(1.16) is Fredholm for $\lambda \in \mathbb{R}$, and it follows from [San02, §3.3] that the densely-defined closed linear operator

$$T(\lambda) : H^1(\mathbb{R}) \longrightarrow L^2(\mathbb{R}), \qquad T(\lambda)u := \frac{du}{dx} - A(\cdot;\lambda)u,$$

associated with (2.3) is also Fredholm. By [San02, Theorem 3.2, Remark 3.3], (2.2) has exponential dichotomies on $\mathbb{R}^+$ and $\mathbb{R}^-$. That is, for each fixed $\lambda \in \mathbb{R}$, on each of the intervals $\mathbb{R}^+$ and $\mathbb{R}^-$ the set of solutions to (2.2) is the direct sum of two subspaces, where one subspace consists solely of solutions that decay (exponentially) backwards in x, and the other of solutions that decay forwards in x. By flowing these subspaces under (2.2), each of these families can be extended to all of $\mathbb{R}$. This leads us to consider the spaces

$$\begin{aligned}\mathbb{E}^u(x,\lambda) &:= \{\xi \in \mathbb{R}^8 : \xi = \mathbf{w}(x;\lambda),\ \mathbf{w} \text{ solves (2.2) and } \mathbf{w}(x;\lambda) \to 0 \text{ as } x \to -\infty\},\\ \mathbb{E}^s(x,\lambda) &:= \{\xi \in \mathbb{R}^8 : \xi = \mathbf{w}(x;\lambda),\ \mathbf{w} \text{ solves (2.2) and } \mathbf{w}(x;\lambda) \to 0 \text{ as } x \to +\infty\},\end{aligned} \tag{2.12}$$

corresponding to the evaluation at $x \in \mathbb{R}$ of the spaces of solutions to (2.2) that decay (exponentially) as $x \to -\infty$ and as $x \to +\infty$, respectively. Following [AGJ90, Cor19], we call these sets the *unstable* and *stable bundles* respectively. For each $x \in \mathbb{R}$ and $\lambda \in \mathbb{R}$, if we consider $\mathbb{U}(\lambda), \mathbb{S}(\lambda), \mathbb{E}^u(x,\lambda), \mathbb{E}^s(x,\lambda)$ as points in the Grassmannian of four-dimensional subspaces of $\mathbb{R}^8$,

$$\mathrm{Gr}_4(\mathbb{R}^8) = \{V \subset \mathbb{R}^8 : \dim V = 4\},$$

which (following [Fur04, HLS17]) we equip with the metric $d(V,U) = \|P_V - P_U\|$, where P_V is the orthogonal projection onto V and $\|\cdot\|$ is any matrix norm, then we have that

$$\lim_{x\to-\infty} \mathbb{E}^u(x,\lambda) = \mathbb{U}(\lambda), \qquad \lim_{x\to+\infty} \mathbb{E}^s(x,\lambda) = \mathbb{S}(\lambda). \tag{2.13}$$

That is, the orthogonal projections onto $\mathbb{E}^u(x,\lambda)$ and $\mathbb{E}^s(x,\lambda)$ converge to those on $\mathbb{U}(\lambda)$ and $\mathbb{S}(\lambda)$ as $x \to -\infty$ and $x \to +\infty$, respectively. This is given in [PSS97, Corollary 2].

The important feature of the system (2.2) that makes it amenable to the Maslov index is that the coefficient matrix $A(x;\lambda)$ is infinitesimally symplectic, i.e.

$$A(x;\lambda)^T J + J A(x;\lambda) = 0, \tag{2.14}$$

which follows from the symmetry of B and $C(x;\lambda)$. This is the motivation for the choice of substitutions (2.1). Consequently, (2.2) induces a flow on the manifold of *Lagrangian planes*. In particular, the stable and unstable bundles of (2.2) are Lagrangian planes of $\mathbb{R}^8$ for all x and all λ. In addition we have that λ_0 is an eigenvalue of N if and only if for any (and hence all) $x \in \mathbb{R}$ we have

$$\mathbb{E}^u(x,\lambda_0) \cap \mathbb{E}^s(x,\lambda_0) \neq \{0\}.$$

In this case we in fact have

$$\dim \mathbb{E}^u(x,\lambda_0) \cap \mathbb{E}^s(x,\lambda_0) = \dim\ker(N - \lambda_0). \tag{2.15}$$

By exploiting homotopy invariance of the Maslov index, we can determine the existence of such intersections by instead analysing the evolution of the unstable bundle $\mathbb{E}^u(x,\lambda_0)$ when $\lambda_0 = 0$. This is explained in Section 3.

3. A symplectic approach to the eigenvalue problem

In this section, we give some background material on the Maslov index before describing the homotopy argument that leads to the lower bound of Theorem 1.2.

3.1. **The Maslov index via higher order crossing forms.**

In this section we follow the discussions in [Arn67, RS93, GPP04a, GPP04b]. Consider $\mathbb{R}^{2n}$ equipped with the symplectic form

$$\omega(u,v) = \langle Ju, v\rangle_{\mathbb{R}^{2n}}, \qquad J = \begin{pmatrix} 0_n & -I_n \\ I_n & 0_n \end{pmatrix}. \tag{3.1}$$

A *Lagrangian subspace* of $\mathbb{R}^{2n}$ is one that is n dimensional and upon which the symplectic form vanishes. We denote the Grassmannian of all Lagrangian subspaces of $\mathbb{R}^{2n}$ by

$$\mathcal{L}(n) := \{\Lambda \subset \mathbb{R}^{2n} : \dim\Lambda = n,\ \omega(u,v) = 0\ \forall\ u,v \in \Lambda\}. \tag{3.2}$$

A *frame* for a Lagrangian subspace Λ of $\mathbb{R}^{2n}$ is a $2n \times n$ matrix with rank n whose columns span Λ. Such a frame may be written in block form by

$$\begin{pmatrix} X \\ Y \end{pmatrix}, \qquad X, Y \in \mathbb{R}^{n \times n},$$

where $X^\top Y = Y^\top X$; the symmetry of $X^\top Y$ follows from the vanishing of (3.1). Such a frame is not unique; right multiplication by any invertible $n \times n$ matrix will yield an alternate frame for Λ. In particular, if X is invertible then an alternate frame is given by

$$\begin{pmatrix} I \\ YX^{-1} \end{pmatrix}, \quad \text{where} \quad \left(YX^{-1}\right)^\top = YX^{-1}.$$

Let $\Lambda : [a, b] \to \mathcal{L}(n)$ be a path in $\mathcal{L}(n)$. Its Maslov index is, roughly speaking, a signed count of its intersections with a certain codimension-one set. More precisely, Arnol'd [Arn67] gave the following definition for non-closed paths satisfying certain conditions.

Fix $V \in \mathcal{L}(n)$. The *train* $\mathcal{T}(V)$ of V is the set of all Lagrangian planes that intersect V nontrivially; it may be decomposed into a family of submanifolds via $\mathcal{T}(V) = \bigcup_{k=1}^n \mathcal{T}_k(V)$, where $\mathcal{T}_k(V) := \{W \in \mathcal{L}(n) : \dim(W \cap V) = k\}$ is the set Lagrangian planes that intersect V in a subspace of dimension k. It is shown in [Arn67] that $\operatorname{codim} \mathcal{T}_k(V) = k(k+1)/2$; in particular, $\operatorname{codim} \mathcal{T}_1(V) = 1$. From the fundamental lemma of [Arn67], $\mathcal{T}_1(V)$ is two-sidedly embedded in $\mathcal{L}(n)$; that is, $\mathcal{T}_1(V)$ is transversely oriented by the velocity field of some (and then of any) one-parameter positive definite Hamiltonian [Arn67, Arn85]. Such a vector field thus defines a 'positive' and a 'negative' side of $\mathcal{T}_1(V)$. Define a *crossing* to be a value $t_0 \in [a, b]$ such that $\Lambda(t_0) \cap V \neq \{0\}$, i.e. $\Lambda(t_0) \in \mathcal{T}(V)$. The *Maslov index* of any continuous curve $\Lambda : [a, b] \to \mathcal{L}(n)$ with endpoints lying off the train and with crossings lying only in $\mathcal{T}_1(V)$ is then defined to be $\nu_+ - \nu_-$, where ν_+ is the number of points of passage of Λ from the negative to the positive side of $\mathcal{T}_1(V)$, and ν_- is defined conversely.

Arnol'd's definition was extended by Robbin and Salamon [RS93] to paths with arbitrary endpoints and with crossings possibly lying in $\mathcal{T}_k(V)$ for $k \geq 2$. This was done by exploiting an identification of the tangent space of $\mathcal{L}(n)$ at some $\Lambda \in \mathcal{L}(n)$ with the space $S^2\Lambda$ of quadratic forms on Λ. This lead to the construction of the crossing form, a quadratic form associated with each crossing whose signature determines local contributions to the Maslov index. The definition given by Robbin and Salamon requires that all crossings are regular, with the definition extended to all continuous Lagrangian paths via homotopy invariance (see Proposition 3.4). As outlined in the introduction, it is desirable to be able to compute the Maslov index directly, without having to use perturbative arguments.

Piccione and Tausk [PT08] provided the means to do exactly that, defining the Maslov index for analytic Lagrangian paths with non-regular crossings via the *partial signatures* of higher order crossing forms. While strictly speaking their definition is given via the fundamental groupoid (see [PT08, §5.2]), and shown to be *computable* via the partial signatures listed below, for our purposes it will suffice to use the latter computational tool as our definition of the Maslov index. This computable formula was previously given by Giambò, Piccione and Portaluri [GPP04b, GPP04a] through the related notion of the spectral flow of an associated family of symmetric bilinear forms (see [GPP04b, Proposition 3.11]); for the details of the equivalence of these definitions, see [PT08, §5.5] (in particular, Proposition 5.5.7 and §4).

The following notions can be found in [PT08, §5.5]. Suppose that $\Lambda : [a, b] \to \mathcal{L}(n)$ is an analytic path of Lagrangian subspaces, and $t = t_0$ is a crossing, that is, $\Lambda(t_0) \cap V \neq \{0\}$. A *$V$-root function* (or simply a *root function* when the choice of V is clear) for Λ at $t = t_0$ is a differentiable mapping $w : [t_0 - \varepsilon, t_0 + \varepsilon] \to \mathbb{R}^{2n}$, $\varepsilon > 0$, such that $w(t) \in \Lambda(t)$ and $w(t_0) \in V$. The *order* of w, $\operatorname{ord}(w)$, is the smallest positive integer k such that $w^{(k)}(t_0) \notin V$.

This allows one to define a sequence of nested subspaces $W_k(\Lambda, V, t_0)$, $k \geq 1$, via

$$W_k(\Lambda, V, t_0) := \{w_0 : \exists \text{ a } V\text{-root function } w \text{ for } \Lambda \text{ with } \mathrm{ord}(w) \geq k \text{ and } w(t_0) = w_0\}$$

(called the *kth degeneracy space*), for which we have

$$W_{k+1}(\Lambda, V, t_0) \subseteq W_k(\Lambda, V, t_0) \text{ for all } k \geq 1, \qquad W_1(\Lambda, V, t_0) = \Lambda(t_0) \cap V. \tag{3.3}$$

(When Λ, V and t_0 are clear, we will simply write W_k.) The first fact in (3.3) follows immediately from the definition, while the second follows from the fact that every V-root function w has $\mathrm{ord}(w) \geq 1$. One can then define a symmetric bilinear form $\mathfrak{m}_{t_0}^{(k)}(\Lambda, V) : W_k \times W_k \longrightarrow \mathbb{R}$ (as in [PT08, Definition 5.5.5]),

$$\mathfrak{m}_{t_0}^{(k)}(\Lambda, V)(w_0, v_0) := \frac{d^k}{dt^k}\,\omega\big(w(t), v_0\big)\Big|_{t=t_0}, \tag{3.4}$$

where w is any V-root function for Λ at $t = t_0$ with $\mathrm{ord}(w) \geq k$ and $w(t_0) = w_0$. (When Λ and V are clear, we will simply write $\mathfrak{m}_{t_0}^{(k)}$.) That $\mathfrak{m}_{t_0}^{(k)}$ is independent of the choice of root function and therefore well-defined follows from [PT08, Corollary 5.5.4]; that it is symmetric follows from [PT08, Lemma 5.5.3]. Using the definition of W_k and $\mathfrak{m}_{t_0}^{(k-1)}$, it is straightforward to show that

$$W_k = \ker \mathfrak{m}_{t_0}^{(k-1)}. \tag{3.5}$$

In light of (3.3), it follows that W_k is simply the subspace of $\Lambda(t_0) \cap V$ upon which the crossing forms up to order $k-1$ are zero.

We define the higher order generalisation of the crossing form defined by Robbin and Salamon to be the quadratic form associated with (3.4).

Definition 3.1. *The* kth-order crossing form *is defined to be*

$$\mathfrak{m}_{t_0}^{(k)}(\Lambda, V)(w_0) := \mathfrak{m}_{t_0}^{(k)}(\Lambda, V)(w_0, w_0) = \frac{d^k}{dt^k}\,\omega\big(w(t), w_0\big)\Big|_{t=t_0} = \omega\left(w^{(k)}(t_0), w_0\right) \tag{3.6}$$

for $w_0 \in W_k$*, where* w *is any* V*-root function for* Λ *at* $t = t_0$ *with* $\mathrm{ord}(w) \geq k$ *and* $w(t_0) = w_0$*.*

To compute higher order crossing forms in the spatial parameter, we will make use of the following fact.

Lemma 3.2. *Let* $\mathbf{Z}(t)$ *be a frame for* $\Lambda(t)$*. There exists a* V*-root function* w *for* Λ *at* $t = t_0$ *with* $\mathrm{ord}(w) \geq k$ *if and only if there exist* k *vectors* $\{h_0, \dots, h_{k-1}\}$ *in* $\mathbb{R}^n$ *such that*

$$\sum_{j=0}^{i} \binom{i}{j} \mathbf{Z}^{(i-j)}(t_0) h_j \in V \quad \textit{for every} \quad i \in \{0, 1, \dots, k-1\}. \tag{3.7}$$

In this case, the k*th order crossing form* (3.6) *is given by*

$$\mathfrak{m}_{t_0}^{(k)}(\Lambda, V)(w_0) = \sum_{j=0}^{k-1} \binom{k}{j} \omega(\mathbf{Z}^{(k-j)}(t_0) h_j, w_0), \qquad w_0 \in W_k. \tag{3.8}$$

Proof. Observe that if w is a V-root function, then we may write $w(t) = \mathbf{Z}(t)h(t) \in \Lambda(t)$ for some smooth $h : [t_0 - \varepsilon, t_0 + \varepsilon] \to \mathbb{R}^n$, and

$$w_i := \frac{d^i}{dt^i} w(t)|_{t=t_0} = \sum_{j=0}^{i} \binom{i}{j} \mathbf{Z}^{(i-j)}(t_0) h^{(j)}(t_0) \in V \text{ for every } i \in \{0, 1, \dots, k-1\}. \tag{3.9}$$

Thus the vectors $h_j := h^{(j)}(t_0)$ satisfy (3.7). Conversely, suppose there exist vectors $\{h_0, \dots, h_{k-1}\}$ such that (3.7) holds. Then

$$w(t) = \mathbf{Z}(t) \sum_{j=0}^{k-1} \frac{(t-t_0)^j}{j!} h_j \tag{3.10}$$

is a V-root function for Λ at $t = t_0$ with $\operatorname{ord}(w) \geq k$, as seen by differentiating w i times for $i \in \{0, 1, \dots, k-1\}$ at t_0. Substituting (3.10) into (3.6) yields (3.8). □

A typical application of this lemma is as follows. We first compute the first order form using equation (3.13) below, where $W_1 = \Lambda(t_0) \cap V$. For $k \geq 2$, the space W_k is given via (3.5). We then find vectors $\{h_0, \dots, h_{k-1}\}$ such that (3.7) holds – in which case w given by (3.10) is a root function – and we compute $\mathfrak{m}_{t_0}^{(k)}$ in (3.6) via (3.8).

In the case that $k = 1$, for notational convenience we will drop the superscript and write $\mathfrak{m}_{t_0}(\Lambda, V)$. Following [RS93], a crossing $t = t_0$ will be called *regular* if $\mathfrak{m}_{t_0}$ is nondegenerate; otherwise, $t = t_0$ will be called *non-regular*. Denote by $n_+(\mathfrak{m}_{t_0}^{(k)})$ and $n_-(\mathfrak{m}_{t_0}^{(k)})$ the number of positive, respectively negative, squares of $\mathfrak{m}_{t_0}^{(k)}$. Following [PT08, GPP04a], we call the collection of integers

$$n_-(\mathfrak{m}_{t_0}^{(k)}), \quad n_+(\mathfrak{m}_{t_0}^{(k)}), \quad \operatorname{sign}(\mathfrak{m}_{t_0}^{(k)}) = n_+(\mathfrak{m}_{t_0}^{(k)}) - n_-(\mathfrak{m}_{t_0}^{(k)}),$$

the *partial signatures* of (3.6). The Maslov index of the Lagrangian path Λ is then given as follows, as in [PT08, Theorem 5.5.9] (up to the convention at the endpoints) and [GPP04b, Proposition 3.11]. As an aside, note that it follows from the arguments in [PT08, §5.5] (namely, Exercise 5.10, Proposition 5.5.7, Lemma 4.3.13 and Proposition 4.3.14) that if Λ is real analytic, then all crossings are isolated.

Definition 3.3. *Suppose $\Lambda : [a, b] \to \mathcal{L}(n)$ is an analytic path of Lagrangian subspaces. The* Maslov index *of Λ is given by*

$$\begin{aligned}\operatorname{Mas}(\Lambda, V; [a, b]) = -\sum_{k\geq 1} n_-\left(\mathfrak{m}_a^{(k)}\right) + \sum_{t_0 \in (a,b)} \left(\sum_{k\geq 1} \operatorname{sign}\left(\mathfrak{m}_{t_0}^{(2k-1)}\right)\right) \\ + \sum_{k\geq 1} \left(n_+\left(\mathfrak{m}_b^{(2k-1)}\right) + n_-\left(\mathfrak{m}_b^{(2k)}\right)\right),\end{aligned} \tag{3.11}$$

where the right hand side has a finite number of nonzero terms.

Notice that at all interior crossings $t_0 \in (a, b)$, only the signatures of the crossing forms of odd order contribute; at the initial point the negative indices of crossing forms of all order contribute; while at the final point, the negative indices of the forms of even order and the positive indices of the forms of odd order contribute. Definition 3.3 is best understood in terms of the formula for the spectral flow of a family of symmetric matrices. Namely, one writes the root function as $w(t) = q(t) + R(t)q(t)$, where $q(t) \in V$, $R(t) : V \to W$ for some $W \in \mathcal{T}_0(V)$, $\Lambda(t) = \operatorname{graph} R(t) = \{q + R(t)q : q \in V\}$ and $\Lambda(t_0) \in \mathcal{T}_0(W)$. One then has a (locally defined) family of symmetric bilinear forms $t \mapsto \omega(R(t)\cdot, \cdot)|_{V\times V}$ for t near t_0. The derivatives of this family at t_0, which coincide with (3.4), then determine the spectral flow of the family, i.e. the net change in the number of non-negative eigenvalues, as t passes through t_0. For further details, we refer the reader to [PT08, Propositions 4.3.9, 4.3.15 and 5.5.7] and [GPP04b, Proposition 2.9, Proposition 3.11].

In [RS93], Robbin and Salamon exploit the two-sided nature of the train $\mathcal{T}(V)$ of V to say that a non-degenerate crossing passes from the 'positive' side to the 'negative' side of the train (or vice versa) according to the signature of the crossing form. (This idea can also be seen in the work of Arnol'd [Arn85].) Through the use of higher order crossing forms, we extend this idea to the case where the crossing form is potentially degenerate. In the case of one-dimensional crossings t_0, i.e. $\Lambda(t_0) \in \mathcal{T}_1(V)$, if the first nondegenerate crossing form is of odd order, then Λ passes through the train. On the other hand, if the first nondegenerate crossing form is of even order, then Λ departs $\mathcal{T}_1(V)$ in the direction in which it arrived.

It is proven in [GPP04b, Corollary 2.11] that

$$\sum_{k\geq 1}\left(n_+(\mathfrak{m}_{t_0}^{(k)}) + n_-(\mathfrak{m}_{t_0}^{(k)})\right) = \dim \Lambda(t_0)\cap V, \tag{3.12}$$

so that by taking sufficiently many higher order crossing forms, a crossing t_0 will always contribute $\dim \Lambda(t_0)\cap V$ summands (the signs of which may offset each other) to the Maslov index.

As pointed out in [GPP04a], Definition 3.3 includes, as a special case, the definition given by Robbin and Salamon in the case that all crossings are regular. To see this, note from the proof of Lemma 3.2 and (3.3) that, if $\mathbf{Z}(t)$ is a frame for $\Lambda(t)$ and $w_0 = \mathbf{Z}(t_0)h_0 \in \Lambda(t_0)\cap V$, then $w(t) = \mathbf{Z}(t)h_0$ is a root function with $\operatorname{ord}(w) \geq 1$ and

$$\mathfrak{m}_{t_0}(\Lambda, V)(w_0) = \omega\left(\mathbf{Z}'(t_0)h_0, \mathbf{Z}(t_0)h_0\right) = \left\langle \mathbf{Z}(t_0)^\top J\mathbf{Z}'(t_0)h_0, h_0\right\rangle_{\mathbb{R}^{2n}}, \tag{3.13}$$

just as in [RS93, Theroem 1.1]. If $\mathfrak{m}_{t_0}$ is nondegenerate, it follows from (3.5) that $W_2 = \{0\}$ and therefore $W_k = \{0\}$ for $k \geq 3$. Hence $\mathfrak{m}_{t_0}^{(k)}$ is trivial for $k \geq 2$, and from (3.12) we have $n_+(\mathfrak{m}_{t_0}) + n_-(\mathfrak{m}_{t_0}) = \dim \Lambda(t_0)\cap V$. The Maslov index of a path $\Lambda : [a,b] \to \mathcal{L}(n)$ with only regular crossings is therefore

$$\operatorname{Mas}(\Lambda, V; [a,b]) = -n_-\left(\mathfrak{m}_a\right) + \sum_{t_0\in(a,b)} \operatorname{sign}\left(\mathfrak{m}_{t_0}\right) + n_+\left(\mathfrak{m}_b\right), \tag{3.14}$$

as per [RS93, §2] (modulo the convention at the endpoints).

We will encounter three types of non-regular crossings for paths $\Lambda : [a,b] \to \mathcal{L}(n)$ in our analysis. The first is a simple interior crossing $t_0 \in (a,b)$ for which the first and second order forms are zero, and the third order form is nondegenerate. In this case

$$\operatorname{Mas}(\Lambda, V; [t_0-\varepsilon, t_0+\varepsilon]) = \operatorname{sign} \mathfrak{m}_{t_0}^{(3)}. \tag{3.15}$$

The second is a two-dimensional interior crossing for which the first order form is degenerate but not identically zero, the second order form is degenerate, and the third order form is nondegenerate. In this case

$$n_+(\mathfrak{m}_{t_0}^{(1)}) + n_-(\mathfrak{m}_{t_0}^{(1)}) + n_+(\mathfrak{m}_{t_0}^{(3)}) + n_-(\mathfrak{m}_{t_0}^{(3)}) = \dim \Lambda(t_0)\cap V, \tag{3.16}$$

and the contribution of the crossing to the Maslov index of the path is

$$\operatorname{Mas}(\Lambda, V; [t_0-\varepsilon, t_0+\varepsilon]) = \operatorname{sign} \mathfrak{m}_{t_0}^{(1)} + \operatorname{sign} \mathfrak{m}_{t_0}^{(3)}. \tag{3.17}$$

The final type of non-regular crossing is one occurring at the initial point $t_0 = a$, for which the first order crossing form is identically zero and the second order form is nondegenerate. In this case $W_2 = W_1 = \Lambda(t_0)\cap V$, and from Definition 3.3 we have, for $\varepsilon > 0$ small enough,

$$\operatorname{Mas}(\Lambda, V; [a, a+\varepsilon]) = -n_-(\mathfrak{m}_a^{(2)}), \tag{3.18}$$

just as in [CCLM23, Proposition 4.15] and [DJ11, Proposition 3.10].

We summarise the important properties of the Maslov index for the current analysis in the following proposition, as in [GPP04b, Lemma 3.8] (see also [RS93, Theorem 2.3]).

Proposition 3.4. *The Maslov index enjoys the following properties:*

(1) (Homotopy invariance.) If two paths $\Lambda_1, \Lambda_2 : [a,b] \longrightarrow \mathcal{L}(n)$ are homotopic with fixed endpoints, then

$$\mathrm{Mas}(\Lambda_1, V; [a,b]) = \mathrm{Mas}(\Lambda_2, \Lambda_0; [a,b]). \tag{3.19}$$

(2) (Additivity under concatenation.) For $\Lambda(t) : [a,c] \longrightarrow \mathcal{L}(n)$ and $a < b < c$,

$$\mathrm{Mas}(\Lambda, V; [a,c]) = \mathrm{Mas}(\Lambda, V; [a,b]) + \mathrm{Mas}(\Lambda, V; [b,c]). \tag{3.20}$$

(3) (Symplectic additivity.) Identify the Cartesian product $\mathcal{L}(n)\times\mathcal{L}(n)$ as a submanifold of $\mathcal{L}(2n)$. If $\Lambda = \Lambda_1 \oplus \Lambda_2 : [a,b] \to \mathcal{L}(2n)$ where $\Lambda_1, \Lambda_2 : [a,b] \to \mathcal{L}(n)$, and $V = V_1 \oplus V_2$ where $V_1, V_2 \in \mathcal{L}(n)$, then

$$\mathrm{Mas}(\Lambda, V; [a,b]) = \mathrm{Mas}(\Lambda_1, V_1; [a,b]) + \mathrm{Mas}(\Lambda_2, V_2; [a,b]). \tag{3.21}$$

(4) (Zero property.) If $\Lambda : [a,b] \longrightarrow \mathcal{T}_k(V)$ for any fixed integer k, then

$$\mathrm{Mas}(\Lambda, V; [a,b]) = 0. \tag{3.22}$$

We will call a crossing $t = t_0$ *positive* if

$$\sum_{k\geq 1} n_+(\mathfrak{m}_{t_0}^{(2k-1)}) = \dim \Lambda(t_0) \cap V, \tag{3.23}$$

and *negative* if

$$\sum_{k\geq 1} n_-(\mathfrak{m}_{t_0}^{(2k-1)}) = \dim \Lambda(t_0) \cap V. \tag{3.24}$$

In light of Definition 3.3, if t_0 is a positive interior crossing, or a positive crossing at the final point $t_0 = b$, then it contributes $\dim \Lambda(t_0) \cap V$ to the Maslov index. Similarly, if t_0 is a negative interior crossing, or a negative crossing at the initial point $t_0 = a$, then its contribution is $-\dim \Lambda(t_0) \cap V$. Note, however, from (3.11), that with this convention, the final crossing $t_0 = b$ may still contribute $\dim \Lambda(b) \cap V$ if it is not positive, and the initial point $t_0 = a$ may still contribute $-\dim \Lambda(a) \cap V$ if it is not negative.

Remark 3.5. In Section 4, we will need to make use of the robustness of one dimensional sign-definite crossings. Suppose then $t_0 \in [a,b]$ is a one-dimensional crossing, i.e. $\Lambda(t_0) \in \mathcal{T}_1(V)$. If t_0 is positive or negative, then the lowest nonzero crossing form is of odd order, and the order of intersection of the curve $t \mapsto \Lambda(t)$ with the codimension-one submanifold $\mathcal{T}_1(V)$ is also odd. It follows that the crossing will persist under small perturbations in the train $\mathcal{T}(V)$. Hence, in a neighbourhood of t_0, Λ will cross nearby trains $\mathcal{T}(W)$ transversely and in the same direction, for all W sufficiently close to V.

3.2. **The Maslov index for Lagrangian pairs.** Suppose now that we have a pair of Lagrangian paths $(\Lambda_1, \Lambda_2) : [a,b] \to \mathcal{L}(n)\times\mathcal{L}(n)$, or a *Lagrangian pair.* Using the symplectic additivity property of Proposition 3.4, it is possible to define the Maslov index of such an object (as in [RS93, Fur04, GPP04b, PT08]), where crossings are values $t_0 \in [a,b]$ such that $\Lambda_1(t_0) \cap \Lambda_2(t_0) \neq \{0\}$. Precisely, one realises the Lagrangian pair as the path $\Lambda_1 \oplus \Lambda_2$ in the space $\mathcal{L}(2n)$ of Lagrangian planes of $\mathbb{R}^{4n}$ equipped with the symplectic form

$$\Omega((u_1,u_2)^\top, (v_1,v_2)^\top) = \omega(u_1,v_1) - \omega(u_2,v_2), \qquad u_1,u_2,v_1,v_2 \in \mathbb{R}^{2n}. \tag{3.25}$$

Crossings of the pair then correspond to intersections of the path $\Lambda_1 \oplus \Lambda_2 : [a,b] \to \mathcal{L}(2n)$ with the diagonal subspace $\triangle = \{(x,x) : x \in \mathbb{R}^{2n}\} \subset \mathbb{R}^{4n}$. The Maslov index of the pair is then defined by

$$\mathrm{Mas}(\Lambda_1, \Lambda_2; [a,b]) := \mathrm{Mas}(\Lambda_1 \oplus \Lambda_2, \triangle; [a,b]). \tag{3.26}$$

The right hand side of (3.26) is computed with Definition 3.3, using the symplectic form (3.6); doing so leads to the following definitions, as in [PT08, Exercise 5.18]. A mapping $(w_1, w_2) : [t_0 - \varepsilon, t_0 + \varepsilon] \to \mathbb{R}^{2n} \times \mathbb{R}^{2n}, \varepsilon > 0$, will be called a *root function pair* for (Λ_1, Λ_2)

at $t = t_0$ if $w_1(t) \in \Lambda_1(t), w_2(t) \in \Lambda_2(t) \in \Lambda_2(t)$ and $w_1(t_0) = w_2(t_0)$. The *order* of (w_1, w_2), $\mathrm{ord}(w_1, w_2)$ is the smallest positive integer k such that $w_1^{(k)}(t_0) \neq w_2^{(k)}(t_0)$. We then define

$$\begin{aligned} W_k(\Lambda_1, \Lambda_2, t_0) := \{w_0 : \exists \text{ a root function pair } (w_1, w_2) \text{ for } (\Lambda_1, \Lambda_2) \\ \text{with } \mathrm{ord}(w_1, w_2) \geq k \text{ and } w_1(t_0) = w_2(t_0) = w_0\}, \end{aligned} \tag{3.27}$$

and when the choice of t_0 is clear we simply write $W_k(\Lambda_1, \Lambda_2)$. As in (3.3), we have $W_{k+1}(\Lambda_1, \Lambda_2, t_0) \subseteq W_k(\Lambda_1, \Lambda_2, t_0)$ for $k \geq 1$ and $W_1 = \Lambda_1(t_0) \cap \Lambda_2(t_0)$. The *kth-order relative crossing form* is then the quadratic form

$$\begin{aligned} \mathfrak{m}_{t_0}^{(k)}(\Lambda_1, \Lambda_2)(w_0) &:= \frac{d^k}{dt^k}\omega\big(w_1(t), w_0\big)\Big|_{t=t_0} - \frac{d^k}{dt^k}\omega\big(w_2(t), w_0\big)\Big|_{t=t_0}, \\ &= \omega\big(w_1^{(k)}(t_0) - w_2^{(k)}(t_0), w_0\big), \end{aligned} \tag{3.28}$$

for $w_0 \in W_k(\Lambda_1, \Lambda_2)$, where (w_1, w_2) is any root function pair for (Λ_1, Λ_2) at t_0 with $\mathrm{ord}(w_1, w_2) \geq k$. Analogous to (3.5), we have $W_k(\Lambda_1, \Lambda_2) = \ker \mathfrak{m}_{t_0}^{(k-1)}(\Lambda_1, \Lambda_2)$. Using (3.28) in Definition 3.3 thus computes the Maslov index of the pair (Λ_1, Λ_2); in the case that $\Lambda_2 = V$ is constant, the computation reduces to the Maslov index of the path Λ_1 with respect to the reference plane V described in Section 3.1.

The Maslov index is invariant for Lagrangian pairs that are *stratum homotopic*; we give a proof of this fact below. The corresponding result for single paths can be found in [RS93, Theorem 2.4]. Suppose the pairs $(\Lambda_1, \Lambda_2) : [a, b] \to \mathcal{L}(n)$ and $(\widetilde{\Lambda}_1, \widetilde{\Lambda}_2) : [a, b] \to \mathcal{L}(n)$ are stratum homotopic, i.e. there exist continuous mappings $H_1, H_2 : [0, 1] \times [a, b] \to \mathcal{L}(n)$ such that

$$\begin{aligned} H_1(0, \cdot) &= \Lambda_1(\cdot), & H_2(0, \cdot) &= \Lambda_2(\cdot) \\ H_1(1, \cdot) &= \widetilde{\Lambda}_1(\cdot), & H_2(1, \cdot) &= \widetilde{\Lambda}_2(\cdot), \end{aligned}$$

for which $\dim(H_1(s, a) \cap H_2(s, a))$ and $\dim(H_1(s, b) \cap H_2(s, b))$ are constant in $s \in [0, 1]$. We then have the following.

Lemma 3.6.

$$\mathrm{Mas}(\Lambda_1, \Lambda_2; [a, b]) = \mathrm{Mas}(\widetilde{\Lambda}_1, \widetilde{\Lambda}_2; [a, b]). \tag{3.29}$$

Proof. Consider the continuous mapping $H = H_1 \oplus H_2 : [0, 1] \times [a, b] \to \mathcal{L}(n) \times \mathcal{L}(n)$. By continuity of H and homotopy invariance (i.e. property (3) of Proposition 3.4), we have

$$\begin{aligned} \mathrm{Mas}(H(0, \cdot), \triangle; [a, b]) + \mathrm{Mas}(H(\cdot, b), \triangle; [0, 1]) \\ - \mathrm{Mas}(H(1, \cdot), \triangle; [a, b]) - \mathrm{Mas}(H(\cdot, a), \triangle; [0, 1]) = 0. \end{aligned} \tag{3.30}$$

Using (3.26) we have

$$\mathrm{Mas}(H(0, \cdot), \triangle; [a, b]) = \mathrm{Mas}(\Lambda_1, \Lambda_2; [a, b]), \quad \mathrm{Mas}(H(1, \cdot), \triangle; [a, b]) = \mathrm{Mas}(\widetilde{\Lambda}_1, \widetilde{\Lambda}_2; [a, b]).$$

By assumption $\dim\left(H(\cdot, a) \cap \triangle\right) = \dim\left(H_1(\cdot, a) \cap H_2(\cdot, a)\right)$ and $\dim\left(H(\cdot, b) \cap \triangle\right) = \dim\left(H_1(\cdot, b) \cap H_2(\cdot, b)\right)$ are constant, so by property (4) of Proposition 3.4 the Maslov indices of the second and fourth terms in (3.30) are zero. Equation (3.29) follows. ☐

3.3. **The Maslov box.** We first discuss the regularity and Lagrangian property of the stable and unstable bundles. Recall $\mathbb{E}^s(x, \lambda)$ and $\mathbb{E}^u(x, \lambda)$ defined in (2.12) for $x \in \mathbb{R}$ and $\lambda \in \mathbb{R}$. We extend $\mathbb{E}^s$ and $\mathbb{E}^u$ to $x = \pm\infty$ by setting

$$\mathbb{E}^s(+\infty, \lambda) := \mathbb{S}(\lambda), \qquad \mathbb{E}^u(-\infty, \lambda) := \mathbb{U}(\lambda), \tag{3.31}$$

and

$$\mathbb{E}^s(-\infty, \lambda) := \lim_{x \to -\infty} \mathbb{E}^s(x, \lambda), \qquad \mathbb{E}^u(+\infty, \lambda) := \lim_{x \to +\infty} \mathbb{E}^u(x, \lambda). \tag{3.32}$$

Thus by (2.13), $\mathbb{E}^s$ and $\mathbb{E}^u$ are continuous on $(-\infty,\infty]\times\mathbb{R}$ and $[-\infty,\infty)\times\mathbb{R}$ respectively. Furthermore, since the right hand side of (2.2) is analytic in λ and x, it follows that the solution spaces $\mathbb{E}^s$ and $\mathbb{E}^u$ are analytic on $(x,\lambda)\in\mathbb{R}\times\mathbb{R}$ (note that $x=\pm\infty$ is excluded). We remark here that the mapping

$$\lambda\mapsto\mathbb{E}^u(+\infty,\lambda)=\lim_{x\to\infty}\mathbb{E}^u(x;\lambda) \tag{3.33}$$

is discontinuous at eigenvalues $\lambda\in\operatorname{Spec}(N)$. Indeed, if $\lambda\notin\operatorname{Spec}(N)$, then $\mathbb{E}^u(+\infty,\lambda)=\mathbb{U}(\lambda)$ (again as points on the Grassmannian $\operatorname{Gr}_4(\mathbb{R}^8)$), while if $\lambda\in\operatorname{Spec}(N)$ is an eigenvalue then $\mathbb{E}^u(+\infty,\lambda)\cap\mathbb{S}(\lambda)\neq\{0\}$, i.e. $\mathbb{E}^u(+\infty,\lambda)\in\mathcal{T}(\mathbb{S}(\lambda))$. Now since $\mathbb{U}(\lambda)\in\mathcal{T}_0(\mathbb{S}(\lambda))$, and $\mathcal{T}_0(\mathbb{S}(\lambda))$ is an open subset of $\mathcal{L}(n)$ with boundary $\mathcal{T}(\mathbb{S}(\lambda))$, it follows that $\mathbb{U}(\lambda)$ is bounded away from $\mathcal{T}(\mathbb{S}(\lambda))$. For more details see the Appendix in [HLS18].

Remark 3.7. The Maslov index is defined for Lagrangian paths over compact intervals. Following [HLS18] we will sometimes compactify $\mathbb{R}$ via the change of variables

$$x=\ln\left(\frac{1+\tau}{1-\tau}\right),\qquad \tau\in[-1,1]. \tag{3.34}$$

(Similar transformations are used in [BCJ+18, AGJ90].) Notationally we will use a hat to indicate such a change has been made, for example,

$$\widehat{\mathbb{E}}^{s,u}(\tau,\cdot):=\mathbb{E}^{s,u}\left(\ln\left(\frac{1+\tau}{1-\tau}\right),\cdot\right),\quad \tau\in[-1,1]. \tag{3.35}$$

In this case, (3.31) implies that $\widehat{\mathbb{E}}^u(-1,\lambda)=\mathbb{U}(\lambda)$ and $\widehat{\mathbb{E}}^s(1,\lambda)=\mathbb{S}(\lambda)$.

Lemma 3.8. *The spaces $\mathbb{E}^u(x;\lambda)$ and $\mathbb{E}^s(x;\lambda)$ are Lagrangian subspaces of $\mathbb{R}^8$ for all $x\in[-\infty,\infty]$ and $\lambda\in\mathbb{R}$.*

Proof. First, recall that $\dim\mathbb{U}(\lambda)=\dim\mathbb{S}(\lambda)=4$ (we showed in (3.31) that $A_\infty(\lambda)$ is hyperbolic with four eigenvalues of positive real part and four of negative real part.) It follows from the continuity of $\mathbb{E}^u$ on $[-\infty,\infty)\times\mathbb{R}$ that $\dim\mathbb{E}^u(x,\lambda)=4$ for all $(x,\lambda)\in[-\infty,\infty)\times\mathbb{R}$. A similar argument shows $\dim\mathbb{E}^s(x,\lambda)=4$ for $(x,\lambda)\in(-\infty,\infty]\times\mathbb{R}$.

Next, for $x\in\mathbb{R}$, let $\mathbf{w}_1(x;\lambda),\mathbf{w}_2(x;\lambda)\in\mathbb{E}^u(x;\lambda)$. We have:

$$\begin{aligned}\omega(\mathbf{w}_1(x;\lambda),\mathbf{w}_2(x;\lambda))&=\langle J\mathbf{w}_1(x;\lambda),\mathbf{w}_2(x;\lambda)\rangle,\\&=\int_{-\infty}^{x}\frac{d}{ds}\langle J\mathbf{w}_1(s;\lambda),\mathbf{w}_2(s;\lambda)\rangle ds,\\&=\int_{-\infty}^{x}\langle JA(s;\lambda)\mathbf{w}_1(s;\lambda),\mathbf{w}_2(s;\lambda)\rangle+\langle J\mathbf{w}_1(s;\lambda),A(s;\lambda)\mathbf{w}_2(s;\lambda)\rangle ds,\\&=\int_{-\infty}^{x}\left\langle\left(A(s;\lambda)^\top J+JA(s;\lambda)\right)\mathbf{w}_1(s;\lambda),\mathbf{w}_2(s;\lambda)\right\rangle ds,\\&=0,\end{aligned}$$

where we used (2.14), i.e. that $A(x;\lambda)$ is infinitesimally symplectic. The proof for $\mathbb{E}^s(x;\lambda)$ is similar, but the integral is taken over $[x,\infty)$. We have shown that $\mathbb{E}^u$ and $\mathbb{E}^s$ are Lagrangian on $\mathbb{R}\times\mathbb{R}$. That this property extends to $x=\pm\infty$ follows the closedness of $\mathcal{L}(n)$ as a submanifold of $\operatorname{Gr}_n(\mathbb{R}^{2n})$. □

We are now ready to give the homotopy argument that leads to the lower bound of Theorem 1.2. We consider the Lagrangian pair

$$\Gamma\ni(x,\lambda)\mapsto(\mathbb{E}^u(x,\lambda),\mathbb{E}^s(\ell,\lambda))\in\mathcal{L}(4)\times\mathcal{L}(4), \tag{3.36}$$

where $\ell \gg 1$ needs to be chosen large enough so that

$$\mathbb{U}(\lambda) \cap \mathbb{E}^s(x,\lambda) = \{0\} \quad \text{for all} \quad x \geq \ell \tag{3.37}$$

(see Remark 3.9). Here $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$, where the Γ_i are the contours

$$\begin{aligned} &\Gamma_1 : x \in [-\infty, \ell], \;\; \lambda = 0, \quad \Gamma_3 : x \in [-\infty, \ell], \;\; \lambda = \lambda_\infty, \\ &\Gamma_2 : x = \ell, \;\; \lambda \in [0, \lambda_\infty], \quad\; \Gamma_4 : x = -\infty, \;\; \lambda = \lambda \in [0, \lambda_\infty], \end{aligned} \tag{3.38}$$

in the λx-plane (see Fig. 1). The set Γ has been dubbed the *Maslov box* [HLS18, Cor19], and the associated homotopy argument (outlined below) can be seen in as far back as the works of Bott [Bot56], Edwards [Edw64], Arnol'd [Arn67] and Duistermaat [Dui76]. Notice that along Γ_1 and Γ_3, the second entry $\mathbb{E}^s(\ell,\lambda)$ of the image of the map in (3.36) is fixed. The Maslov index of (3.36) along these pieces thus reduces to the Maslov index for a single path with respect to a fixed reference plane. Along Γ_2 and Γ_4, however, we have a genuine Lagrangian pair.

Crossings of (3.36) are points $(x,\lambda) \in \Gamma$ such that

$$\mathbb{E}^u(x,\lambda) \cap \mathbb{E}^s(\ell,\lambda) \neq \{0\}.$$

Recalling that λ is an eigenvalue of N if and only if $\mathbb{E}^u(x,\lambda) \cap \mathbb{E}^s(x,\lambda) \neq \{0\}$ for all $x \in \mathbb{R}$, it follows that the λ-values of the crossings along Γ_2 (where $x = \ell$) are exactly the eigenvalues of N. In particular, since $0 \in \operatorname{Spec}(N)$ there will be a crossing at $(x,\lambda) = (0,\ell)$. From Hypothesis 1.1 we have $\ker(L_-) = \operatorname{span}\{\phi\}$ and $\ker(L_+) = \operatorname{span}\{\phi'\}$. Denoting the corresponding solutions of (2.2) by

$$\boldsymbol{\phi}(x) := \begin{pmatrix} 0 \\ \phi''(x) + \sigma_2\phi(x) \\ 0 \\ -\phi(x) \\ 0 \\ -\phi'(x) \\ 0 \\ \phi'''(x) \end{pmatrix}, \qquad \boldsymbol{\varphi}(x) := \begin{pmatrix} \phi'''(x) + \sigma_2\phi'(x) \\ 0 \\ \phi'(x) \\ 0 \\ \phi''(x) \\ 0 \\ \phi''''(x) \\ 0 \end{pmatrix}, \tag{3.39}$$

(obtained from (2.1) with $v = \phi$ and $u = \phi'$ respectively), we therefore have

$$\mathbb{E}^u(x;0) \cap \mathbb{E}^s(x;0) = \operatorname{span}\{\boldsymbol{\phi}(x), \boldsymbol{\varphi}(x)\} \quad \text{for all} \;\; x \in \mathbb{R}. \tag{3.40}$$

Remark 3.9. That the path (3.33) is discontinuous in λ prohibits taking Γ_2 to be at $x = +\infty$. Taking Γ_2 to be at $x = \ell$ for ℓ large enough avoids this issue. Chen and Hu [CH07] showed that by taking ℓ large enough so that (3.37) holds, the Maslov index of (3.36) along Γ_1 is independent of the choice of ℓ. For more details, see [CH07, Cor19].

Crossings along Γ_1, i.e. points $(x,\lambda) = (x_0, 0)$ such that

$$\mathbb{E}^u(x_0,0) \cap \mathbb{E}^s(\ell,0) \neq \{0\}, \tag{3.41}$$

are called *conjugate points*. Recall that when $\lambda = 0$ the eigenvalue equations (1.14) decouple to give $L_+u = 0$ and $L_-v = 0$. Similarly, the first order system (2.2) decouples into two independent systems for the u and v variables. In Section 4 the eigenvalue problems for the operators L_+ and L_- will be written as first order systems; the stable and unstable bundles for the L_+ system will be denoted by $\mathbb{E}^s_+(x,\lambda)$ and $\mathbb{E}^u_+(x,\lambda)$, respectively, while the stable and unstable bundles for the L_- system will be denoted by $\mathbb{E}^{s,u}_-(x,\lambda)$. As a result of the decoupling at $\lambda = 0$ we have

$$\mathbb{E}^u(x,0) = \mathbb{E}^u_+(x,0) \oplus \mathbb{E}^u_-(x,0) \qquad \text{and} \qquad \mathbb{E}^s(x,0) = \mathbb{E}^s_+(x,0) \oplus \mathbb{E}^s_-(x,0) \tag{3.42}$$

for all $x \in \mathbb{R}$, so that

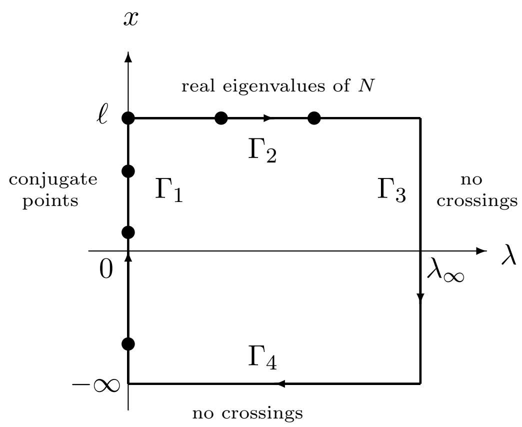


Figure 1. Maslov box in the λx-plane, with edges oriented in a clockwise fashion. The crossing at the top left corner $(0, \ell)$ corresponds to the zero eigenvalue of N. Noting that $\lambda \in \mathbb{R}$ is a spectral parameter, and therefore lives on the real axis in $\mathbb{C}$, it is natural to place λ on the horizontal axis.

$$\{x \in \mathbb{R} : \mathbb{E}^u(x,0) \cap \mathbb{E}^s(\ell,0) \neq \{0\}\} = \\ \{x \in \mathbb{R} : \mathbb{E}^u_+(x,0) \cap \mathbb{E}^s_+(\ell,0) \neq \{0\}\} \cup \{x \in \mathbb{R} : \mathbb{E}^u_-(x,0) \cap \mathbb{E}^s_-(\ell,0) \neq \{0\}\}. \quad (3.43)$$

(The precise notion of the direct sums in (3.42) will be given in Section 5.) When dealing with conjugate points, we will see that it suffices to use the stable subspace $\mathbb{S}(0)$ (instead of $\mathbb{E}^s(\ell,0)$) as the reference plane to do computations. That $\mathbb{S}(0) = \mathbb{S}_+(0) \oplus \mathbb{S}_-(0)$, where $\mathbb{S}_\pm(0)$ is the stable subspace of the asymptotic first order system for the eigenvalue problem for $L_\pm$, leads to the following classification of conjugate points.

Definition 3.10. *An* L_+ conjugate point *is a point* $(x,\lambda) = (x_0, 0)$ *such that* $\mathbb{E}^u_+(x_0,0) \cap \mathbb{S}_+(0) \neq \{0\}$*. An* L_- conjugate point *is similarly defined via* $\mathbb{E}^u_-(x_0,0) \cap \mathbb{S}_-(0) \neq \{0\}$*.*

Since the solid rectangle $\mathcal{M} = [-\infty, \ell] \times [0, \lambda_\infty]$ is contractible and the map (3.36) with domain $\mathcal{M}$ is continuous, the image of the boundary $\partial\mathcal{M} = \Gamma$ of $\mathcal{M}$ in $\mathcal{L}(4) \times \mathcal{L}(4)$ is homotopic to a fixed point. From homotopy invariance (Proposition 3.4), it follows that

$$\operatorname{Mas}(\mathbb{E}^u(\cdot,\cdot), \mathbb{E}^s(\cdot,\cdot); \Gamma) = 0. \quad (3.44)$$

By additivity under concatenation, we can decompose the left hand side into the contributions coming from the constituent sides of the Maslov box, i.e.

$$\operatorname{Mas}(\mathbb{E}^u(\cdot,0), \mathbb{E}^s(\ell,0); [-\infty,\ell]) + \operatorname{Mas}(\mathbb{E}^u(\ell,\cdot), \mathbb{E}^s(\ell,\cdot); [0,\lambda_\infty]) \\ - \operatorname{Mas}(\mathbb{E}^u(\cdot,\lambda_\infty), \mathbb{E}^s(\ell,\lambda_\infty); [-\infty,\ell]) - \operatorname{Mas}(\mathbb{E}^u(-\infty,\cdot), \mathbb{E}^s(\ell,\cdot); [0,\lambda_\infty]) = 0. \quad (3.45)$$

Note we have included minus signs for the last two terms in order to be consistent with the clockwise orientation of the Maslov box (see Fig. 1). We will show in Section 5 that in fact these last two Maslov indices are zero. Defining

$$\mathfrak{c} := \operatorname{Mas}(\mathbb{E}^u(\cdot,0), \mathbb{E}^s(\ell,0); [\ell-\varepsilon,\ell]) + \operatorname{Mas}(\mathbb{E}^u(\ell,\cdot), \mathbb{E}^s(\ell,\cdot); [0,\varepsilon]), \quad 0 < \varepsilon \ll 1, \quad (3.46)$$

to be the contribution to the Maslov index of the crossing $(x,\lambda) = (\ell,0)$ at the top left corner of the Maslov box, it follows once more from additivity under concatenation that

$$\operatorname{Mas}(\mathbb{E}^u(\cdot,0), \mathbb{E}^s(\ell,0); [-\infty,\ell-\varepsilon]) + \mathfrak{c} + \operatorname{Mas}(\mathbb{E}^u(\ell,\cdot), \mathbb{E}^s(\ell,\cdot); [\varepsilon,\lambda_\infty]) = 0. \quad (3.47)$$

We will compute the first term of (3.47) by counting L_+ and L_- conjugate points. By bounding $n_+(N)$ from below by the absolute value of the third term in (3.47), computing $\mathfrak{c}$ and rearranging, we will arrive at the statement of Theorem 1.2. Before doing so, we turn to the computation of the Morse indices of L_+ and L_- via the Maslov index.

4. Spectral counts for L_+ and L_- via conjugate points

In this section we focus on the spectral problems for the fourth-order selfadjoint operators L_+ and L_-. Specifically, we prove that the Morse index of each operator is equal to the respective number of conjugate points on $\mathbb{R}$. Similar results for certain classes of selfadjoint fourth order operators may be found in [How21, How23].

Theorem 4.1. *The number of positive eigenvalues of L_+ is equal to the number of L_+-conjugate points, counted with multiplicity, on $\mathbb{R}$:*

$$P = \sum_{x\in\mathbb{R}} \dim\left(\mathbb{E}^u_+(x,0)\cap\mathbb{S}_+(0)\right). \tag{4.1}$$

A similar assertion holds for L_-.

We will prove the proposition using a homotopy argument involving the Maslov box, similar to that described in Section 3.3. We will focus primarily on the spectral count for L_+; the spectral count for L_- follows similarly with minor adjustments. In order to set the argument up, we introduce the first order systems for the L_+ and L_- eigenvalue problems and their associated stable and unstable bundles. In what follows, we use a subscript $+$ or $-$ to indicate that objects pertain to the eigenvalue problem for L_+ or L_-.

The eigenvalue equation for L_+,

$$-u'''' - \sigma_2 u'' - \beta u + 3\phi^2 u = \lambda u, \qquad u\in H^4(\mathbb{R}), \tag{4.2}$$

can be reduced to the following first order system via the u substitutions in (2.1),

$$\begin{pmatrix} u_1\\ u_2\\ u_3\\ u_4\end{pmatrix}' = \begin{pmatrix} 0 & 0 & \sigma_2 & 1\\ 0 & 0 & 1 & 0\\ 1 & -\sigma_2 & 0 & 0\\ -\sigma_2 & \alpha(x)-\lambda & 0 & 0\end{pmatrix}\begin{pmatrix} u_1\\ u_2\\ u_3\\ u_4\end{pmatrix}, \tag{4.3}$$

where $\alpha(x) = 3\phi(x)^2 - \beta + 1$. Similar to (2.2), we write this system as

$$\mathbf{u}_x = A_+(x,\lambda)\mathbf{u}, \tag{4.4}$$

where $\mathbf{u} = (u_1,u_2,u_3,u_4)^\top$ and

$$A_+(x,\lambda) = \begin{pmatrix} 0 & B_+\\ C_+(x,\lambda) & 0\end{pmatrix},\quad B_+ = \begin{pmatrix} \sigma_2 & 1\\ 1 & 0\end{pmatrix},\quad C_+(x,\lambda) = \begin{pmatrix} 1 & -\sigma_2\\ -\sigma_2 & \alpha(x)-\lambda\end{pmatrix}.$$

Likewise, the eigenvalue equation for L_-,

$$-v'''' - \sigma_2 v'' - \beta v + \phi^2 v = \lambda v, \qquad v\in H^4(\mathbb{R}), \tag{4.5}$$

can be reduced to the following first order system via the v substitutions in (2.1),

$$\begin{pmatrix} v_1\\ v_2\\ v_3\\ v_4\end{pmatrix}' = \begin{pmatrix} 0 & 0 & -\sigma_2 & 1\\ 0 & 0 & 1 & 0\\ -1 & -\sigma_2 & 0 & 0\\ -\sigma_2 & \eta(x)+\lambda & 0 & 0\end{pmatrix}\begin{pmatrix} v_1\\ v_2\\ v_3\\ v_4\end{pmatrix}. \tag{4.6}$$

where $\eta(x) = -\phi(x)^2 + \beta - 1$. We write this as

$$\mathbf{v}_x = A_-(x,\lambda)\mathbf{v}, \tag{4.7}$$

where $\mathbf{v} = (v_1,v_2,v_3,v_4)^\top$ and

$$A_-(x,\lambda) = \begin{pmatrix} 0 & B_-\\ C_-(x,\lambda) & 0\end{pmatrix},\quad B_- = \begin{pmatrix} -\sigma_2 & 1\\ 1 & 0\end{pmatrix},\quad C_-(x,\lambda) = \begin{pmatrix} -1 & -\sigma_2\\ -\sigma_2 & \eta(x)+\lambda\end{pmatrix}.$$

The coefficient matrices $A_\pm(x,\lambda)$ are infinitesimally symplectic, satisfying equation (2.14). In order to be consistent with (2.2) at $\lambda = 0$, we have used the same substitutions (2.1) to reduce (4.2) and (4.5) to (4.3) and (4.6) respectively. Consequently, λ appears with a different sign in (4.3) and (4.6), due to the substitutions for u_2 and u_3 in (2.1) having different signs to those for v_2 and v_3 in the same equation. This will be the reason for the difference in sign of the Maslov indices in Lemma 4.2.

Under the assumptions (1.10),(1.11), for all λ lying outside of the essential spectrum in (2.10), the asymptotic matrices

$$A_+(\lambda) := \lim_{x\to\pm\infty} A_+(x,\lambda), \qquad A_-(\lambda) := \lim_{x\to\pm\infty} A_-(x,\lambda),$$

each have two eigenvalues with negative real part and two with positive real part. We denote the associated stable and unstable subspaces by $\mathbb{S}_\pm(\lambda)$ and $\mathbb{U}_\pm(\lambda)$ respectively. Reasoning as in Section 3.3, associated with each of the systems (4.3) and (4.6) are the stable and unstable bundles,

$$\begin{aligned}
\mathbb{E}^u_+(x,\lambda) &:= \{\xi \in \mathbb{R}^4 : \xi = \mathbf{u}(x;\lambda),\ \mathbf{u} \text{ solves (4.3) and } \mathbf{u}(x;\lambda)\to 0 \text{ as } x\to-\infty\},\\
\mathbb{E}^s_+(x,\lambda) &:= \{\xi \in \mathbb{R}^4 : \xi = \mathbf{u}(x;\lambda),\ \mathbf{u} \text{ solves (4.3) and } \mathbf{u}(x;\lambda)\to 0 \text{ as } x\to+\infty\},\\
\mathbb{E}^u_-(x,\lambda) &:= \{\xi \in \mathbb{R}^4 : \xi = \mathbf{v}(x;\lambda),\ \mathbf{v} \text{ solves (4.6) and } \mathbf{v}(x;\lambda)\to 0 \text{ as } x\to-\infty\},\\
\mathbb{E}^s_-(x,\lambda) &:= \{\xi \in \mathbb{R}^4 : \xi = \mathbf{v}(x;\lambda),\ \mathbf{v} \text{ solves (4.6) and } \mathbf{v}(x;\lambda)\to 0 \text{ as } x\to+\infty\}.
\end{aligned} \tag{4.8}$$

When considered as points on the Grassmannian $\mathrm{Gr}_2(\mathbb{R}^4)$, these bundles converge to the asymptotic stable and unstable subspaces as follows,

$$\begin{aligned}
\lim_{x\to-\infty} \mathbb{E}^u_+(x,\lambda) &= \mathbb{U}_+(\lambda), & \lim_{x\to+\infty} \mathbb{E}^s_+(x,\lambda) &= \mathbb{S}_+(\lambda),\\
\lim_{x\to-\infty} \mathbb{E}^u_-(x,\lambda) &= \mathbb{U}_-(\lambda), & \lim_{x\to+\infty} \mathbb{E}^s_-(x,\lambda) &= \mathbb{S}_-(\lambda).
\end{aligned}$$

That $\mathbb{E}^u_+(x,\lambda), \mathbb{E}^u_-(x,\lambda), \mathbb{E}^s_+(x,\lambda), \mathbb{E}^s_-(x,\lambda)$ are Lagrangian subspaces of $\mathbb{R}^4$, with the mappings $(x,\lambda) \mapsto \mathbb{E}^u_\pm(x,\lambda)$ being continuous on $[-\infty,\infty)\times\mathbb{R}$ and $(x,\lambda)\mapsto \mathbb{E}^{u,s}_\pm(x,\lambda)$ analytic on $\mathbb{R}\times\mathbb{R}$, follows from the same arguments as in Section 3.3. We omit the proofs.

Let us now consider the path

$$\Gamma \ni (x,\lambda) \mapsto (\mathbb{E}^u_+(x,\lambda), \mathbb{E}^s_+(\ell,\lambda)) \in \mathcal{L}(2)\times\mathcal{L}(2), \tag{4.9}$$

where Γ is given in (3.38) (see Fig. 1). Crossings of (4.9) along Γ_2 now represent eigenvalues of L_+. Homotopy invariance and additivity under concatenation implies that

$$\begin{aligned}
&\mathrm{Mas}(\mathbb{E}^u_+(\cdot,0), \mathbb{E}^s_+(\ell,0); [-\infty,\ell]) + \mathrm{Mas}(\mathbb{E}^u_+(\ell,\cdot), \mathbb{E}^s_+(\ell,\cdot); [0,\lambda_\infty])\\
&\quad - \mathrm{Mas}(\mathbb{E}^u_+(\cdot,\lambda_\infty), \mathbb{E}^s_+(\ell,\lambda_\infty); [-\infty,\ell]) - \mathrm{Mas}(\mathbb{E}^u_+(-\infty,\cdot), \mathbb{E}^s_+(\ell,\cdot); [0,\lambda_\infty]) = 0.
\end{aligned} \tag{4.10}$$

To prove Theorem 4.1, we will show that the last two terms in the right hand side of (4.10) are zero, and that every crossing along Γ_2 is positive. We also show that along Γ_1, we can interchange the reference plane $\mathbb{E}^s_+(\ell,0)$ with $\mathbb{S}_+(0)$ over a modified interval, i.e.

$$\mathrm{Mas}(\mathbb{E}^u_+(\cdot,0), \mathbb{E}^s_+(\ell,0); [-\infty,\ell]) = \mathrm{Mas}(\mathbb{E}^u_+(\cdot,0), \mathbb{S}_+(0); [-\infty,\infty]), \tag{4.11}$$

and that all crossings of the latter path are negative. By showing that the corner crossing $(x,\lambda) = (\ell,0)$ has two contributions (the arrival along Γ_1 and departure along Γ_2) to the Maslov index of the path (4.9) that cancel each other out, Theorem 4.1 will then follow. The proof for the L_- problem will be similar.

4.1. **Computing the Maslov index along Γ_1.** In order to compute the right hand side of (4.11), we will need a real frame for $\mathbb{S}_\pm(0)$ with which we can compute crossing forms. To that end, note that the asymptotic matrices $A_\pm(0)$ satisfy $\mathrm{Spec}(A_+(0)) = \mathrm{Spec}(A_-(0)) = \{\pm\mu_1, \pm\mu_2\}$, where

$$\mu_1 = \frac{\sqrt{-\sigma_2 - \sqrt{1-4\beta}}}{\sqrt{2}}, \qquad \mu_2 = \frac{\sqrt{-\sigma_2 + \sqrt{1-4\beta}}}{\sqrt{2}}. \tag{4.12}$$

Under the assumption (1.10), we have $\mu_2 = \bar{\mu}_1$ whenever $\beta \geq 1/4$ (for both $\sigma_2 = 1$ and $\sigma_2 = -1$), and $\mu_1, \mu_2 \in \mathbb{R}$ when $\sigma_2 = -1$ and $0 < |\beta| \leq 1/4$. The eigenvectors corresponding to $-\mu_1$ and $-\mu_2$ are given by

$$\mathbf{p}_1 = \begin{pmatrix} \mu_2^2 \\ -1 \\ \mu_1 \\ \mu_1^3 \end{pmatrix}, \ \mathbf{p}_2 = \begin{pmatrix} \mu_1^2 \\ -1 \\ \mu_2 \\ \mu_2^3 \end{pmatrix}, \qquad \text{and} \qquad \mathbf{m}_1 = \begin{pmatrix} \mu_2^2 \\ 1 \\ -\mu_1 \\ \mu_1^3 \end{pmatrix}, \ \mathbf{m}_2 = \begin{pmatrix} \mu_1^2 \\ 1 \\ -\mu_2 \\ \mu_2^3 \end{pmatrix}, \tag{4.13}$$

so that

$$\ker\left(A_+(0) + \mu_i\right) = \mathrm{span}\{\mathbf{p}_i\}, \quad \ker\left(A_-(0) + \mu_i\right) = \mathrm{span}\{\mathbf{m}_i\}, \qquad i = 1, 2. \tag{4.14}$$

Notice that the vectors $\mathbf{p}_i, \mathbf{m}_i$ for $i = 1, 2$ are complex-valued if $\beta > 1/4$. We collect these vectors into the columns of two frames, which we denote with 2×2 blocks P_i, M_i, $i = 1, 2$, via

$$\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} \coloneqq \begin{pmatrix} \mu_2^2 & \mu_1^2 \\ -1 & -1 \\ \mu_1 & \mu_2 \\ \mu_1^3 & \mu_2^3 \end{pmatrix}, \qquad \begin{pmatrix} M_1 \\ M_2 \end{pmatrix} \coloneqq \begin{pmatrix} \mu_2^2 & \mu_1^2 \\ 1 & 1 \\ -\mu_1 & -\mu_2 \\ \mu_1^3 & \mu_2^3 \end{pmatrix}. \tag{4.15}$$

All of the matrices P_i, M_i are invertible under (1.10) and (1.11). Right multiplying each frame in (4.15) by the inverse of its upper 2×2 block yields the following *real* frame for $\mathbb{S}_\pm(0)$,

$$\mathbf{S}_\pm = \begin{pmatrix} I \\ S_\pm \end{pmatrix}, \qquad S_\pm = \frac{1}{\sqrt{2\sqrt{\beta} - \sigma_2}} \begin{pmatrix} \mp 1 & \sigma_2 - \sqrt{\beta} \\ \sigma_2 - \sqrt{\beta} & \pm(\sqrt{\beta}\sigma_2 + \beta - 1) \end{pmatrix}, \tag{4.16}$$

where $S_+ = P_2 P_1^{-1}$ and $S_- = M_2 M_1^{-1}$.

An important relation exists between $S_\pm$ and the blocks of the asymptotic matrix $A_\pm(0)$ that will be needed in our analysis. Define $C_\pm(x) \coloneqq C_\pm(x, 0)$. Focusing on the L_+ problem, because the columns of the frame (P_1, P_2) are eigenvectors of $A_+(0)$, we have

$$\begin{pmatrix} 0 & B_+ \\ \widehat{C}_+ & 0 \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} D, \qquad D = \mathrm{diag}\{-\mu_1, -\mu_2\}, \tag{4.17}$$

where

$$\widehat{C}_+ = \lim_{x \to \pm\infty} C_+(x) = \begin{pmatrix} \pm 1 & -\sigma_2 \\ -\sigma_2 & \mp(\beta - 1) \end{pmatrix}. \tag{4.18}$$

That is, $B_+ P_2 = P_1 D$ and $\widehat{C}_+ P_1 = P_2 D$. It follows that

$$\widehat{C}_+ = P_2 D_+ P_1^{-1} = \left(P_2 P_1^{-1}\right)\left(P_1 D_+ P_2^{-1}\right)\left(P_2 P_1^{-1}\right) = S_+ B_+ S_+. \tag{4.19}$$

It can be similarly shown that

$$\widehat{C}_- = S_- B_- S_-, \tag{4.20}$$

where $\widehat{C}_- \coloneqq \lim_{x \to \pm\infty} C_-(x)$.

We are ready to state our first intermediate result towards the proof of Theorem 4.1: monotonicity of the paths $x \mapsto (\mathbb{E}_\pm^u(x, 0), \mathbb{S}_\pm(0))$. In what follows, $\langle \cdot, \cdot \rangle$ denotes the Euclidean dot product.

Lemma 4.2. *Each crossing $x_0 \in \mathbb{R}$ of the Lagrangian path $x \mapsto (\mathbb{E}_+^u(x,0), \mathbb{S}_+(0))$ is negative. Thus*

$$\operatorname{Mas}(\mathbb{E}_+^u(\cdot,0), \mathbb{S}_+(0); [-\infty,\infty)) = -\sum_{x\in\mathbb{R}} \dim\left(\mathbb{E}_+^u(x,0) \cap \mathbb{S}_+(0)\right). \tag{4.21}$$

Similarly, each crossing $x = x_0 \in \mathbb{R}$ of $x \mapsto \left(\mathbb{E}_-^u(x,0), \mathbb{S}_-(0)\right)$ is positive, and we have

$$\operatorname{Mas}(\mathbb{E}_-^u(\cdot,0), \mathbb{S}_-(0); [-\infty,\infty)) = \sum_{x\in\mathbb{R}} \dim\left(\mathbb{E}_-^u(x,0) \cap \mathbb{S}_-(0)\right). \tag{4.22}$$

Remark 4.3. In the above lemma (and throughout), by having the domain of the Lagrangian paths $x \mapsto (\mathbb{E}_\pm^u(\cdot,0), \mathbb{S}_\pm(0))$ as $x \in [-\infty,\infty)$, we mean that $\tau \in [-1, 1-\varepsilon]$ for the compactified path $\tau \mapsto \widehat{\mathbb{E}}_+^u(\tau,0)$ for some small $\varepsilon > 0$ (see Remark 3.7). We emphasise that the final point of the path, $\tau = +1$ ($x = \infty$), which is always a conjugate point because $\mathbb{E}_\pm^u(+\infty,0) \in \mathcal{T}_1(\mathbb{S}_\pm(0))$ on account of Hypothesis 1.1, is excluded.

Proof. We begin with the L_+ problem. Denote a frame for the unstable bundle $\mathbb{E}_+^u(x,0)$ by

$$\mathbf{U}(x) = \begin{pmatrix} X(x) \\ Y(x) \end{pmatrix}, \qquad X(x), Y(x) \in \mathbb{R}^{2\times 2}. \tag{4.23}$$

In what follows, $x = x_0 \in \mathbb{R}$ is a conjugate point, i.e. $\mathbb{E}_+^u(x_0,0) \cap \mathbb{S}_+(0) \neq \{0\}$, and we will denote

$$\mathbf{U}_0 := \mathbf{U}(x_0), \quad X_0 := X(x_0), \quad Y_0 := Y(x_0).$$

We momentarily assume the following.

Hypothesis 4.4. At every crossing $x_0 \in \mathbb{R}$, $\phi(x_0) \neq 0$.

We begin by computing the first order crossing form, i.e. (3.6) with x as the independent variable and $k = 1$. To that end, recall from (3.3) that

$$W_1 = \mathbb{E}_+^u(x,0) \cap \mathbb{S}_+(0).$$

Any $w_0 \in W_1$ can therefore be written

$$w_0 = \mathbf{U}_0 h_0 = \begin{pmatrix} X_0 \\ Y_0 \end{pmatrix} h_0 = \begin{pmatrix} I \\ S_+ \end{pmatrix} k_0 \in \mathbb{E}_+^u(x,0) \cap \mathbb{S}_+(0), \tag{4.24}$$

for some $h_0, k_0 \in \mathbb{R}^2$. To compute the first order form, as in (3.13), we need to write down $\mathbf{U}'(x_0)$; since the columns of $\mathbf{U}(x)$ satisfy (4.3), we have

$$X'(x) = B_+ Y(x), \qquad Y'(x) = C_+(x) X(x), \tag{4.25}$$

(recall that $C_+(x) := C_+(x,0)$). Now let us define

$$\widetilde{C}_+(x) := C_+(x) - \widehat{C}_+ = \begin{pmatrix} 0 & 0 \\ 0 & 3\phi(x)^2 \end{pmatrix}, \tag{4.26}$$

and recalling $\widehat{C}_+$ from (4.18) and using (4.19), we observe that

$$C_+(x) - S_+ B_+ S_+ = \left(\widehat{C}_+ + \widetilde{C}_+(x)\right) - S_+ B_+ S_+ = \widetilde{C}_+(x). \tag{4.27}$$

Letting $k_0 = (a_0, b_0)^\top$, and using (4.24), (4.25) and (4.27), we compute (3.13):

$$\begin{aligned} \mathfrak{m}_{x_0}(\mathbb{E}_+^u(\cdot,0), \mathbb{S}_+(0))(w_0) = \omega\left(\mathbf{U}'(x_0) h_0, w_0\right) &= \left\langle \begin{pmatrix} -Y'(x_0) \\ X'(x_0) \end{pmatrix} h_0, \begin{pmatrix} I \\ S_+ \end{pmatrix} k_0 \right\rangle, \\ &= -\left\langle \left(Y'(x_0) - S_+ X'(x_0)\right) h_0, k_0 \right\rangle, \\ &= -\left\langle \left(C_+(x_0) - S_+ B_+ S_+\right) k_0, k_0 \right\rangle, \\ &= -\widetilde{C}_+(x_0) k_0 = -3\phi(x_0)^2 b_0^2. \end{aligned} \tag{4.28}$$

Under Hypothesis 4.4, we conclude that if there exist root functions w such that $w_0 = \mathbf{S}_+ k_0$ where $k_0 = (a_0, b_0)^\top, b_0 \neq 0$, then $n_-(\mathfrak{m}_{x_0}) = 1$. If, however, there exist root functions w such that $b_0 = 0$, then the form $\mathfrak{m}_{x_0}$ is degenerate, and we need to compute higher order crossing forms. We split the analysis into two cases; in what follows, we suppress x-dependence of C_+ to simplify the notation.

Case 1: $\dim \mathbb{E}^u_+(x_0, 0) \cap \mathbb{S}_+(0) = 1$.

Let $\mathbf{s}_1, \mathbf{s}_2$ denote the first and second columns of the frame $\mathbf{S}_+$ given in (4.16). If for some fixed a_0 and $b_0 \neq 0$ we have

$$W_1 = \mathbb{E}^u_+(x_0, 0) \cap \mathbb{S}_+(0) = \operatorname{span}\{a_0 \mathbf{s}_1 + b_0 \mathbf{s}_2\}, \tag{4.29}$$

then from (4.28) we see that $n_-(\mathfrak{m}_{x_0}) = 1 = \dim \mathbb{E}^u_+(x_0, 0) \cap \mathbb{S}_+(0)$, and the crossing x_0 is negative. On the other hand, if

$$W_1 = \mathbb{E}^u_+(x_0, 0) \cap \mathbb{S}_+(0) = \operatorname{span}\{\mathbf{s}_1\}, \tag{4.30}$$

so that $b_0 = 0$, then $\mathfrak{m}_{x_0} = 0$, and we need to compute higher order crossing forms. For the rest of Case 1, we assume (4.30), so that the vector k_0 in (4.24) is given by $k_0 = (a_0, 0)^\top$ (where $a_0 \neq 0$ for a nontrivial intersection.)

Since the crossing form is zero on this (one-dimensional) crossing, we know from (3.5) that

$$W_2 = \ker \mathfrak{m}_{x_0} = W_1 = \operatorname{span}\{\mathbf{s}_1\}. \tag{4.31}$$

According to Lemma 3.2, assuming h_0 satisfies (4.24) with $b_0 = 0$, i.e.

$$\mathbf{U}_0 h_0 = \begin{pmatrix} X_0 \\ Y_0 \end{pmatrix} h_0 = \begin{pmatrix} I \\ S_+ \end{pmatrix} k_0, \qquad k_0 = \begin{pmatrix} a_0 \\ 0 \end{pmatrix}, \tag{4.32}$$

we now need to find a vector $h_1 \in \mathbb{R}^2$ such that

$$\mathbf{U}'(x_0) h_0 + \mathbf{U}_0 h_1 = \begin{pmatrix} X'(x_0) \\ Y'(x_0) \end{pmatrix} h_0 + \begin{pmatrix} X_0 \\ Y_0 \end{pmatrix} h_1 \in \mathbb{S}_+(0). \tag{4.33}$$

Since (I, S_+) is a frame for $\mathbb{S}_+(0)$, this just means

$$Y'(x_0) h_0 + Y_0 h_1 = S_+ \left(X'(x_0) h_0 + X_0 h_1 \right). \tag{4.34}$$

Thus, using (4.32), (4.25), and that

$$C_+ k_0 = S_+ B_+ S_+ k_0, \tag{4.35}$$

(see (4.26), (4.27)), it follows from (4.34) that

$$\left(Y_0 - S_+ X_0\right) h_1 = -\left(Y'(x_0) h_0 - S_+ X'(x_0)\right) h_0 = -(C_+ - S_+ B_+ S_+) k_0 = 0. \tag{4.36}$$

In order to write down h_1, we first note from (4.32) that

$$\left(Y_0 - S_+ X_0\right) h_0 = 0. \tag{4.37}$$

Next, we observe that $\mathbb{E}^u_+(x_0, 0) \cap \mathbb{S}_+(0) \cong \ker\left(Y_0 - S_+ X_0\right)$, where the bijective correspondence is given by $h_0 \leftrightarrow \mathbf{U}_0 h_0$. To see this, note that any $\mathbf{U}_0 h_0 \in \mathbb{E}^u_+(x_0, 0)$ is also in $\mathbb{S}_+(0)$ if and only if $Y_0 h_0 = S_+ X_0 h_0$, i.e. $h_0 \in \ker(Y_0 - S_+ X_0)$. Under our assumption (4.30), it follows that $\dim \ker\left(Y_0 - S_+ X_0\right) = 1$. Hence, (4.36) and (4.37) imply that

$$h_1 = \alpha h_0, \qquad \alpha \in \mathbb{R}. \tag{4.38}$$

We are ready to compute the second order form using Lemma 3.2. With W_2 given by (4.31) and h_0, h_1 given by (4.32) and (4.38), we compute:

$$\begin{aligned}
\mathfrak{m}^{(2)}_{x_0}(\mathbb{E}^u_+(\cdot, 0), \mathbb{S}_+(0))(w_0) &= \omega(\mathbf{U}''(x_0) h_0 + 2\mathbf{U}'(x_0) h_1, w_0), \\
&= \left\langle \begin{pmatrix} -Y''(x_0) \\ X''(x_0) \end{pmatrix} h_0, \begin{pmatrix} I \\ S_+ \end{pmatrix} k_0 \right\rangle + 2 \left\langle \begin{pmatrix} -Y'(x_0) \\ X'(x_0) \end{pmatrix} h_1, \begin{pmatrix} I \\ S_+ \end{pmatrix} k_0 \right\rangle,
\end{aligned}$$

$$= -\left\langle \left(Y''(x_0) - S_+X''(x_0)\right) h_0, k_0\right\rangle - 2\left\langle \left(Y'(x_0) - S_+X'(x_0)\right) h_1, k_0\right\rangle. \qquad (4.39)$$

Differentiating (4.25) yields

$$X''(x) = B_+C_+X(x), \qquad Y''(x) = C'_+X(x) + C_+B_+Y(x). \qquad (4.40)$$

Using (4.32), it follows that

$$X''(x_0)h_0 = B_+C_+k_0, \qquad Y''(x_0)h_0 = C'_+k_0 + C_+B_+S_+k_0 = C_+B_+S_+k_0, \qquad (4.41)$$

where we used that

$$C'_+k_0 = \widetilde{C}'_+k_0 = \begin{pmatrix} 0 & 0 \\ 0 & 6\phi(x)\phi'(x) \end{pmatrix} \begin{pmatrix} a_0 \\ 0 \end{pmatrix} = 0. \qquad (4.42)$$

Using (4.41), the symmetry of C_+ and $S_+B_+S_+$ and (4.35), the first term of (4.39) becomes

$$\begin{aligned} -\left\langle \left(Y''(x_0) - S_+X''(x_0)\right) h_0, k_0\right\rangle &= -\left\langle C_+B_+S_+k_0, k_0\right\rangle + \left\langle S_+B_+C_+k_0, k_0\right\rangle, \\ &= -\left\langle S_+B_+S_+B_+S_+k_0, k_0\right\rangle + \left\langle S_+B_+S_+B_+S_+k_0, k_0\right\rangle = 0. \end{aligned}$$

For the second term of (4.39), using (4.25), the symmetry of C_+ and $S_+B_+S_+$, (4.35) and (4.36), we obtain

$$\begin{aligned} -2\left\langle \left(Y'(x_0) - S_+X'(x_0)\right) h_1, k_0\right\rangle &= -2\left\langle C_+X_0h_1, k_0\right\rangle + 2\left\langle S_+B_+Y_0h_1, k_0\right\rangle, \\ &= 2\left\langle S_+B_+\left(Y_0 - S_+X_0\right) h_1, k_0\right\rangle = 0. \end{aligned}$$

Hence both terms in (4.39), are zero, that is,

$$\mathfrak{m}^{(2)}_{x_0}(\mathbb{E}^u_+(\cdot,0), \mathbb{S}_+(0)) = 0,$$

and we need to compute the third order crossing form.

The last calculation shows that $W_3 = \ker \mathfrak{m}^{(2)}_{x_0} = W_2 = W_1$. As per Lemma 3.2, assuming h_0 and h_1 are given by (4.32) and (4.38), we need to find a vector $h_2 \in \mathbb{R}^2$ such that $\mathbf{U}''(x_0)h_0 + 2\mathbf{U}'(x_0)h_1 + \mathbf{U}(x_0)h_2 \in \mathbb{S}_+(0)$, that is,

$$Y''(x_0)h_0 + 2Y'(x_0)h_1 + Y_0h_2 = S_+\Big(X''(x_0)h_0 + 2X'(x_0)h_1 + X_0h_2\Big). \qquad (4.43)$$

Rearranging, this becomes

$$\left(Y_0 - S_+X_0\right) h_2 = -\left(Y''(x_0) - S_+X''(x_0)\right) h_0 - 2\left(Y'(x_0) - S_+X'(x_0)\right) h_1. \qquad (4.44)$$

Using (4.41), (4.25), (4.32) and (4.35) in the previous equation yields

$$\left(Y_0 - S_+X_0\right) h_2 = -(C_+ - S_+B_+S_+)B_+S_+k_0 + 2(S_+B_+Y_0 - C_+X_0)h_1. \qquad (4.45)$$

With $w_0 \in W_3 = W_2 = W_1$, and h_0, h_1 and h_2 given by (4.32), (4.38) and (4.45), we are ready to compute the third order form. Using Lemma 3.2, similar to (4.39) we arrive at

$$\begin{aligned} \mathfrak{m}^{(3)}_{x_0}(\mathbb{E}^u_+(\cdot,0), \mathbb{S}_+(0))(w_0) &= \omega(\mathbf{U}'''(x_0)h_0 + 3\mathbf{U}''(x_0)h_1 + 3\mathbf{U}'(x_0)h_2, w_0), \\ &= -\left\langle Y'''(x_0) - S_+X'''(x_0)h_0, k_0\right\rangle - 3\left\langle \left(Y''(x_0) - S_+X''(x_0)\right) h_1, k_0\right\rangle \\ &\quad - 3\left\langle \left(Y'(x_0) - S_+X'(x_0)\right) h_2, k_0\right\rangle. \end{aligned} \qquad (4.46)$$

Differentiating (4.40) yields

$$\begin{aligned} X'''(x) &= B_+C'_+X(x) + B_+C_+B_+Y(x), \\ Y'''(x) &= C''_+X(x) + 2C'_+B_+Y(x) + C_+B_+C_+X(x). \end{aligned} \qquad (4.47)$$

For the first term of (4.46), using (4.47) at x_0, (4.32) and (4.35), as well as that $C''_+k_0 = C'_+k_0 = 0$, we have

$$\begin{aligned} -\left\langle Y'''(x_0) - S_+X'''(x_0)h_0, k_0\right\rangle &= -\left\langle C''_+k_0 + 2C'_+B_+S_+k_0 + C_+B_+C_+k_0, k_0\right\rangle \\ &\quad + \left\langle S_+B_+C_+B_+S_+k_0, k_0\right\rangle, \\ &= \left\langle S_+B_+\left(C_+ - S_+B_+S_+\right) B_+S_+k_0, k_0\right\rangle. \end{aligned} \qquad (4.48)$$

For the second term of (4.46), combining (4.40) at x_0, (4.35) and $C_+' k_0 = 0$, we find that

$$\begin{aligned} -3\left\langle\left(Y''(x_0) - S_+X''(x_0)\right)h_1, k_0\right\rangle &= -3\left\langle C_+B_+Y_0h_1, k_0\right\rangle + 3\left\langle S_+B_+C_+X_0h_1, k_0\right\rangle, \\ &= 3\big\langle S_+B_+\left(C_+X_0 - S_+B_+Y_0\right)h_1, k_0\big\rangle, \\ &= 3\big\langle S_+B_+\left(C_+ - S_+B_+S_+\right)X_0h_1, k_0\big\rangle \\ &\qquad + 3\big\langle S_+B_+S_+B_+\left(S_+X_0 - Y_0\right)h_1, k_0\big\rangle, \\ &= 0. \end{aligned} \tag{4.49}$$

In the last line we used that $h_1 \in \ker\left(Y_0 - S_+X_0\right)$, as well as that $h_1 = \alpha h_0$ for some $\alpha \in \mathbb{R}$ and hence $X_0h_1 = \alpha X_0h_0 = \alpha k_0$, so that

$$\left(C_+ - S_+B_+S_+\right)X_0h_1 = \alpha\left(C_+ - S_+B_+S_+\right)k_0 = 0. \tag{4.50}$$

For the last term of (4.46), using (4.25) and (4.45), we find that

$$\begin{aligned} -3\left\langle\left(Y'(x_0) - S_+X'(x_0)\right)h_2, k_0\right\rangle &= -3\left\langle C_+X_0h_2, k_0\right\rangle + 3\left\langle S_+B_+Y_0h_2, k_0\right\rangle, \\ &= 3\left\langle S_+B_+\left(Y_0 - S_+X_0\right)h_2, k_0\right\rangle, \\ &= -3\left\langle S_+B_+\left(C_+ - S_+B_+S_+\right)B_+S_+k_0, k_0\right\rangle \\ &\qquad - 6\langle S_+B_+\left(C_+X_0 - S_+B_+Y_0\right)h_1, k_0\rangle. \end{aligned} \tag{4.51}$$

The second term of the last line is minus two times the quantity in (4.49), and is therefore zero. Combining the first term with (4.48), we arrive at

$$\mathfrak{m}^{(3)}_{x_0}(\mathbb{E}^u_+(\cdot,0), \mathbb{S}_+(0))(w_0) = -2\left\langle S_+B_+\left(C_+ - S_+B_+S_+\right)B_+S_+k_0, k_0\right\rangle = -\frac{6a_0^2\phi(x_0)^2}{2\sqrt{\beta} - \sigma_2}.$$

It follows from our assumptions (1.10), (1.11) that $2\sqrt{\beta} - \sigma_2 > 0$. Hence, under Hypothesis 4.4, we have $n_-(\mathfrak{m}^{(3)}_{x_0}) = 1 = \dim \mathbb{E}^u_+(x_0, 0) \cap \mathbb{S}_+(0)$, and the crossing is negative.

Case 2: $\dim \mathbb{E}^u_+(x_0, 0) \cap \mathbb{S}_+(0) = 2$.

In this case, $W_1 = \mathbb{E}^u_+(x, 0) = \mathbb{S}_+(0)$, and hence $\mathbf{U}_0$ and $\mathbf{S}_+$ are frames for the same Lagrangian plane. Therefore $\mathbf{U}_0 = \mathbf{S}_+M$ for some 2×2 invertible matrix M, so that $X_0 = M$ and

$$Y_0 = S_+X_0. \tag{4.52}$$

Evaluating the first order form on $W_1 = \mathbb{E}^u_+(x, 0) = \mathbb{S}_+(0)$, we already saw that $n_-(\mathfrak{m}_{x_0}) = 1$. Since $\ker \mathfrak{m}_{x_0} \neq \{0\}$, we again compute higher order crossing forms; the proof is similar to Case 1 with the following changes. For the second order crossing form, we have

$$W_2 = \ker \mathfrak{m}_{x_0} = \operatorname{span}\{\mathbf{s}_1\}. \tag{4.53}$$

Since $Y_0 = S_+X_0$, equation (4.36) indicates that any $h_1 \in \mathbb{R}^2$ will satisfy (4.33). With h_1 free and h_0 given by (4.32), the computation of the second order crossing form is unchanged, owing to (4.52). For the third order crossing form, we'll have $W_3 = W_2$, while using (4.52) in (4.45) shows that

$$(S_+B_+Y_0 - C_+X_0)h_1 = \frac{1}{2}(C_+ - S_+B_+S_+)B_+S_+k_0. \tag{4.54}$$

We conclude that *any* $h_2 \in \mathbb{R}^2$ will satisfy (4.43), *provided* h_1 satisfies (4.54). With such h_0, h_1, h_2, the computation of the third order form is similar; the first term of (4.46) again gives (4.48), the last term of (4.46) vanishes due to (4.52) (see the second line of (4.51)), while for the second term of (4.46) we now have, using (4.54),

$$\begin{aligned} -3\left\langle\left(Y''(x_0) - S_+X''(x_0)\right)h_1, k_0\right\rangle &= 3\big\langle S_+B_+\left(C_+X_0 - S_+B_+Y_0\right)h_1, k_0\big\rangle, \\ &= -\frac{3}{2}\big\langle S_+B_+\left(C_+ - S_+B_+S_+\right)B_+S_+k_0, k_0\big\rangle. \end{aligned} \tag{4.55}$$

Adding (4.48) and (4.55) together, we find

$$\mathfrak{m}_{x_0}^{(3)}(\mathbb{E}_+^u(\cdot,0),\mathbb{S}_+(0))(w_0) = -\frac{1}{2}\left\langle S_+B_+\left(C_+ - S_+B_+S_+\right)B_+S_+k_0,k_0\right\rangle = -\frac{3\phi(x_0)^2a_0^2}{2(2\sqrt{\beta}-\sigma_2)}.$$

Arguing as in Case 1, we conclude that $n_-(\mathfrak{m}_{x_0}^{(3)}) = 1$. Hence, $n_-(\mathfrak{m}_{x_0}^{(1)}) + n_-(\mathfrak{m}_{x_0}^{(3)}) = 2 = \dim \mathbb{E}_+^u(x_0,0)\cap\mathbb{S}_+(0)$, and the crossing is negative.

The proof for the L_- problem is similar, the main difference being the change in sign of the crossing forms. In particular, using the same arguments as those above show that the first order crossing form is given by

$$\mathfrak{m}_{x_0}(\mathbb{E}_-^u(\cdot,0),\mathbb{S}_-(0))(w_0) = -\left\langle (C_- - S_-B_-S_-)k_0,k_0\right\rangle = \phi(x_0)^2b_0^2,$$

the second order form $\mathfrak{m}_{x_0}^{(2)}(\mathbb{E}_-^u(\cdot,0),\mathbb{S}_-(0))(w_0) = 0$, while in the case of a one-dimensional intersection the third order form is given by

$$\mathfrak{m}_{x_0}^{(3)}(\mathbb{E}_-^u(\cdot,0),\mathbb{S}_-(0))(w_0) = -2\left\langle S_-B_-\left(C_- - S_-B_-S_-\right)B_-S_-k_0,k_0\right\rangle = \frac{2a_0^2\phi(x_0)^2}{2\sqrt{\beta}-\sigma_2},$$

and in the two-dimensional case,

$$\mathfrak{m}_{x_0}^{(3)}(\mathbb{E}_-^u(\cdot,0),\mathbb{S}_-(0))(w_0) = -\frac{1}{2}\left\langle S_+B_+\left(C_+ - S_+B_+S_+\right)B_+S_+k_0,k_0\right\rangle = \frac{\phi(x_0)^2a_0^2}{2(2\sqrt{\beta}-\sigma_2)}.$$

We omit the details. Thus, in all cases we have $n_-(\mathfrak{m}_{x_0})+n_-(\mathfrak{m}_{x_0}^{(3)}) = \dim \mathbb{E}_+^u(x_0,0)\cap\mathbb{S}_+(0)$, and the crossings are positive.

If Hypothesis 4.4 fails, i.e. a crossing x_0 is such that $\phi(x_0) = 0$, then the first order crossing form is identically zero, and in Cases 1 and 2 described above, the third order form is also identically zero. Thus, we need to compute higher order crossing forms. The complete proof of monotonicity in this case is cumbersome, and we leave it to the appendix; see Appendix A. □

Remark 4.5. Theorem 4.1 will also hold in the case of any power-law fourth-order NLS equation, i.e. (1.21) for any $p\in\mathbb{N}$. In this case, $L_\pm$ are given by

$$L_- = -\partial_x^4 - \sigma_2\partial_x^2 - \beta + \phi^{2p}, \qquad L_+ = -\partial_x^4 - \sigma_2\partial_x^2 - \beta + (2p+1)\phi^{2p},$$

and the crossing forms $\mathfrak{m}_{x_0}^{(k)}$ for $k = 1,2,3$ will be the same as those in the proof of Theorem 4.1, but scaled by a positive constant, and with $\phi(x_0)^2$ replaced by $\phi(x_0)^{2p}$. The signs are therefore preserved.

Our next task is to show (4.11).

Lemma 4.6. *For ℓ large enough, we have*

$$\operatorname{Mas}(\mathbb{E}_+^u(\cdot,0),\mathbb{E}_+^s(\ell,0);[-\infty,\ell]) = \operatorname{Mas}(\mathbb{E}_+^u(\cdot,0),\mathbb{S}_+(0);[-\infty,\infty]). \tag{4.56}$$

A similar statement holds for the L_- problem.

Proof. We show that the Lagrangian pairs in the left and right hand sides of (4.56) are stratum homotopic. To do so, it will be convenient to compactify $\mathbb{R}$ via the change of variables in Remark 3.7. Thus, defining

$$\widehat{\mathbb{E}}_\pm^{s,u}(\tau,0) := \mathbb{E}_\pm^{s,u}\left(\ln\left(\frac{1+\tau}{1-\tau}\right),0\right), \tag{4.57}$$

(4.56) is equivalent to

$$\operatorname{Mas}(\widehat{\mathbb{E}}_+^u(\cdot,0),\widehat{\mathbb{E}}_+^s(\tau_\ell,0);[-1,\tau_\ell]) = \operatorname{Mas}(\widehat{\mathbb{E}}_+^u(\cdot,0),\widehat{\mathbb{E}}_+^s(1,0);[-1,1]), \tag{4.58}$$

where $\ell = \ln((1+\tau_\ell)/(1-\tau_\ell))$, i.e. $\tau_\ell = (e^\ell - 1)/(e^\ell + 1)$, and we used that $\widehat{\mathbb{E}}^s_+(1,0) = \mathbb{E}^s_+(+\infty, 0) \coloneqq \mathbb{S}_+(0)$ and Rescaling further, we can map $[-1,1]$ to $[-1,\tau_\ell]$ via

$$g(\tau) = \left(\frac{1+\tau_\ell}{2}\right)\tau + \left(\frac{\tau_\ell - 1}{2}\right),$$

where $g(-1) = -1$ and $g(1) = \tau_\ell$. This allows us to write both Lagrangian paths in (4.58) over $[-1,1]$, i.e.

$$\operatorname{Mas}(\widehat{\mathbb{E}}^u_+(g(\cdot),0), \widehat{\mathbb{E}}^s_+(\tau_\ell,0); [-1,1]) = \operatorname{Mas}(\widehat{\mathbb{E}}^u_+(\cdot,0), \widehat{\mathbb{E}}^s_+(1,0); [-1,1]). \tag{4.59}$$

To prove (4.59), we set

$$\Lambda_1(s,\tau) \coloneqq \widehat{\mathbb{E}}^u_+(\tau + (g(\tau)-\tau)s, 0), \qquad \Lambda_2(s,\tau) \coloneqq \widehat{\mathbb{E}}^s_+(1 + (\tau_\ell - 1)s, 0), \tag{4.60}$$

noting that Λ_2 is independent of τ, and that both mappings $(s,\tau) \mapsto \Lambda_{1,2}(s,\tau)$ are continuous on $[0,1]\times[-1,1]$. In addition,

$$\Lambda_1(s,-1) = \widehat{\mathbb{E}}^u_+(-1,0) = \mathbb{U}_+(0), \qquad \Lambda_2(s,-1) = \widehat{\mathbb{E}}^s_+(1+(\tau_\ell-1)s, 0),$$

where we used that $g(-1) = -1$. Since $\mathbb{U}_+(0) \cap \mathbb{E}^s_+(x,0) = \{0\}$ for all $x \geq \ell$ (see (3.37)) and $\mathbb{U}_+(0) \cap \mathbb{S}_+(0) = \{0\}$, we have $\mathbb{U}_+(0) \cap \widehat{\mathbb{E}}^s_+(\tau,0) = \{0\}$ for all $\tau \in [\tau_\ell, 1]$, and hence

$$\Lambda_1(s,-1) \cap \Lambda_2(s,-1) = \{0\}$$

for all $s \in [0,1]$. Furthermore,

$$\Lambda_1(s,1) = \widehat{\mathbb{E}}^u_+(1+(\tau_\ell-1)s,0), \qquad \Lambda_2(s,1) = \widehat{\mathbb{E}}^s_+(1+(\tau_\ell-1)s,0),$$

and therefore

$$\dim \Lambda_1(s,1) \cap \Lambda_2(s,1) = 1$$

for all $s \in [0,1]$ by Hypothesis 1.1. Equation (4.59) (and thus (4.56)) now follows from Lemma 3.6. ☐

Finally, we show that the crossings occurring at the final points of each of the paths in (4.11) (guaranteed by Hypothesis 4.4) have the same contribution to their respective Maslov indices. Hence, we can exclude the final point of each path in (4.11). This will complete the computation of the Maslov index along Γ_1, i.e. the first term of (4.10). We note that some care is needed when dealing with the final crossing of the pair $x \mapsto \big(\mathbb{E}^u_+(x,0), \mathbb{S}_+(0)\big)$, which is obtained in the limit as $x \to +\infty$. Lemma 4.2 therefore does not apply, since the root functions used in the crossing form calculations either blow up to infinity or decay to zero asymptotically.

Lemma 4.7. *For $\varepsilon > 0$ small enough and ℓ large enough, we have*

$$\operatorname{Mas}(\mathbb{E}^u_+(\cdot,0), \mathbb{E}^s_+(\ell,0); [-\infty, \ell-\varepsilon]) = \operatorname{Mas}(\mathbb{E}^u_+(\cdot,0), \mathbb{S}_+(0); [-\infty,\infty)). \tag{4.61}$$

A similar statement holds for the L_- problem.

Proof. By additivity under concatenation (see Proposition 3.4), we can write (4.59) as

$$\begin{aligned}\operatorname{Mas}(\widehat{\mathbb{E}}^u_+(g(\cdot),0), \widehat{\mathbb{E}}^s_+(\tau_\ell,0); [-1,1-\varepsilon]) + \operatorname{Mas}(\widehat{\mathbb{E}}^u_+(g(\cdot),0), \widehat{\mathbb{E}}^s_+(\tau_\ell,0); [1-\varepsilon,1])\\ = \operatorname{Mas}(\widehat{\mathbb{E}}^u_+(\cdot,0), \widehat{\mathbb{E}}^s_+(1,0); [-1,1-\varepsilon]) + \operatorname{Mas}(\widehat{\mathbb{E}}^u_+(\cdot,0), \widehat{\mathbb{E}}^s_+(1,0); [1-\varepsilon,1])\end{aligned} \tag{4.62}$$

for $\varepsilon > 0$ small. Hypothesis 1.1 implies that $\widehat{\mathbb{E}}^u_+(\tau,0) \in \mathcal{T}_1(\widehat{\mathbb{E}}^s_+(\tau,0))$ for $\tau = \tau_\ell, 1$, i.e. that there exists a (one-dimensional) crossing at the final point of each of the paths

$$\tau \mapsto \left(\widehat{\mathbb{E}}^u_+(g(\tau),0), \widehat{\mathbb{E}}^s_+(\tau_\ell,0)\right), \quad \tau \mapsto \left(\widehat{\mathbb{E}}^u_+(\tau,0), \mathbb{S}_+(0)\right), \quad \tau \in [-1,1]. \tag{4.63}$$

Both of these crossings are isolated. Indeed, undoing the scaling by g one sees that the first path $\tau \mapsto \widehat{\mathbb{E}}^u_+(\tau,0)$ is analytic at $\tau_\ell < 1$. On the other hand, isolation of the final crossing of the second path in (4.63) follows from analyticity of $\tau \mapsto \widehat{\mathbb{E}}^u_+(\tau,0)$ for $\tau \in (-1,1)$, monotonicity of $\tau \mapsto \widehat{\mathbb{E}}^u_+(\tau,0)$ with respect to $\mathcal{T}(\mathbb{S}_+(0))$, and the fact that $\widehat{\mathbb{E}}^u_+(\tau,0)$ approaches $\mathbb{S}_+(0)$ as $\tau \to 1^-$ (by assumption). Hence we can choose $\varepsilon > 0$ small enough so that $\tau = 1$ is the *only* crossing in the interval $[1-\varepsilon, 1]$ for the paths in (4.63). With this choice, we now claim that

$$\operatorname{Mas}(\widehat{\mathbb{E}}^u_+(g(\cdot),0), \widehat{\mathbb{E}}^s_+(\tau_\ell,0); [1-\varepsilon,1]) = \operatorname{Mas}(\widehat{\mathbb{E}}^u_+(\cdot,0), \widehat{\mathbb{E}}^s_+(1,0); [1-\varepsilon,1]) = 0, \tag{4.64}$$

i.e. that the conjugate points occurring at the final points of each of the paths in (4.63) do not contribute to their respective Maslov indices. Assuming the claim, by (4.62) we have

$$\operatorname{Mas}(\widehat{\mathbb{E}}^u_+(g(\cdot),0), \widehat{\mathbb{E}}^s_+(\tau_\ell,0); [-1,1-\varepsilon]) = \operatorname{Mas}(\widehat{\mathbb{E}}^u_+(\cdot,0), \widehat{\mathbb{E}}^s_+(1,0); [-1,1-\varepsilon]).$$

Recalling Remark 4.3, this is exactly (4.61) (for a different but still arbitrarily small ε).

It remains to prove (4.64). To that end, note that the paths $\tau \mapsto \widehat{\mathbb{E}}^u_+(g(\tau),0)$ and $\tau \mapsto \widehat{\mathbb{E}}^u_+(\tau,0)$ are arbitrarily close to one another: for any given $\delta > 0$ small, we can choose $\ell = \ln((2-\delta)/\delta)$ so that $\tau_\ell = 1-\delta$, in which case

$$|g(\tau) - \tau| = \left(\frac{1-\tau_\ell}{2}\right)(\tau+1) \le \delta,$$

uniformly for $\tau \in [-1,1]$. In addition, for large enough ℓ the trains $\mathcal{T}(\widehat{\mathbb{E}}^s_+(\tau_\ell,0))$ and $\mathcal{T}(\mathbb{S}_+(0))$ are arbitrarily small perturbations of one another. Since the final crossings of (4.63) are both one-dimensional by Hypothesis 1.1, we conclude that the curves $\tau \mapsto \widehat{\mathbb{E}}^u_+(g(\tau),0)$ and $\tau \mapsto \widehat{\mathbb{E}}^u_+(\tau,0)$ approach $\mathcal{T}_1(\widehat{\mathbb{E}}^s_+(\tau_\ell,0))$ and $\mathcal{T}_1(\mathbb{S}_+(0))$, respectively, from the same direction as $\tau \to 1^-$. This proves the first equality in (4.64). It follows from Lemma 4.2 and Remark 3.5 that the crossing at $\tau_\ell < 1$ of the path $\tau \mapsto \left(\widehat{\mathbb{E}}^u_+(\tau,0), \widehat{\mathbb{E}}^s_+(\tau_\ell,0)\right)$ is negative, i.e. the crossing at $\tau = 1$ of the first path in (4.63) is negative. In line with Definition 3.3, if the final crossing is one-dimensional and transverse, its contribution to the Maslov index is $+1$ if the path arrives at the train in the positive direction, and zero otherwise. Hence $\operatorname{Mas}(\widehat{\mathbb{E}}^u_+(g(\cdot),0), \widehat{\mathbb{E}}^s_+(\tau_\ell,0); [1-\varepsilon,1]) = 0$, and (4.64) follows. The proof for the L_- problem is similar. ☐

4.2. **Computing the Maslov index along Γ_2.** In the following we prove monotonicity of the paths $\lambda \mapsto (\mathbb{E}^u_\pm(\ell,\lambda), \mathbb{E}^s_\pm(\ell,\lambda))$.

Lemma 4.8. *Each crossing of the path of Lagrangian pairs $\lambda \mapsto (\mathbb{E}^u_+(\ell,\lambda), \mathbb{E}^s_+(\ell,\lambda))$ is positive. Thus,*

$$\operatorname{Mas}(\mathbb{E}^u_+(\ell,\cdot), \mathbb{E}^s_+(\ell,\cdot); [\varepsilon,\lambda_\infty]) = P \tag{4.65}$$

for $\varepsilon > 0$ small enough. Similarly, each crossing of the path $\lambda \mapsto (\mathbb{E}^u_-(\ell,\lambda), \mathbb{E}^s_-(\ell,\lambda))$ is negative, and we have

$$\operatorname{Mas}(\mathbb{E}^u_-(\ell,\cdot), \mathbb{E}^s_-(\ell,\cdot); [\varepsilon,\lambda_\infty]) = -Q. \tag{4.66}$$

Proof. We begin with the statements pertaining to L_+. We proceed by computing the (first-order) relative crossing form (3.28) at each crossing $\lambda = \lambda_0$, given here by

$$\mathfrak{m}_{\lambda_0}(\mathbb{E}^u_+(\ell,\cdot), \mathbb{E}^s_+(\ell,\cdot))(w_0) = \frac{d}{d\lambda}\omega(p(\lambda),w_0)\big|_{\lambda=\lambda_0} - \frac{d}{d\lambda}\omega(q(\lambda),w_0)\big|_{\lambda=\lambda_0}, \tag{4.67}$$

where $w_0 \in W_1(\mathbb{E}^u_+(\ell,\cdot), \mathbb{E}^s_+(\ell,\cdot), \lambda_0) = \mathbb{E}^u_+(\ell,\lambda_0) \cap \mathbb{E}^s_+(\ell,\lambda_0)$ and (p,q) is a root function pair for $\left(\mathbb{E}^u_+(\ell,\cdot), \mathbb{E}^s_+(\ell,\cdot)\right)$ with $p(\lambda_0) = q(\lambda_0) = w_0$. We compute each of the terms on the right hand side separately.

For the first, recall that $p(\lambda) \in \mathbb{E}^u_+(\ell,\lambda)$ for $\lambda \in [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]$ with $\varepsilon > 0$ small. From the definition of $\mathbb{E}^u_+(\ell,\lambda)$, it follows that there exists a one-parameter family of solutions

$\lambda \mapsto \mathbf{u}(\cdot;\lambda)$ to (4.3) satisfying $\mathbf{u}(x;\lambda) \to 0$ as $x \to -\infty$, such that $\mathbf{u}(\ell;\lambda) = p(\lambda)$ and $\mathbf{u}(\ell;\lambda_0) = w_0$. Now

$$\begin{aligned}
\omega\Big(\frac{d}{d\lambda}\mathbf{u}(\ell,\lambda),\mathbf{u}(\ell,\lambda)\Big) &= \int_{-\infty}^{\ell} \partial_x\, \omega(\partial_\lambda \mathbf{u}(x;\lambda),\mathbf{u}(x;\lambda))\,dx,\\
&= \int_{-\infty}^{\ell} \omega\left(\partial_\lambda\big[A_+(x;\lambda)\mathbf{u}(x;\lambda)\big],\mathbf{u}(x;\lambda)\right) + \omega(\partial_\lambda \mathbf{u}(x;\lambda), A_+(x;\lambda)\mathbf{u}(x;\lambda))\,dx,\\
&= \int_{-\infty}^{\ell} \omega(\partial_\lambda\left(A_+(x;\lambda)\right)\mathbf{u}(x;\lambda),\mathbf{u}(x;\lambda)) + \omega(A_+(x;\lambda)\partial_\lambda \mathbf{u}(x;\lambda),\mathbf{u}(x;\lambda))\\
&\qquad\qquad + \omega(\partial_\lambda \mathbf{u}(x;\lambda), A_+(x;\lambda)\mathbf{u}(x;\lambda))\,dx,\\
&= \int_{-\infty}^{\ell} \omega(\partial_\lambda\left(A_+(x;\lambda)\right)\mathbf{u}(x;\lambda),\mathbf{u}(x;\lambda))\\
&\qquad\qquad + \big\langle [A_+(x;\lambda)^\top J + JA_+(x;\lambda)]\partial_\lambda \mathbf{u}(x;\lambda),\mathbf{u}(x;\lambda)\big\rangle\,dx,\\
&= \int_{-\infty}^{\ell} \omega(\partial_\lambda\left(A_+(x;\lambda)\right)\mathbf{u}(x;\lambda),\mathbf{u}(x;\lambda))\,dx,
\end{aligned} \tag{4.68}$$

where we used that $\lim_{x\to-\infty}\mathbf{u}(x;\lambda) = 0$ in the first line and (2.14) in the last line. Since

$$\partial_\lambda A_+(x;\lambda) = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 \end{pmatrix}, \tag{4.69}$$

and $\mathbf{u} = (u_1,u_2,u_3,u_4)^\top$, evaluating the last line of (4.68) at $\lambda = \lambda_0$ we have

$$\frac{d}{d\lambda}\omega(p(\lambda),w_0)\big|_{\lambda=\lambda_0} = \omega\Big(\frac{d}{d\lambda}\mathbf{u}(\ell,\lambda),\mathbf{u}(\ell,\lambda)\Big)\big|_{\lambda=\lambda_0} = \int_{-\infty}^{\ell} u_2(x;\lambda_0)^2\,dx. \tag{4.70}$$

For the second term of (4.67), we consider $q(\lambda) \in \mathbb{E}_+^s(\ell,\lambda)$ for $\lambda \in [\lambda_0-\varepsilon,\lambda_0+\varepsilon]$ with $\varepsilon > 0$ small. For the same w_0 used to compute the first term of (4.67), there exists a one-parameter family of solutions $\lambda \mapsto \widetilde{\mathbf{u}}(\cdot;\lambda)$ to (4.3) satisfying $\widetilde{\mathbf{u}}(x;\lambda) \to 0$ as $x \to +\infty$, such that $\widetilde{\mathbf{u}}(\ell;\lambda) = q(\lambda)$ and $\widetilde{\mathbf{u}}(\ell;\lambda_0) = w_0$. Arguing as previously, but now using the decay at $+\infty$, we have

$$\frac{d}{d\lambda}\omega(q(\lambda),w_0)\big|_{\lambda=\lambda_0} = \omega\Big(\frac{d}{d\lambda}\widetilde{\mathbf{u}}(\ell;\lambda),\widetilde{\mathbf{u}}(\ell;\lambda)\Big)\big|_{\lambda=\lambda_0} = -\int_{\ell}^{\infty} \widetilde{u}_2(x;\lambda_0)^2\,dx \tag{4.71}$$

(where $\widetilde{\mathbf{u}} = (\widetilde{u}_1,\widetilde{u}_2,\widetilde{u}_3,\widetilde{u}_4)^\top$). Importantly, by uniqueness of solutions we have $\widetilde{\mathbf{u}}(\cdot;\lambda_0) = \mathbf{u}(\cdot;\lambda_0)$, so that the integrands in (4.71) and (4.70) are the same. Therefore, (4.67) becomes

$$\mathfrak{m}_{\lambda_0}(\mathbb{E}_+^u(\ell,\cdot),\mathbb{E}_+^s(\ell,\cdot))(w_0) = \int_{-\infty}^{\infty} u_2(x;\lambda_0)^2\,dx > 0. \tag{4.72}$$

Thus $n_+(\mathfrak{m}_{\lambda_0}) = \dim \mathbb{E}_+^u(\ell,\lambda_0)\cap\mathbb{E}_+^s(\ell,\lambda_0)$, and all crossings are positive. It follows that the Maslov index counts the number of crossings (up to dimension) of the Lagrangian pair $\lambda \mapsto (\mathbb{E}_+^u(\ell,\lambda),\mathbb{E}_+^s(\ell,\lambda))$, $\lambda \in [\varepsilon,\lambda_\infty]$, for $\varepsilon > 0$ small enough. But this is precisely a count of the number of positive eigenvalues of L_+ up to multiplicity, i.e. equation (4.65) holds.

For the path $\lambda \mapsto (\mathbb{E}_-^u(\ell,\lambda),\mathbb{E}_-^s(\ell,\lambda))$, $\lambda \in [0,\lambda_\infty]$ the argument is similar, where now the Maslov index counts, with negative sign, the number of crossings along Γ_2. The sign change results from the fact that λ now appears with positive sign in the first order system (4.6), so that

$$\partial_\lambda A_-(x;\lambda) = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{4.73}$$

The associated crossing form will then be negative, and by similar reasoning equation (4.66) holds. ☐

4.3. **Computing the Maslov index along Γ_3 and Γ_4.** The following lemma shows that there are no crossings along Γ_3 and Γ_4.

Lemma 4.9. *We have $\mathbb{E}^u_+(x,\lambda_\infty)\cap\mathbb{E}^s_+(\ell,\lambda_\infty)=\{0\}$ for all $x\in\mathbb{R}$, provided both $\lambda_\infty>0$ and $\ell>0$ are large enough. In addition, $\mathbb{U}_+(\lambda)\cap\mathbb{E}^s_+(\ell,\lambda)=\{0\}$ for all $\lambda\geq 0$ provided $\ell>0$ is large enough. Therefore*

$$\operatorname{Mas}(\mathbb{E}^u_+(\cdot,\lambda_\infty),\mathbb{E}^s_+(\ell,\lambda_\infty);[-\infty,\ell])=\operatorname{Mas}(\mathbb{U}_+(\cdot),\mathbb{E}^s_+(\ell,\cdot);[0,\lambda_\infty])=0. \tag{4.74}$$

Similar statements hold for the paths $x\mapsto\big(\mathbb{E}^u_-(x,\lambda_\infty),\mathbb{E}^s_-(\ell,\lambda_\infty)\big)$ and $\lambda\mapsto\big(\mathbb{U}_+(\lambda),\mathbb{E}^s_+(\ell,\lambda)\big)$.

Proof. The strategy of the following proof mirrors the one given in [Cor19, §4] (see also [AGJ90, §3 and §5.B]).

For the first statement, we begin by noting that $\operatorname{Spec}(L_+)$ is bounded from above. To see this, note that we can write

$$L_+=\mathcal{A}+\mathcal{V},\qquad \mathcal{A}=-\partial_{xxxx}-\sigma_2\partial_{xx},\quad \mathcal{V}=-\beta+3\phi(x)^2, \tag{4.75}$$

where $\operatorname{dom}(\mathcal{A})=H^4(\mathbb{R})$, so that $\mathcal{A}=\mathcal{A}^*$ is selfadjoint and $\mathcal{V}$ is bounded and symmetric on $L^2(\mathbb{R})$. It can be shown that $\mathcal{A}$ has no point spectrum, and moreover that $\operatorname{Spec}(\mathcal{A})=\operatorname{Spec}_{\rm ess}(\mathcal{A})=(-\infty,1/4]$ if $\sigma_2=1$, and $\operatorname{Spec}(\mathcal{A})=\operatorname{Spec}_{\rm ess}(\mathcal{A})=(-\infty,0]$ if $\sigma=-1$. Consequently, by virtue of [Kat80, Theorem V.4.10, p.291] we have

$$\operatorname{dist}\left(\operatorname{Spec}(L_+),\operatorname{Spec}(\mathcal{A})\right)\leq\|\mathcal{V}\|, \tag{4.76}$$

so that $\operatorname{Spec}(L_+)\subseteq(-\infty,\|\mathcal{V}\|]$. It then follows that $\mathbb{E}^u_+(\ell,\lambda)\cap\mathbb{E}^s_+(\ell,\lambda)=\{0\}$ for all $\lambda>\|\mathcal{V}\|$.

Next, we claim that there exists a $\lambda_\infty>\|\mathcal{V}\|$ such that

$$\mathbb{E}^u_+(x,\lambda)\cap\mathbb{S}_+(\lambda)=\{0\} \tag{4.77}$$

for all $x\in\mathbb{R}$ and all $\lambda\geq\lambda_\infty$. Once this is shown, it follows that there exists an $\ell_\infty\gg 1$ such that

$$\mathbb{E}^u_+(x,\lambda_\infty)\cap\mathbb{E}^s_+(\ell,\lambda_\infty)=\{0\} \tag{4.78}$$

for all $x\in\mathbb{R}$ and all $\ell\geq\ell_\infty$, because $\lim_{x\to\infty}\mathbb{E}^s_+(x,\lambda)=\mathbb{S}_+(\lambda)$. It remains to prove the claim. We mimic the proof of [Cor19, Lemma 4.1]. Consider then the change of variables:

$$y=\lambda^{1/4}x,\qquad \widetilde{u}_1=u_1,\quad \widetilde{u}_2=\lambda^{1/2}u_2,\quad \widetilde{u}_3=\lambda^{1/4}u_3,\quad \widetilde{u}_4=\lambda^{-1/4}u_4, \tag{4.79}$$

under which the system (4.3) becomes

$$\frac{d}{dy}\begin{pmatrix}\widetilde{u}_1\\\widetilde{u}_2\\\widetilde{u}_3\\\widetilde{u}_4\end{pmatrix}=\begin{pmatrix}0&0&\frac{\sigma_2}{\sqrt{\lambda}}&1\\0&0&1&0\\1&-\frac{\sigma_2}{\sqrt{\lambda}}&0&0\\-\frac{\sigma_2}{\sqrt{\lambda}}&\frac{\alpha\left(\frac{y}{\sqrt[4]{\lambda}}\right)}{\lambda}-1&0&0\end{pmatrix}\begin{pmatrix}\widetilde{u}_1\\\widetilde{u}_2\\\widetilde{u}_3\\\widetilde{u}_4\end{pmatrix} \tag{4.80}$$

(recall that $\alpha\left(\frac{y}{\sqrt[4]{\lambda}}\right)=3\phi(\frac{y}{\sqrt[4]{\lambda}})^2-\beta+1$). Taking $y\to\pm\infty$, the asymptotic system for (4.80) is given by

$$\frac{d}{dy}\begin{pmatrix}\widetilde{u}_1\\\widetilde{u}_2\\\widetilde{u}_3\\\widetilde{u}_4\end{pmatrix}=\begin{pmatrix}0&0&\frac{\sigma_2}{\sqrt{\lambda}}&1\\0&0&1&0\\1&-\frac{\sigma_2}{\sqrt{\lambda}}&0&0\\-\frac{\sigma_2}{\sqrt{\lambda}}&\frac{-\beta+1}{\lambda}-1&0&0\end{pmatrix}\begin{pmatrix}\widetilde{u}_1\\\widetilde{u}_2\\\widetilde{u}_3\\\widetilde{u}_4\end{pmatrix}. \tag{4.81}$$

Denote the stable and unstable subspaces for (4.81) by $\widetilde{\mathbb{S}}_+(\lambda)$ and $\widetilde{\mathbb{U}}_+(\lambda)$ respectively, and denote the unstable bundle of (4.80) by $\widetilde{\mathbb{E}}^u_+(y,\lambda)$. Then, we have

$$\mathbb{E}^u_+(x,\lambda)\cap\mathbb{S}_+(\lambda)=\{0\} \iff \widetilde{\mathbb{E}}^u_+(\lambda^{1/4}x,\lambda)\cap\widetilde{\mathbb{S}}_+(\lambda)=\{0\}, \tag{4.82}$$

since from (4.79) one has $\widetilde{\mathbb{E}}^u_+(\lambda^{1/4}x,\lambda)=M\cdot\mathbb{E}^u(x,\lambda)$ and $\widetilde{\mathbb{S}}_+(\lambda)=M\cdot\mathbb{S}_+(\lambda)$, where $M=\operatorname{diag}\{1,\lambda^{1/2},\lambda^{1/4},\lambda^{-1/4}\}$ is nonsingular and " $\cdot$ " is the induced action of M on $\mathbb{R}^4$.

Both the nonautonomous system (4.80) and the autonomous system (4.81) induce flows on $\mathrm{Gr}_2(\mathbb{R}^4)$, the Grassmannian of two dimensional subspaces of $\mathbb{R}^4$. For the flow associated with (4.81), it is known [AGJ90] that $\widetilde{\mathbb{U}}_+(\lambda)$, the invariant subspace associated with eigenvalues of positive real part, is an attracting fixed point. Thus, since $\mathcal{L}(2)\subset\mathrm{Gr}_2(\mathbb{R}^4)$, there exists a trapping region $\mathcal{R}\subset\mathcal{L}(2)$ containing $\widetilde{\mathbb{U}}_+(\lambda)$. By taking λ large enough, we can ensure that the flow induced by (4.80) is as close as we like to that induced by (4.81), because $\phi\left(\frac{y}{\sqrt[4]{\lambda}}\right)^2/\lambda$ – the nonautonomous part of (4.80) – is close to zero. It follows that $\mathcal{R}\subset\mathcal{L}(2)$ is also a trapping region for (4.80). Furthermore, we can choose $\mathcal{R}$ small enough such that $\mathbb{V}\cap\widetilde{\mathbb{S}}_+(\lambda)=\{0\}$ for all $\mathbb{V}\in\mathcal{R}$, uniformly for λ large enough. To see this, note that clearly $\widetilde{\mathbb{S}}_+(\lambda)\cap\widetilde{\mathbb{U}}_+(\lambda)=\{0\}$, while taking $\lambda\to+\infty$ in (4.81) yields

$$\frac{d}{dy}\begin{pmatrix}\widetilde{u}_1\\ \widetilde{u}_2\\ \widetilde{u}_3\\ \widetilde{u}_4\end{pmatrix}=\begin{pmatrix}0&0&0&1\\0&0&1&0\\1&0&0&0\\0&-1&0&0\end{pmatrix}\begin{pmatrix}\widetilde{u}_1\\ \widetilde{u}_2\\ \widetilde{u}_3\\ \widetilde{u}_4\end{pmatrix}, \tag{4.83}$$

which has stable and unstable subspaces $\widetilde{\mathbb{S}}_{+\infty}$ and $\widetilde{\mathbb{U}}_{+\infty}$ with respective frames $(I,-W)$ and (I,W), where

$$W=\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}.$$

Thus, in the limit we also have $\widetilde{\mathbb{S}}_{+\infty}\cap\widetilde{\mathbb{U}}_{+\infty}=\{0\}$, so we can choose $\mathcal{R}$ as stated. Finally, we note that if $\lambda>\|\mathcal{V}\|$ so that $\lambda\notin\mathrm{Spec}(L_+)$, then by [AGJ90, Lemma 3.7] we have $\lim_{y\to\infty}\widetilde{\mathbb{E}}^u_+(y,\lambda)=\widetilde{\mathbb{U}}_+(\lambda)$. All in all, we conclude that for any $\lambda=\lambda_\infty>\|\mathcal{V}\|$ large enough, the trajectory $\widetilde{\mathbb{E}}^u_+(\cdot,\lambda_\infty):[-\infty,\infty]\to\mathcal{L}(2)$, which starts and finishes at $\widetilde{\mathbb{U}}_+(\lambda_\infty)$, will remain inside $\mathcal{R}$ and thus always be disjoint from $\widetilde{\mathbb{S}}_+(\lambda_\infty)$. This proves the claim.

For the second statement of the lemma, the facts that $\mathbb{U}_+(\lambda)\cap\mathbb{S}_+(\lambda)=\{0\}$ and $\lim_{x\to\infty}\mathbb{E}^s_+(x,\lambda)=\mathbb{S}_+(\lambda)$ imply that there exists an $\ell_0\gg1$ such that $\mathbb{U}_+(\lambda)\cap\mathbb{E}^s_+(x,\lambda)=\{0\}$ for all $x\geq\ell_0$. Taking $\ell>\ell_0$ gives the result. □

4.4. **Proof of Theorem 4.1.** In what follows, we choose $\ell>0$ and $\lambda_\infty>0$ large enough so that the statements of Lemma 4.9 hold.

Proof of Theorem 4.1. By homotopy invariance and additivity under concatenation, we have

$$\begin{aligned}&\mathrm{Mas}(\mathbb{E}^u_+(\cdot,0),\mathbb{E}^s_+(\ell,0);[-\infty,\ell])+\mathrm{Mas}(\mathbb{E}^u_+(\ell,\cdot),\mathbb{E}^s_+(\ell,\cdot);[0,\lambda_\infty])\\&\quad-\mathrm{Mas}(\mathbb{E}^u_+(\cdot,\lambda_\infty),\mathbb{E}^s_+(\ell,\lambda_\infty);[-\infty,\ell])-\mathrm{Mas}(\mathbb{E}^u_+(-\infty,\cdot),\mathbb{E}^s_+(\ell,\cdot);[0,\lambda_\infty])=0.\end{aligned} \tag{4.84}$$

From Lemma 4.9 the third and fourth terms on the left hand side vanish. Again using the concatenation property, we find that

$$\begin{aligned}&\mathrm{Mas}(\mathbb{E}^u_+(\cdot,0),\mathbb{E}^s_+(\ell,0);[-\infty,\ell-\varepsilon])+\mathrm{Mas}(\mathbb{E}^u_+(\cdot,0),\mathbb{E}^s_+(\ell,0);[\ell-\varepsilon,\ell])\\&\quad+\mathrm{Mas}(\mathbb{E}^u_+(\ell,\cdot),\mathbb{E}^s_+(\ell,\cdot);[0,\varepsilon])+\mathrm{Mas}(\mathbb{E}^u_+(\ell,\cdot),\mathbb{E}^s_+(\ell,\cdot);[\varepsilon,\lambda_\infty])=0\end{aligned} \tag{4.85}$$

where $\varepsilon > 0$ is small. The second and third terms of (4.85) represent the contributions to the Maslov index from the conjugate point $(x, \lambda) = (\ell, 0)$ at the top left corner of the Maslov box in the x and λ directions respectively. From (4.64), Lemma 4.8 and Definition 3.3 we have

$$\mathrm{Mas}(\mathbb{E}^u_+(\cdot, 0), \mathbb{E}^s_+(\ell, 0); [\ell - \varepsilon, \ell]) = \mathrm{Mas}(\mathbb{E}^u_+(\ell, \cdot), \mathbb{E}^s_+(\ell, \cdot); [0, \varepsilon]) = 0. \tag{4.86}$$

Lemmas 4.2 and 4.7 imply that

$$\mathrm{Mas}(\mathbb{E}^u_+(\cdot, 0), \mathbb{E}^s_+(\ell, 0); [-\infty, \ell - \varepsilon]) = -\sum_{x \in \mathbb{R}} \dim\left(\mathbb{E}^u_+(x, 0) \cap \mathbb{S}_+(0)\right). \tag{4.87}$$

The previous three equations along with Lemma 4.8 now yield (4.1).

The proof for the Morse index of the L_- operator is similar. This time, crossings along Γ_1 are positive, while crossings along Γ_2 are negative. Arguing as we did for (4.64), we have

$$\mathrm{Mas}(\mathbb{E}^u_-(\cdot, 0), \mathbb{E}^s_-(\ell, 0); [\ell - \varepsilon, \ell]) = 1, \tag{4.88}$$

and from Lemma 4.8 and Definition 3.3 we have

$$\mathrm{Mas}(\mathbb{E}^u_-(\ell, \cdot), \mathbb{E}^s_-(\ell, \cdot); [0, \varepsilon]) = -1. \tag{4.89}$$

The contributions (4.88) and (4.89) coming from the corner crossing $(x, \lambda) = (\ell, 0)$ thus cancel each other out. Applying the same homotopy argument as we did for L_+ yields the formula for Q in the proposition. □

5. Proofs of the main results

We now return to the computation of the Maslov indices appearing on the left hand side of (3.45). After computing each, we provide the proofs of Theorems 1.2 and 1.6. We begin with Γ_1.

Lemma 5.1. $\mathrm{Mas}(\mathbb{E}^u(\cdot, 0), \mathbb{E}^s(\ell, 0); [-\infty, \ell - \varepsilon]) = Q - P$, *where* $\varepsilon > 0$ *is small.*

Proof. Recall that when $\lambda = 0$ the eigenvalue equations (1.13) decouple. Consequently, the equations for the u and v components in the first order system (2.2) also decouple. Hence, for each $x \in \mathbb{R}$,

$$\mathbb{E}^u(x, 0) = \mathbb{E}^u_+(x, 0) \oplus \mathbb{E}^u_-(x, 0), \tag{5.1}$$

in the sense that for any $\mathbf{w} \in \mathbb{E}^u(x, 0)$ we have

$$\mathbf{w} = \begin{pmatrix} u_1 \\ 0 \\ u_2 \\ 0 \\ u_3 \\ 0 \\ u_4 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ v_1 \\ 0 \\ v_2 \\ 0 \\ v_3 \\ 0 \\ v_4 \end{pmatrix}, \tag{5.2}$$

where $\mathbf{u} = (u_1, u_2, u_3, u_4)^\top \in \mathbb{E}^u_+(x, 0)$ and $\mathbf{v} = (v_1, v_2, v_3, v_4)^\top \in \mathbb{E}^u_-(x, 0)$. By the same reasoning, for the reference plane we have

$$\mathbb{E}^s(\ell, 0) = \mathbb{E}^s_+(\ell, 0) \oplus \mathbb{E}^s_-(\ell, 0). \tag{5.3}$$

Now using property (3) of Proposition 3.4, we have

$$\begin{aligned} \mathrm{Mas}(\mathbb{E}^u(\cdot, 0), \mathbb{E}^s(\ell, 0); [-\infty, \ell - \varepsilon]) &= \mathrm{Mas}(\mathbb{E}^u_+(\cdot, 0), \mathbb{E}^s_+(\ell, 0); [-\infty, \ell - \varepsilon]) \\ &\quad + \mathrm{Mas}(\mathbb{E}^u_-(\cdot, 0), \mathbb{E}^s_-(\ell, 0); [-\infty, \ell - \varepsilon]), \end{aligned} \tag{5.4}$$

and the result follows combining equations (4.87) and (4.1) (and the accompanying statements for L_-). ☐

Next, we show that there are no crossings along Γ_3 and Γ_4.

Lemma 5.2. *There exists $\ell_1 \gg 1$ such that $\mathbb{E}^u(x, \lambda_\infty) \cap \mathbb{E}^s(\ell, \lambda_\infty) = \{0\}$ for all $x \in \mathbb{R}$ and all $\ell \geq \ell_1$, provided $\lambda_\infty > 0$ is large enough. Therefore, for all $\ell \geq \ell_1$,*

$$\mathrm{Mas}(\mathbb{E}^u(\cdot, \lambda_\infty), \mathbb{E}^s(\ell, \lambda_\infty); [-\infty, \ell]) = 0.$$

In addition, $\mathbb{U}(\lambda) \cap \mathbb{E}^s(\ell, \lambda) = \{0\}$ for all $\lambda \geq 0$ provided $\ell > 0$ is large enough. Consequently,

$$\mathrm{Mas}(\mathbb{U}(\cdot), \mathbb{E}^s(\ell, \cdot); [0, \lambda_\infty]) = 0.$$

Proof. For the first assertion, note that N is a bounded perturbation of a skew-selfadjoint operator, so that its spectrum lies in a vertical strip around the imaginary axis in the complex plane. More precisely, we have that

$$iN = \widetilde{\mathcal{A}} + \widetilde{\mathcal{V}}, \quad \widetilde{\mathcal{A}} = i \begin{pmatrix} 0 & \partial_{xxxx} + \sigma_2 \partial_{xx} \\ -\partial_{xxxx} - \sigma_2 \partial_{xx} & 0 \end{pmatrix}, \quad \widetilde{\mathcal{V}} = i \begin{pmatrix} 0 & \beta - \phi^2 \\ -\beta + 3\phi^2 & 0, \end{pmatrix} \tag{5.5}$$

where, with $\mathrm{dom}(\widetilde{\mathcal{A}}) = \mathrm{dom}(N)$, $\widetilde{\mathcal{A}}^* = \widetilde{\mathcal{A}}$ is selfadjoint in $L^2(\mathbb{R})$ and $\widetilde{\mathcal{V}}$ is bounded. Now using [Kat80, Remark 3.2, p.208] and [Kat80, eq. (3.16), p.272], we conclude that

$$\zeta \in \mathrm{Spec}(\widetilde{\mathcal{A}} + \widetilde{\mathcal{V}}) \implies |\mathrm{Im}(\zeta)| \leq \|\widetilde{\mathcal{V}}\|. \tag{5.6}$$

By the spectral mapping theorem, $\mathrm{Spec}(iN) = i\,\mathrm{Spec}(N)$. It follows that

$$\lambda \in \mathrm{Spec}(N) \implies |\mathrm{Re}(\lambda)| \leq \|\widetilde{\mathcal{V}}\|. \tag{5.7}$$

Thus, for all $\lambda > \|\widetilde{\mathcal{V}}\|$ we have $\mathbb{E}^u(\ell, \lambda) \cap \mathbb{E}^s(\ell, \lambda) = \{0\}$.

The proof now follows from the same arguments used to prove the first assertion in Lemma 4.9. Namely, via the change of variables (4.79) along with

$$\widetilde{v}_1 = v_1, \quad \widetilde{v}_2 = \lambda^{1/2} v_2, \quad \widetilde{v}_3 = \lambda^{1/4} v_3, \quad \widetilde{v}_4 = \lambda^{-1/4} v_4 \tag{5.8}$$

we can rewrite (2.2) as

$$\frac{d}{dy} \begin{pmatrix} \widetilde{u}_1 \\ \widetilde{v}_1 \\ \widetilde{u}_2 \\ \widetilde{v}_2 \\ \widetilde{u}_3 \\ \widetilde{v}_3 \\ \widetilde{u}_4 \\ \widetilde{v}_4 \end{pmatrix} = \left(\begin{array}{cccc|cccc} & & & & \frac{\sigma_2}{\sqrt{\lambda}} & 0 & 1 & 0 \\ & 0 & & & 0 & -\frac{\sigma_2}{\sqrt{\lambda}} & 0 & 1 \\ & & & & 1 & 0 & 0 & 0 \\ & & & & 0 & 1 & 0 & 0 \\ \hline 1 & 0 & -\frac{\sigma_2}{\sqrt{\lambda}} & 0 & & & & \\ 0 & -1 & 0 & -\frac{\sigma_2}{\sqrt{\lambda}} & & & & \\ -\frac{\sigma_2}{\sqrt{\lambda}} & 0 & \frac{\alpha(x)}{\lambda} & 1 & & 0 & & \\ 0 & -\frac{\sigma_2}{\sqrt{\lambda}} & 1 & \frac{\eta(x)}{\lambda} & & & & \end{array}\right) \begin{pmatrix} \widetilde{u}_1 \\ \widetilde{v}_1 \\ \widetilde{u}_2 \\ \widetilde{v}_2 \\ \widetilde{u}_3 \\ \widetilde{v}_3 \\ \widetilde{u}_4 \\ \widetilde{v}_4 \end{pmatrix}. \tag{5.9}$$

Again, the flow of the associated asymptotic system is close to that of (5.9) for large λ. From the transversality of the four dimensional stable and unstable subspaces of the limiting system of (5.9) as $\lambda \to \infty$, i.e.

$$\frac{d}{dy}\begin{pmatrix}\widetilde{u}_1\\ \widetilde{v}_1\\ \widetilde{u}_2\\ \widetilde{v}_2\\ \widetilde{u}_3\\ \widetilde{v}_3\\ \widetilde{u}_4\\ \widetilde{v}_4\end{pmatrix} = \left(\begin{array}{cccc|cccc} & & & & 0 & 0 & 1 & 0\\ & 0 & & & 0 & 0 & 0 & 1\\ & & & & 1 & 0 & 0 & 0\\ & & & & 0 & 1 & 0 & 0\\ \hline 1 & 0 & 0 & 0 & & & & \\ 0 & -1 & 0 & 0 & & 0 & & \\ 0 & 0 & 0 & 1 & & & & \\ 0 & 0 & 1 & 0 & & & & \end{array}\right)\begin{pmatrix}\widetilde{u}_1\\ \widetilde{v}_1\\ \widetilde{u}_2\\ \widetilde{v}_2\\ \widetilde{u}_3\\ \widetilde{v}_3\\ \widetilde{u}_4\\ \widetilde{v}_4\end{pmatrix}, \tag{5.10}$$

one can show that there exists a $\lambda_\infty > \|\widetilde{\mathcal{V}}\|$ such that $\mathbb{E}^u(x,\lambda)$ and $\mathbb{S}(\lambda)$ are transverse for all $x \in \mathbb{R}$ and all $\lambda \geq \lambda_\infty$. Hence $\mathbb{E}^u(x,\lambda)$ and $\mathbb{E}^s(\ell,\lambda_\infty)$ are transverse for all $x \in \mathbb{R}, \ell \geq \ell_\infty$ and $\lambda \geq \lambda_\infty$. The second assertion follows from the same arguments used to prove the second assertion in Lemma 4.9. □

For the proof of Theorem 1.2, it remains to compute the contribution to the Maslov index from the conjugate point $(x,\lambda) = (\ell, 0)$, i.e.

$$\mathfrak{c} := \mathrm{Mas}(\mathbb{E}^u(\cdot,0), \mathbb{E}^s(\ell,0); [\ell-\varepsilon,\ell]) + \mathrm{Mas}(\mathbb{E}^u(\ell,\cdot), \mathbb{E}^s(\ell,\cdot); [0,\varepsilon]). \tag{5.11}$$

For the first term in (5.11), i.e. the arrival along Γ_1, again using property (3) of Proposition 3.4 and equations (4.86) and (4.88), we have

$$\begin{aligned}\mathrm{Mas}(\mathbb{E}^u(\cdot,0), \mathbb{E}^s(\ell,0); [\ell-\varepsilon,\ell]) &= \mathrm{Mas}(\mathbb{E}^u_+(\cdot,0), \mathbb{E}^s_+(\ell,0); [\ell-\varepsilon,\ell]) \\ &\qquad + \mathrm{Mas}(\mathbb{E}^u_-(\cdot,0), \mathbb{E}^s_-(\ell,0); [\ell-\varepsilon,\ell]), \\ &= 1.\end{aligned} \tag{5.12}$$

For the second term in (5.11), i.e. the departure along Γ_2, we compute crossing forms. To that end, suppose $\lambda = \lambda_0 \in [0,\lambda_\infty]$ (not necessarily zero) is a crossing of the Lagrangian pair $\lambda \mapsto (\mathbb{E}^u(\ell,\lambda), \mathbb{E}^s(\ell,\lambda))$. The first-order relative crossing form (3.28) is given by

$$\mathfrak{m}_{\lambda_0}(\mathbb{E}^u(\ell,\cdot), \mathbb{E}^s(\ell,\cdot))(w_0) = \frac{d}{d\lambda}\omega(r(\lambda), w_0)\big|_{\lambda=\lambda_0} - \frac{d}{d\lambda}\omega(s(\lambda), w_0)\big|_{\lambda=\lambda_0}, \tag{5.13}$$

where $w_0 \in W_1(\mathbb{E}^u(\ell,\cdot), \mathbb{E}^s(\ell,\cdot), \lambda_0) = \mathbb{E}^u(\ell,\lambda_0) \cap \mathbb{E}^s(\ell,\lambda_0)$ and (r,s) is a root function pair for $(\mathbb{E}^u(\ell,\cdot), \mathbb{E}^s(\ell,\cdot))$ with $r(\lambda_0) = s(\lambda_0) = w_0$. As in the proof of Lemma 4.8, we compute each of these terms separately. For the first, noting that $r(\lambda) \in \mathbb{E}^u(\ell,\lambda)$ for $\lambda \in [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]$, it follows from the definition of $\mathbb{E}^u(\ell,\lambda)$ that there exists a one-parameter family of solutions $\lambda \mapsto \mathbf{w}(\cdot;\lambda)$ to (2.2), such that $\mathbf{w}(x;\lambda) \to 0$ as $x \to -\infty$, $\mathbf{w}(\ell;\lambda) = r(\lambda)$ and $\mathbf{w}(\ell;\lambda_0) = w_0$. Thus

$$\frac{d}{d\lambda}\omega(r(\lambda), w_0)\big|_{\lambda=\lambda_0} = \omega\Big(\frac{d}{d\lambda}\mathbf{w}(\ell,\lambda), \mathbf{w}(\ell,\lambda)\Big)\big|_{\lambda=\lambda_0},$$

and a calculation similar to (4.68) with

$$\partial_\lambda A(x;\lambda) = \begin{pmatrix} 0_4 & 0_4 \\ A_{21} & 0_4 \end{pmatrix}, \qquad A_{21} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{5.14}$$

and $\mathbf{w} = (u_1, v_2, u_2, v_2, u_3, v_3, u_4, v_4)^\top$ yields

$$\frac{d}{d\lambda}\omega(r(\lambda), w_0)\big|_{\lambda=\lambda_0} = -2\int_{-\infty}^{\ell} u_2(x;\lambda_0)v_2(x;\lambda_0)\,dx.$$

For the second term in (5.13) and the same fixed w_0, associated with $s(\lambda) \in \mathbb{E}^s(\ell,\lambda)$ is a family of solutions $\lambda \mapsto \widetilde{\mathbf{w}}(\cdot;\lambda)$ to (2.2) such that $\widetilde{\mathbf{w}}(x;\lambda) \to 0$ as $x \to +\infty$, $\widetilde{\mathbf{w}}(\ell;\lambda) = s(\lambda)$

and $\widetilde{\mathbf{w}}(\ell;\lambda_0) = w_0$. Arguing as for the first term of (5.13), but now using the decay at $+\infty$, we have

$$\frac{d}{d\lambda}\omega(s(\lambda), w_0)\big|_{\lambda=\lambda_0} = \omega\Big(\frac{d}{d\lambda}\widetilde{\mathbf{w}}(\ell;\lambda), \widetilde{\mathbf{w}}(\ell;\lambda)\Big)\Big|_{\lambda=\lambda_0} = 2\int_\ell^\infty \widetilde{u}_2(x;\lambda_0)\widetilde{v}_2(x;\lambda_0)\,dx.$$

Using uniqueness of solutions as in the proof of Lemma 4.8, we conclude

$$\mathfrak{m}_{\lambda_0}(\mathbb{E}^u(\ell,\cdot), \mathbb{E}^s(\ell,\cdot))(w_0) = -2\int_{-\infty}^{\infty} u_2(x;\lambda_0)v_2(x;\lambda_0)\,dx. \tag{5.15}$$

Remark 5.3. The form (5.15) is *not* sign definite, and therefore the Maslov index does not afford an exact count of the crossings of the path $\lambda \mapsto (\mathbb{E}^u(\ell,\cdot), \mathbb{E}^s(\ell,\cdot))$ for $\lambda \in [0, \lambda_\infty]$. This will be the reason for the inequality (and not equality) (1.19) in Theorem 1.2.

We now evaluate the form (5.15) at $\lambda_0 = 0$, where $W_1((\mathbb{E}^u(\ell,\cdot), \mathbb{E}^s(\ell,\cdot), 0) = \mathbb{E}^u(\ell,0) \cap \mathbb{E}^s(\ell,0)$. Recalling (3.40), we have $\mathbb{E}^u(\ell,0)\cap\mathbb{E}^s(\ell,0) = \mathrm{span}\{\boldsymbol{\phi}(\ell), \boldsymbol{\varphi}(\ell)\}$, where $\boldsymbol{\phi}$ and $\boldsymbol{\varphi}$ are given in (3.39). Hence, it suffices to evaluate (5.15) on $w_0 = \boldsymbol{\phi}(\ell)k_1 + \boldsymbol{\varphi}(\ell)k_2$ for some $k_1, k_2 \in \mathbb{R}$. Evidently, the family $\mathbf{w}(\cdot;0) = \widetilde{\mathbf{w}}(\cdot;0)$ described above is given by $\mathbf{w}(x;0) = \boldsymbol{\phi}(x)k_1 + \boldsymbol{\varphi}(x)k_2$, so that $u_2(x;0) = \phi'(x)k_1$ and $v_2(x;0) = -\phi(x)k_2$. Hence

$$\mathfrak{m}_{\lambda_0}(\mathbb{E}^u(\ell,\cdot), \mathbb{E}^s(\ell,\cdot))(w_0) = 2\int_{-\infty}^{\infty}(\phi' k_1)(\phi k_2)\,dx = \left(\int_{-\infty}^{\infty}\frac{d}{dx}\phi^2\,dx\right)k_1k_2 = 0, \tag{5.16}$$

since $\phi \in H^4(\mathbb{R})$. That is, the relative crossing form $\mathfrak{m}_{\lambda_0}$ in (5.13) is identically zero at $\lambda_0 = 0$, and the conjugate point $(\ell, 0)$ is non-regular in the λ direction. Therefore, $W_2(\mathbb{E}^u(\ell,\cdot), \mathbb{E}^s(\ell,\cdot), 0) = \ker\mathfrak{m}_{\lambda_0}(\mathbb{E}^u(\ell,\cdot), \mathbb{E}^s(\ell,\cdot)) = \mathbb{E}^u(\ell,0)\cap\mathbb{E}^s(\ell,0)$, and we need to compute higher order crossing forms.

To that end, in this case the second-order relative crossing form (3.28) at $\lambda_0 = 0$ is given by

$$\mathfrak{m}^{(2)}_{\lambda_0}(\mathbb{E}^u(\ell,\cdot), \mathbb{E}^s(\ell,\cdot))(w_0) = \frac{d^2}{d\lambda^2}\omega(r(\lambda), w_0)\big|_{\lambda=0} - \frac{d^2}{d\lambda^2}\omega(s(\lambda), w_0)\big|_{\lambda=0}, \tag{5.17}$$

where $w_0 \in \mathbb{E}^u(\ell,0)\cap\mathbb{E}^s(\ell,0)$ and (r,s) is a root function pair for $(\mathbb{E}^u(\ell,\cdot), \mathbb{E}^s(\ell,\cdot))$ with $r(0) = s(0) = w_0$ and $\dot r(0) = \dot s(0)$. (Dot denotes $d/d\lambda$.) For the first term of (5.17), we again have a one-parameter family $\lambda \to \mathbf{w}(\cdot;\lambda)$ decaying to zero as $x \to -\infty$ such that $\mathbf{w}(\ell;\lambda) = r(\lambda)$ and $\mathbf{w}(\ell;0) = w_0$. Now

$$\begin{aligned}
\omega\Big(\frac{d^2}{d\lambda^2}\mathbf{w}(\ell,\lambda), \mathbf{w}(\ell,\lambda)\Big) &= \int_{-\infty}^{\ell}\partial_x\,\omega(\partial_{\lambda\lambda}\mathbf{w}(x;\lambda), \mathbf{w}(x;\lambda))dx,\\
&= \int_{-\infty}^{\ell}\omega\left(\partial_{\lambda\lambda}\left[A(x;\lambda)\mathbf{w}(x;\lambda)\right], \mathbf{w}(x;\lambda)\right) + \omega\left(\partial_{\lambda\lambda}\mathbf{w}(x;\lambda), A(x;\lambda)\mathbf{w}(x;\lambda)\right)dx,\\
&= \int_{-\infty}^{\ell}\omega(A_{\lambda\lambda}(x;\lambda)\mathbf{w}(x;\lambda), \mathbf{w}(x;\lambda)) + 2\,\omega(A_\lambda(x;\lambda)\partial_\lambda\mathbf{w}(x;\lambda), \mathbf{w}(x;\lambda))\\
&\qquad + \omega(A(x;\lambda)\partial_{\lambda\lambda}\mathbf{w}(x;\lambda), \mathbf{w}(x;\lambda)) + \omega\left(\partial_{\lambda\lambda}\mathbf{w}(x;\lambda), A(x;\lambda)\mathbf{w}(x;\lambda)\right)dx,\\
&= \int_{-\infty}^{\ell}\langle[A(x;\lambda)^\top J + JA(x;\lambda)]\partial_{\lambda\lambda}\mathbf{w}(x;\lambda), \mathbf{w}(x;\lambda)\rangle\\
&\qquad\qquad + 2\,\omega(A_\lambda(x;\lambda)\partial_\lambda\mathbf{w}(x;\lambda), \mathbf{w}(x;\lambda))\,dx,\\
&= 2\int_{-\infty}^{\ell}\omega(A_\lambda(x;\lambda)\partial_\lambda\mathbf{w}(x;\lambda), \mathbf{w}(x;\lambda))dx,
\end{aligned} \tag{5.18}$$

where we used (2.14) and $A_{\lambda\lambda}(x;\lambda) = 0$. Using (5.14) and evaluating at $\lambda = 0$, we find

$$\frac{d^2}{d\lambda^2}\omega(r(\lambda), w_0) = \omega\Big(\frac{d^2}{d\lambda^2}\mathbf{w}(\ell,\lambda), \mathbf{w}(\ell,\lambda)\Big)\Big|_{\lambda=0},$$

$$= -2 \int_{-\infty}^{\ell} u_2(x;0)\partial_\lambda v_2(x;0) + v_2(x;0)\partial_\lambda u_2(x;0)\, dx. \tag{5.19}$$

For the second term in (5.17), we have a family $\lambda \to \widetilde{\mathbf{w}}(\cdot;\lambda)$ decaying to zero as $x \to +\infty$ such that $\widetilde{\mathbf{w}}(\ell;\lambda) = s(\lambda)$ and $\widetilde{\mathbf{w}}(\ell;0) = w_0$, and a similar argument used to arrive at (5.19) shows

$$\mathfrak{m}^{(2)}_{\lambda_0}(\mathbb{E}^s(\ell,\cdot),\mathbb{E}^u(\ell,0))(q) = 2\int_{\ell}^{\infty} \widetilde{u}_2(x;0)\partial_\lambda \widetilde{v}_2(x;0) + \widetilde{v}_2(x;0)\partial_\lambda \widetilde{u}_2(x;0)\, dx. \tag{5.20}$$

By uniqueness of solutions we have $\mathbf{w}(\cdot;0) = \widetilde{\mathbf{w}}(\cdot;0)$. Furthermore, since $\dot{r}(0) = \dot{s}(0)$,

$$\partial_\lambda \mathbf{w}(\ell;0) = \dot{r}(0) = \dot{s}(0) = \partial_\lambda \widetilde{\mathbf{w}}(\ell;0). \tag{5.21}$$

Now, both $\partial_\lambda \mathbf{w}(\cdot;0)$ and $\partial_\lambda \widetilde{\mathbf{w}}(\cdot;0)$ solve the inhomogeneous differential equation

$$\frac{d}{dx}\left(\partial_\lambda \mathbf{w}\right) = A\left(\partial_\lambda \mathbf{w}\right) + A_\lambda\left(\boldsymbol{\phi}\, k_1 + \boldsymbol{\varphi}\, k_2\right), \tag{5.22}$$

obtained by differentiating (2.3) with respect to λ and evaluating at $\lambda = 0$, and using that $\mathbf{w}(\cdot;0) = \boldsymbol{\phi}\, k_1 + \boldsymbol{\varphi}\, k_2$. (Note that $k_1, k_2 \in \mathbb{R}$ are determined by the fixed vector w_0, where $w_0 = \mathbf{w}(\ell;0) = \boldsymbol{\phi}(\ell)k_1 + \boldsymbol{\varphi}(\ell)k_2$.) It follows from (5.21) and uniqueness of solutions of (5.22) that indeed $\partial_\lambda \mathbf{w}(x;0) = \partial_\lambda \widetilde{\mathbf{w}}(x;0)$ for *all* $x \in \mathbb{R}$. Collecting (5.19) and (5.20) together, (5.17) then becomes

$$\mathfrak{m}^{(2)}_{\lambda_0}(\mathbb{E}^u(\ell,\cdot),\mathbb{E}^s(\ell,\cdot))(w_0) = -2\int_{-\infty}^{\infty} u_2(x;0)\partial_\lambda v_2(x;0) + v_2(x;0)\partial_\lambda u_2(x;0)\, dx. \tag{5.23}$$

We need to understand the function $\partial_\lambda \mathbf{w}(\cdot;0)$. Notice that it solves the inhomogeneous equation (5.22) if and only if its third and fourth entries $\partial_\lambda u_2(\cdot;0)$ and $\partial_\lambda v_2(\cdot;0)$ solve

$$N\begin{pmatrix} \partial_\lambda u_2(\cdot;0) \\ -\partial_\lambda v_2(\cdot;0) \end{pmatrix} = \begin{pmatrix} \phi_x\, k_1 \\ -\phi\, k_2 \end{pmatrix}. \tag{5.24}$$

This follows from differentiating the eigenvalue equation (1.14) with respect to λ, evaluating at $\lambda = 0$ and making the substitutions (as in (2.1))

$$\partial_\lambda u(\cdot;0) = \partial_\lambda u_2(\cdot;0), \quad \partial_\lambda v(\cdot;0) = -\partial_\lambda v_2(\cdot;0), \quad u(\cdot;0) = \phi_x\, k_1, \quad v(\cdot;0) = -\phi\, k_2.$$

Both equations in (5.24),

$$\begin{aligned} -L_-\partial_\lambda v_2(\cdot;0) &= -\phi_x\, k_1, \\ L_+\partial_\lambda u_2(\cdot;0) &= -\phi\, k_2, \end{aligned} \tag{5.25}$$

are solvable by virtue of the Fredholm alternative, since $\langle \phi', \phi\rangle_{L^2(\mathbb{R})} = 0$ and hence $\phi_x \in \ker(L_-)^\perp$ and $\phi \in \ker(L_+)^\perp$. Denoting by $\widehat{v}$ and $\widehat{u}$ any solutions to

$$-L_- v = \phi_x \qquad \text{and} \qquad L_+ u = \phi \tag{5.26}$$

in $H^4(\mathbb{R})$ respectively (note the sign change in both equations from (5.25)), (5.23) becomes

$$\mathfrak{m}^{(2)}_{\lambda_0}(\mathbb{E}^u(\ell,\cdot),\mathbb{E}^s(\ell,\cdot))(w_0) = 2\left(\int_{-\infty}^{\infty} \phi_x\, \widehat{v}\, dx\right)k_1^2 - 2\left(\int_{-\infty}^{\infty} \phi\, \widehat{u}\, dx\right)k_2^2, \tag{5.27}$$

recalling that $u_2 = \phi_x k_1$ and $v_2 = -\phi\, k_2$.

Having computed the form, we count the number of negative squares. Using (3.18), and defining $\mathcal{I}_1$ and $\mathcal{I}_2$ as in (1.18), we find that

$$\operatorname{Mas}(\mathbb{E}^u(\ell,\cdot),\mathbb{E}^s(\ell,\cdot);[0,\varepsilon]) = -n_-(\mathfrak{m}^{(2)}_{\lambda_0}) = \begin{cases} 0 & \mathcal{I}_1 > 0,\, \mathcal{I}_2 < 0, \\ -1 & \mathcal{I}_1\mathcal{I}_2 > 0, \\ -2 & \mathcal{I}_1 < 0,\, \mathcal{I}_2 > 0. \end{cases} \tag{5.28}$$

Recalling the definition of $\mathfrak{c}$ in (5.11) and using (5.12) yields the following.

Lemma 5.4. *The value of* $\mathfrak{c}$ *is given by*

$$\mathfrak{c} = \begin{cases} 1 & \mathcal{I}_1 > 0,\, \mathcal{I}_2 < 0, \\ 0 & \mathcal{I}_1\mathcal{I}_2 > 0, \\ -1 & \mathcal{I}_1 < 0,\, \mathcal{I}_2 > 0. \end{cases} \tag{5.29}$$

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. By homotopy invariance and additivity under concatenation, we have

$$\begin{aligned} &\operatorname{Mas}(\mathbb{E}^u(\cdot,0),\mathbb{E}^s(\ell,0);[-\infty,\ell]) + \operatorname{Mas}(\mathbb{E}^u(\ell,\cdot),\mathbb{E}^s(\ell,\cdot);[0,\lambda_\infty]) \\ &\qquad - \operatorname{Mas}(\mathbb{E}^u(\cdot,\lambda_\infty),\mathbb{E}^s(\ell,\lambda_\infty);[-\infty,\ell]) - \operatorname{Mas}(\mathbb{E}^u(-\infty,\cdot),\mathbb{E}^s(\ell,\cdot);[0,\lambda_\infty]) = 0. \end{aligned}$$

By Lemma 5.2 the last two terms on the left hand side vanish. Recalling the definition of $\mathfrak{c}$ from (3.46) and using the concatenation property once more,

$$\operatorname{Mas}(\mathbb{E}^u(\cdot,0),\mathbb{E}^s(\ell,0);[-\infty,\ell-\varepsilon]) + \mathfrak{c} + \operatorname{Mas}(\mathbb{E}^u(\ell,\cdot),\mathbb{E}^s(\ell,\cdot);[\varepsilon,\lambda_\infty]) = 0. \tag{5.30}$$

Since the Maslov index counts *signed* crossings, the number of crossings along Γ_2 for $\lambda > 0$ is bounded from below by the absolute value of the Maslov index of this piece, i.e.

$$n_+(N) \geq |\operatorname{Mas}(\mathbb{E}^u(\ell,\cdot),\mathbb{E}^s(\ell,\cdot);[\varepsilon,\lambda_\infty])|. \tag{5.31}$$

Combining (5.30) and (5.31) with Lemma 5.1, the inequality (1.19) follows. The statement of the theorem then follows from the computation of $\mathfrak{c}$ in Lemma 5.4. □

Remark 5.5. In practice, it is more tractable to compute P and Q via Theorem 4.1. Thus, an alternate form of (1.19) is given by

$$n_+(N) \geq \left| \sum_{x\in\mathbb{R}} \dim\left(\mathbb{E}^u_+(x,0)\cap\mathbb{S}_+(0)\right) - \sum_{x\in\mathbb{R}} \dim\left(\mathbb{E}^u_-(x,0)\cap\mathbb{S}_-(0)\right) - \mathfrak{c} \right|. \tag{5.32}$$

We conclude with the proof of Theorem 1.6, for which we will need the following lemma. The first assertion gives a sufficient condition for monotonicity of the Maslov index along Γ_2, and is adapted from [CCLM23, Lemma 5.1]. The second assertion is given in [CCLM23, Lemma 5.2].

Lemma 5.6. *If* L_- *is a nonpositive operator, then each crossing* $\lambda = \lambda_0 > 0$ *of the path* $\lambda \mapsto (\mathbb{E}^u(\ell,\lambda),\mathbb{E}^s(\ell,\lambda))$ *is positive. Moreover, in this case* $\operatorname{Spec}(N) \subset \mathbb{R}\cup i\mathbb{R}$.

Proof. If $\lambda = \lambda_0$ is a crossing then the eigenvalue equations

$$-L_-v = \lambda_0 u, \qquad L_+u = \lambda_0 v \tag{5.33}$$

are satisfied for some $\tilde{u}, \tilde{v} \in H^4(\mathbb{R})$. Notice that $\lambda_0 > 0$ necessitates that *both* $\tilde{u}$ and $\tilde{v}$ are nontrivial.

Solving the first equation in (5.33) yields $\tilde{v} = \alpha\phi + \tilde{v}_\perp$ for some $\alpha \in \mathbb{R}$, where $\ker(L_-) = \operatorname{span}\{\phi\}$ and $\tilde{v}_\perp \in \ker(L_-)^\perp$. Therefore

$$\langle L_-\tilde{v},\tilde{v}\rangle_{L^2(\mathbb{R})} = \langle L_-(\alpha\phi+\tilde{v}_\perp),\alpha\phi+\tilde{v}_\perp\rangle_{L^2(\mathbb{R})} = \langle L_-\tilde{v}_\perp,\tilde{v}_\perp\rangle_{L^2(\mathbb{R})} < 0 \tag{5.34}$$

because L_- is nonpositive and $\hat{v}_\perp \in \ker(L_-)^\perp$. Now analysing the crossing form (5.15) for the path $\lambda \mapsto (\mathbb{E}^u(\ell,\lambda),\mathbb{E}^s(\ell,\lambda))$, where $v_2 = -\tilde{v}$ and $u_2 = \tilde{u}$, we have

$$\mathfrak{m}_{\lambda_0}(\mathbb{E}^u(\ell,\cdot),\mathbb{E}^s(\ell,\cdot))(q) = -\frac{2}{\lambda_0}\int_{-\infty}^{\infty}(\lambda_0\, u_2)\, v_2\, dx = -\frac{2}{\lambda_0}\langle L_-\tilde{v},\tilde{v}\rangle_{L^2(\mathbb{R})} > 0,$$

which was to be proven. The second statement may be proven using similar arguments as in the proof of [CCLM23, Lemma 5.1]. Namely, we can rewrite (1.14) as the selfadjoint eigenvalue problem

$$(-L_-|_{X_c})^{1/2}\,\Pi L_+\Pi\,(-L_-|_{X_c})^{1/2}\,w = \lambda^2 w, \tag{5.35}$$

where $X_c = \ker(L_-)^\perp$, Π is the orthogonal projection in $L^2(\mathbb{R})$ onto X_c, $(-L_-|_{X_c})^{1/2}$ is well-defined because $-L_-$ is nonnegative, and $w = (-L_-|_{X_c})^{1/2}\,\Pi v$. It follows that $\lambda^2 \in \mathbb{R}$. For more on the equivalence of (1.14) with (5.35), see [CCLM23, Lemma 3.21]. We omit the details here. □

Proof of Theorem 1.6. If $Q = 0$ then it follows from Lemma 5.6 that

$$\operatorname{Mas}(\mathbb{E}^u(\ell,\cdot),\mathbb{E}^s(\ell,\cdot);[\varepsilon,\lambda_\infty]) = n_+(N) \tag{5.36}$$

for ε small enough. Using this and Lemma 5.1 in (5.30), we obtain

$$n_+(N) = P - Q - \mathfrak{c} = 1 - \mathfrak{c}. \tag{5.37}$$

For the evaluation of $\mathfrak{c}$, using (5.26) we can write

$$\mathcal{I}_1 = \int_{-\infty}^{\infty} \phi_x\,\widehat{v}\,dx = -\int_{-\infty}^{\infty} (L_-\widehat{v})\,\widehat{v}\,dx, \tag{5.38}$$

so that if $Q = 0$ then $\mathcal{I}_1 \geq 0$. An argument similar to (5.34) shows that in fact $\mathcal{I}_1 > 0$. Lemma 5.4 now yields the value of $\mathfrak{c}$. In particular, if $\mathcal{I}_2 > 0$, then $\mathfrak{c} = 0$ and $n_+(N) = 1$. On the other hand, if $\mathcal{I}_2 < 0$, then $\mathfrak{c} = 1$ and $n_+(N) = 0$; by the second assertion of Lemma 5.6, this means $\operatorname{Spec}(N) \subset i\mathbb{R}$. □

6. Application: spectral stability of the Karlsson and Höök solution

In this section we apply our theory to confirm the spectral stability of the Karlsson and Höök solution (1.8), i.e.

$$\phi_{\mathrm{KH}}(x) = \sqrt{\frac{3}{10}}\,\operatorname{sech}^2\left(\frac{x}{2\sqrt{5}}\right), \tag{6.1}$$

which solves (1.5) for $\beta = 4/25, \sigma_2 = -1$. Note that in this case $\operatorname{Spec}_{\mathrm{ess}}(L_\pm) = (-\infty, -4/25)$. While orbital stability was proven in [NP15], the following serves to showcase how our analytical results may be implemented for a given standing wave.

In Fig. 2, we have plotted the set of points in the λx-plane where the unstable bundle nontrivially intersects the stable subspace of the asymptotic system for each of the eigenvalue problems for L_+ and L_-. More precisely, for L_+ problem we have the following construction (the construction for the L_- problem is similar). We denote the eigenvalues of the λ-dependent asymptotic matrix $A_+(\lambda)$ by $\{\pm\mu_1(\lambda), \pm\mu_2(\lambda)\}$ and the corresponding stable and unstable eigenvectors by $\mathbf{p}_{1,2}(\lambda)$ and $\mathbf{u}_{1,2}(\lambda)$ respectively, so that for $i = 1, 2$,

$$\ker\left(A_+(\lambda) + \mu_i(\lambda)\right) = \operatorname{span}\{\mathbf{p}_i(\lambda)\}, \qquad \ker\left(A_+(\lambda) - \mu_i(\lambda)\right) = \operatorname{span}\{\mathbf{q}_i(\lambda)\}$$

(similar to (4.12)–(4.14)). Collecting the vectors $\mathbf{s}_{1,2}$ in the columns of a 4×2 frame and right multiplying by the inverse of the resulting upper 2×2 block yields the following real frame for $\mathbb{S}_+(\lambda)$,

$$\begin{pmatrix} I \\ S_+(\lambda) \end{pmatrix}, \qquad S_+(\lambda) \in \mathbb{R}^{2\times 2}. \tag{6.2}$$

Next, we require a frame for the unstable bundle, suitably initialised. Making similar manipulations as above on the scaled unstable eigenvectors $e^{-\mu_1(\lambda)\ell}\mathbf{u}_1(\lambda), e^{-\mu_2(\lambda)\ell}\mathbf{u}_2(\lambda)$ yields a real frame for the unstable subspace, which we denote by

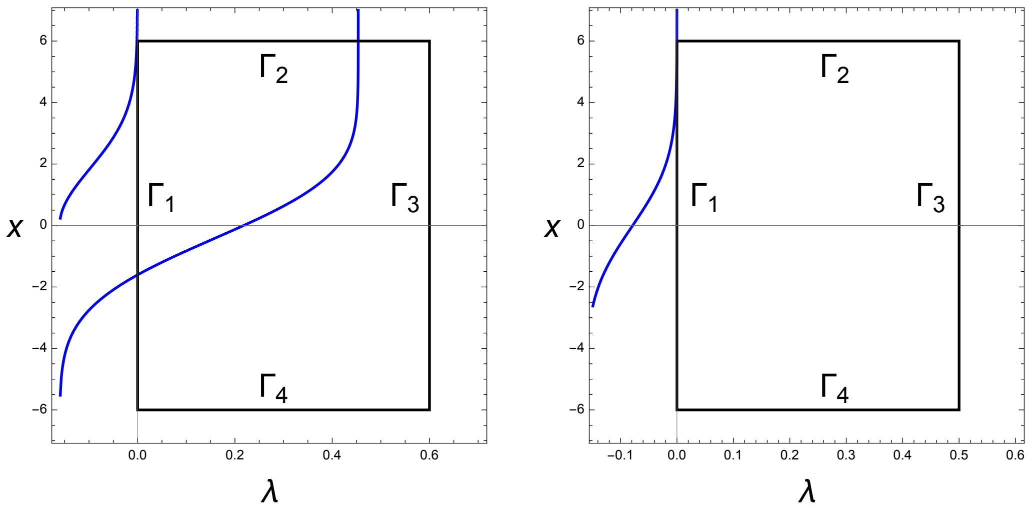
$$\begin{pmatrix} I \\ U_+(\lambda) \end{pmatrix}, \qquad U_+(\lambda) \in \mathbb{R}^{2\times 2}. \tag{6.3}$$

A frame for the unstable bundle is then given by solutions to (4.3) initialised at (6.3). We denote this frame by $\mathbf{U}_+(x,\lambda) = (X_+(x,\lambda), Y_+(x,\lambda))$. The *eigenvalue curves* are then given by the locus of points (λ, x) such that

$$\det \begin{pmatrix} X_+(x,\lambda) & I \\ Y_+(x,\lambda) & S_+(\lambda) \end{pmatrix} = \det(S_+(\lambda)X_+(x,\lambda) - Y_+(x,\lambda)) = 0, \tag{6.4}$$

see Fig. 2. (The name follows from the fact that each point (λ, x) on such curves represents an eigenvalue λ of the operator $L_\pm$ with domain $\mathrm{dom}(L_\pm) = \{u \in H^4(-\infty, s) : (u''(s) + \sigma_2 u(s), \mp u(s), \mp u'(s), u'''(s)) \in \mathbb{S}_\pm(\lambda)\}$.) The intersections of the eigenvalue curves with Γ_1 (where $\lambda = 0$) thus represent conjugate points, while the crossings along Γ_2 (where $x = \ell$) represent the eigenvalues of $L_\pm$.

Remark 6.1. Strictly speaking, as per our analysis we should be using $\mathbb{E}^s_\pm(x,\ell)$ for some large ℓ as our reference plane. However, for the purposes of graphical illustration, it suffices to use $\mathbb{S}_\pm(\lambda)$, the train of which is an arbitrarily small perturbation of the train of $\mathbb{E}^s_\pm(x,\ell)$ (for large enough ℓ.)



(a) L_+ eigenvalue curves. (b) L_- eigenvalue curves.

Figure 2. L_+ and L_- eigenvalue curves and Maslov box for the Karlsson and Höök solution ϕ_{KH}, where $\beta = 4/25$, $\sigma_2 = -1$ and $\ell = 7$. In both cases, an eigenvalue curve is asymptotic to the line $\lambda = 0$ (but never crosses).

From Fig. 2, we conclude that there is one conjugate point for the L_+ problem, while there are none for the L_- problem (the eigenvalue curve in each subfigure which is asymptotic to $\lambda = 0$ never crosses $\lambda = 0$). Theorem 4.1 correctly predicts the number of crossings along Γ_2 in both cases, i.e. $P = 1, Q = 0$. By Theorem 1.6, spectral stability of the Karlsson and Höök solution is thus determined by the sign of $\mathcal{I}_2$. Using the theory developed in [Alb92], it was shown in [NP15, Remark 4.6] that $\mathcal{I}_2 < 0$. Hence ϕ_{KH} is spectrally stable.

7. Concluding remarks

Theorem 1.2 remains true for pure quartic solitons, i.e. standing wave solutions to (1.1) with $\beta_2 = 0$. In this case, non-dimensionalising (1.1) with the transformations (assuming $\gamma > 0$ and $\beta_4 < 0$)

$$\psi = \sqrt{\frac{24\gamma}{|\beta_4|}}\Psi, \qquad z = \frac{24}{|\beta_4|}x,$$

leads to (1.3) (after interchanging z back to x) with $\sigma_2 = 0$. Thus, our proof of Theorem 1.2 can be applied to PQSs simply by setting $\sigma_2 = \sigma_2^2 = 0$ (for example, in (2.2) one has $\alpha(x) = 3\phi(x)^2 - \beta$ and $\eta(x) = -\phi(x)^2 + \beta$). The conditions (1.10),(1.11) are now replaced with $\beta > 0$; this ensures that both the origin in (1.7) is hyperbolic, and the essential spectra of $L_\pm$ are confined to the negative half line, $\mathrm{Spec}_{\mathrm{ess}}(L_\pm) \subset \mathbb{R}^-$, so that P and Q are well-defined. The crossing form calculations in Section 4 and Section 5 are identical; setting $\sigma_2 = 0$ and requiring that $\beta > 0$ preserves the signatures of the forms.

In this work, similar to the analyses in [Cor19, HLS18, How23], the central objects are Lagrangian pairs comprising the unstable and stable bundles. For the fourth order selfadjoint eigenvalue problems discussed in Section 4, where we considered the pairs

$$\Gamma \ni (x,\lambda) \mapsto (\mathbb{E}^u_\pm(x,\lambda), \mathbb{E}^s_\pm(\ell,\lambda)) \in \mathcal{L}(2) \times \mathcal{L}(2), \tag{7.1}$$

we showed that monotonicity holds in both the spectral and spatial parameters. Another possible choice for the second entry of the pair in (7.1), as in [BJP24, How23], is to use the (fixed) *sandwich* plane with frame

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix},$$

with respect to which we expect the unstable bundles $\mathbb{E}^u_\pm(x,\lambda)$ will be monotonic in both x and λ. It should then be possible to argue as in [BCJ$^+$18, BJP24], by first considering the problem on the half line $(-\infty, \ell]$ for large ℓ, and then using asymptotic arguments similar to those in [SS00] to recover a Morse–Maslov theorem on the full domain $\mathbb{R}$. However, for the eigenvalue problem for N, it is unclear how using the sandwich reference plane (of $\mathbb{R}^8$) would affect the existence of a crossing at the top left corner of the Maslov box, the contribution of which is significant in the lower bound of Theorem 1.2.

Generally speaking, a rigorous proof of spectral stability or instability using the Maslov index is a two-step process. The first involves proving some kind of monotonicity result *assuming* the existence of conjugate points, allowing one to relate a certain spectral index of a linear operator to a count of the conjugate points. The second involves proving the existence (or non-existence) of conjugate points given a particular stationary state. This can either be done numerically [BJ22, BJP24], or even analytically (see, for example, the beautiful argument in [BCJ$^+$18] which exploits the reversibility symmetry of the underlying PDE). Practically, our lower bound would be useful in the following cases: (1) using Corollary 1.5 to prove instability by showing $|P - Q| \geq 2$, and (2) when either $P = 0$ or $Q = 0$. As pointed out in Section 5, in the latter case our lower bound becomes an exact count, $n_+(N) = |P - Q - \mathfrak{c}|$, and spectral stability or instability follows by showing that $|P - Q - \mathfrak{c}| = 0$ or $|P - Q - \mathfrak{c}| \geq 1$, respectively.

Finally, we comment on the case when either $\mathcal{I}_1 = 0$ or $\mathcal{I}_2 = 0$. In this case, the second order form (5.27) is degenerate, and one proceeds by computing the third order crossing form. The Fredholm Alternative dictates that the algebraic multiplicity of $0 \in \mathrm{Spec}(N)$

increases by one for each quantity $\mathcal{I}_1, \mathcal{I}_2$ that vanishes. Similar to (5.27), any third order form will be given by the $L_2(\mathbb{R})$ inner products, akin to $\mathcal{I}_1$ and $\mathcal{I}_2$, of functions in $\ker(N)$ and $\ker(N^3)\setminus\ker(N^2)$. Due to the Hamiltonian structure of N, which implies an even symmetry in $\mathrm{Spec}(N)$, we expect that all higher order crossing forms of odd order will be identically zero, and the contribution to the Maslov index will come from the negative index of any nondegenerate even order forms.

A. Appendix: Removal of Hypothesis 4.4.

In order to complete the proof of Lemma 4.2, i.e. monotonicity in the spatial parameter of the paths $x \mapsto (\mathbb{E}^u_\pm(x,0), \mathbb{S}_\pm(0))$, we need to account for the case when Hypothesis 4.4 fails, i.e. $\phi(x_0) = 0$. In this case, all of the crossing forms computed in the proof of Lemma 4.2 are identically zero, and we need to compute higher order crossing forms. The key observation here is that since ϕ solves the standing wave equation (1.5), a fourth order ordinary differential equation, for every $x \in \mathbb{R}$ at least one of the elements of the set $\{\phi(x), \phi'(x), \phi''(x), \phi'''(x)\}$ is nonzero. As we will see, it will follow that computing sufficiently many higher order crossing forms will yield enough nonzero summands in the left hand side of (3.12), i.e. such that

$$\sum_{k\geq 1}\left(n_+(\mathfrak{m}^{(k)}_{x_0}) + n_-(\mathfrak{m}^{(k)}_{x_0})\right) = \dim \mathbb{E}^u_\pm(x_0,0)\cap\mathbb{S}_\pm(0). \tag{A.1}$$

Similar to the proof of Lemma 4.2, we separate the analyses depending on the nature of the intersection $\mathbb{E}^u_\pm(x_0,0)\cap\mathbb{S}_\pm(0)$, and focus on the L_+ problem only; the proof for the L_- problem is similar. We further split the analysis of each of these cases into three subcases based on the lowest nonzero derivative of ϕ at x_0. Since the calculations are involved and similar to those in the proof of Lemma 4.2, in the interest of expediency we present only the main results.

The following facts are needed for the results listed thereafter. If $\phi(x_0) = 0$ and $\phi'(x_0) \neq 0$, then it follows from (4.27) and (4.26) that

$$C_+ = S_+B_+S_+, \qquad C_+' = 0, \qquad C''(x_0) = \begin{pmatrix} 0 & 0 \\ 0 & 6\phi'(x_0)^2\end{pmatrix}. \tag{A.2}$$

If $\phi(x_0) = \phi'(x_0) = 0$ and $\phi''(x_0)$, then

$$C_+ = S_+B_+S_+, \qquad C_+''' = C_+'' = C_+' = 0, \qquad C_+^{(4)}(x_0) = \begin{pmatrix} 0 & 0 \\ 0 & 18\phi''(x_0)^2\end{pmatrix}. \tag{A.3}$$

Finally, if $\phi^{(k)}(x_0) = 0$ for $0 \leq k \leq 2$ and $\phi'''(x_0) \neq 0$, then $C_+ = S_+B_+S_+$ and

$$C_+^{(5)} = C_+^{(4)} = C_+''' = C_+'' = C_+' = 0, \qquad C_+^{(6)}(x_0) = \begin{pmatrix} 0 & 0 \\ 0 & 60\phi'''(x_0)^2\end{pmatrix}. \tag{A.4}$$

Expressions for derivatives of $X(x)$ and $Y(x)$ at x_0 up to order 9 are required, and can be simplified using (A.2) – (A.4) where appropriate. The vectors h_i are found using (3.7); in particular, assuming the set $\{h_0, \dots, h_{k-2}\}$ satisfies (3.7), then $\{h_0, \dots, h_{k-2}, h_{k-1}\}$ solves

$$(Y_0 - S_+X_0)h_{k-1} = -\sum_{i=0}^{k-2}\binom{k-1}{i}\left(Y^{(k-1-i)}(x_0) - S_+X^{(k-1-i)}(x_0)\right)h_i. \tag{A.5}$$

In this case the forms $\mathfrak{m}^{(k)}_{x_0}$ are computed via (3.8), which amounts to

$$\mathfrak{m}^{(k)}_{x_0}(\mathbb{E}^u_+(\cdot,0), \mathbb{S}_+(0))(w_0) = -\sum_{i=0}^{k-1}\binom{k}{i}\left\langle\left(Y^{(k-i)}(x_0) - S_+X^{(k-i)}(x_0)\right)h_i, k_0\right\rangle. \tag{A.6}$$

Case 1: $\dim \mathbb{E}_+^u(x_0,0)\cap\mathbb{S}_+(0) = 1, b_0 \neq 0$. In this case (4.29) holds, i.e. $W_1 = \mathbb{E}_+^u(x_0,0)\cap \mathbb{S}_+(0) = \text{span}\{a_0\mathbf{s}_1 + b_0\mathbf{s}_2\}$ for some fixed a_0 and $b_0 \neq 0$.

Case (i): $\phi(x_0) = 0, \phi'(x_0) \neq 0$. An identical calculation to that in the proof of (4.2) shows that $h_1 \in \ker(Y_0 - S_+X_0)$ and $\mathfrak{m}_{x_0}^{(2)} = 0$. Using (A.2), it can be shown that $h_2 \in \ker(Y_0 - S_+X_0)$ and

$$\mathfrak{m}_{x_0}^{(3)}(\mathbb{E}_+^u(\cdot,0),\mathbb{S}_+(0))(w_0) = -\left\langle C_+''k_0, k_0\right\rangle = -6[\phi'(x_0)]^2b_0^2 < 0. \tag{A.7}$$

Case (ii): $\phi(x_0) = 0, \phi'(x_0) = 0, \phi''(x_0) \neq 0$. We now have $\mathfrak{m}_{x_0}^{(k)} = 0$ for $k = 1,2,3$. Using that $C_+'''(x_0) = C_+''(x_0) = 0$, it can be shown that $h_3 \in \ker(Y_0 - S_+X_0)$. With $h_k \in \ker(Y_0 - S_+X_0)$ for $k = 1,2,3$, one finds $\mathfrak{m}_{x_0}^{(4)} = 0$. For the fifth order form, one additionally has $h_4 \in \ker(Y_0 - S_+X_0)$ and

$$\mathfrak{m}_{x_0}^{(5)}(\mathbb{E}_+^u(\cdot,0),\mathbb{S}_+(0))(w_0) = -\left\langle C_+^{(4)}k_0, k_0\right\rangle = -18\left[\phi''(x_0)\right]^2 b_0^2 < 0. \tag{A.8}$$

Case (iii): $\phi(x_0) = 0, \phi'(x_0) = \phi''(x_0) = 0, \phi'''(x_0) \neq 0$. Now $\mathfrak{m}_{x_0}^{(k)} = 0$ for $k = 1,2,3,4,5$. Using $C_+^{(5)}(x_0) = C_+^{(4)}(x_0) = 0$, it can be shown that $h_5 \in \ker(Y_0 - S_+X_0)$ and $\mathfrak{m}_{x_0}^{(6)} = 0$. Moreover, one finds $h_6 \in \ker(Y_0 - S_+X_0)$ and

$$\mathfrak{m}_{x_0}^{(7)}(\mathbb{E}_+^u(\cdot,0),\mathbb{S}_+(0))(w_0) = -\left\langle C_+^{(6)}k_0, k_0\right\rangle = -60\left[\phi'''(x_0)\right]^2 b_0^2 < 0. \tag{A.9}$$

Case 2: $\dim \mathbb{E}_+^u(x_0,0)\cap\mathbb{S}_+(0) = 1, b_0 = 0$. In this case (4.30) holds, i.e. $W_1 = \mathbb{E}_+^u(x_0,0)\cap \mathbb{S}_+(0) = \text{span}\{\mathbf{s}_1\}$, and for any $i \geq 1$, we have $C_+^{(i)}k_0 = 0$.

Case (i): $\phi(x_0) = 0, \phi'(x_0) \neq 0$. As in the proof of Lemma 4.2, we have $\mathfrak{m}_{x_0}^{(k)} = 0$ for $k = 1,2,3$. Using (A.2) and $h_k \in \ker(Y_0 - S_+X_0)$ for $k = 1,2$, one finds that $h_3 \in \ker(Y_0 - S_+X_0)$ and $\mathfrak{m}_{x_0}^{(4)} = 0$. With these h_k one then finds that h_4 satisfies

$$(Y_0 - S_+X_0)h_4 = -3C_+''B_+Y_0h_0 = -3C_+''B_+S_+k_0, \tag{A.10}$$

and

$$\mathfrak{m}_{x_0}^{(5)}(\mathbb{E}_+^u(\cdot,0),\mathbb{S}_+(0))(w_0) = -3\cdot 4\left\langle S_+B_+C_+''B_+S_+k_0, k_0\right\rangle = -3\cdot 4\left(\frac{6\phi'(x_0)^2a_0^2}{2\sqrt{\beta}-\sigma_2}\right) < 0.$$

Case (ii): $\phi(x_0) = \phi'(x_0) = 0, \phi''(x_0) \neq 0$. Now $C_+''' = C_+'' = C_+' = 0$, and $\mathfrak{m}_{x_0}^{(k)} = 0$ for $k = 1,2,3,4,5$. With $h_k \in \ker(Y_0 - S_+X_0)$ for $k = 1,2,3,4$, one finds that $h_5 \in \ker(Y_0 - S_+X_0)$ and $\mathfrak{m}_{x_0}^{(6)} = 0$. With these h_k one finds that h_6 satisfies

$$(Y_0 - S_+X_0)h_6 = -5C_+^{(4)}B_+S_+k_0, \tag{A.11}$$

and

$$\mathfrak{m}_{x_0}^{(7)}(\mathbb{E}_+^u(\cdot,0),\mathbb{S}_+(0))(w_0) = -5\cdot 6\left\langle S_+B_+C_+^{(4)}B_+S_+k_0, k_0\right\rangle = -5\cdot 6\left(\frac{18\phi''(x_0)^2a_0^2}{2\sqrt{\beta}-\sigma_2}\right) < 0.$$

Case (iii): $\phi(x_0) = \phi'(x_0) = \phi''(x_0) = 0, \phi'''(x_0) \neq 0$. Now $C_+^{(i)} = 0$ for $1 \leq i \leq 5$, and $\mathfrak{m}_{x_0}^{(k)} = 0$ for all $1 \leq k \leq 7$. With $h_k \in \ker(Y_0 - S_+X_0)$ for $1 \leq k \leq 6$, one finds that $h_7 \in \ker(Y_0 - S_+X_0)$ and $\mathfrak{m}_{x_0}^{(8)} = 0$. With these h_k one finds that h_8 satisfies

$$(Y_0 - S_+X_0)h_8 = -7C_+^{(6)}B_+S_+k_0, \tag{A.12}$$

and

$$\mathfrak{m}_{x_0}^{(9)}(\mathbb{E}_+^u(\cdot,0),\mathbb{S}_+(0))(w_0) = -7\cdot 8\left\langle S_+B_+C_+^{(6)}B_+S_+k_0, k_0\right\rangle = -7\cdot 8\left(\frac{60\phi'''(x_0)^2a_0^2}{2\sqrt{\beta}-\sigma_2}\right) < 0.$$

Case 3: $\dim \mathbb{E}^u_+(x_0,0)\cap\mathbb{S}_+(0) = 2$. Now we have $W_1 = \mathbb{E}^u_+(x_0,0) = \mathbb{S}_+(0)$, and $Y_0 = S_+X_0$. At the outset, there is no restriction on the vector $k_0 = (a,b)^\top$.

Case (i): $\phi(x_0) = 0, \phi'(x_0) \neq 0$. As in the proof of Lemma 4.2, we have $C''_+ \neq 0$ and $\mathfrak{m}^{(k)}_{x_0} = 0$ for $k = 1,2$. It can be shown that now $h_1, h_2 \in \ker(Y_0 - S_+X_0)$ are arbitrary, and the third order form is given by (A.7). so that $n_-(\mathfrak{m}^{(3)}_{x_0}) = 1$. Next, using $W_4 = \ker \mathfrak{m}^{(3)}_{x_0} = \operatorname{span}\{\mathbf{s}_1\}$, one finds that h_3 is free provided $X_0h_0 = k_0 \in \ker C''_+$, i.e. $k_0 = (a_0,0)^\top$, in which case $\mathfrak{m}^{(4)}_{x_0} = 0$. For the fifth order form, with $k_0 = (a,0)^\top$, one finds that h_4 is free provided h_1 satisfies

$$C''_+X_0h_1 = -\frac{3}{4}C''_+B_+S_+k_0, \tag{A.13}$$

in which case

$$\mathfrak{m}^{(5)}_{x_0}(\mathbb{E}^u_+(\cdot,0),\mathbb{S}_+(0))(w_0) = -\frac{3}{4}\left\langle S_+B_+C''_+B_+S_+k_0, k_0\right\rangle = -\frac{3}{4}\left(\frac{6\phi'(x_0)^2a_0^2}{2\sqrt{\beta}-\sigma_2}\right) < 0, \tag{A.14}$$

so that $n_-(\mathfrak{m}^{(3)}_{x_0}) + n_-(\mathfrak{m}^{(5)}_{x_0}) = 2$.

Case (ii): $\phi(x_0) = \phi'(x_0) = 0, \phi''(x_0) \neq 0$. Now $C'''_+ = C''_+ = C'_+ = 0$, and $\mathfrak{m}^{(k)}_{x_0} = 0$ for $1 \le k \le 4$. With $W_k = W_1 = \mathbb{E}^u_+(x_0,0) = \mathbb{S}_+(0)$ for $1 \le k \le 5$, we find that $\mathfrak{m}^{(5)}_{x_0}$ is given by (A.8), thus $n_-(\mathfrak{m}^{(5)}_{x_0}) = 1$. Next, using $W_6 = \ker \mathfrak{m}^{(5)}_{x_0} = \operatorname{span}\{\mathbf{s}_1\}$, one finds that h_5 is free provided $X_0h_0 = k_0 \in \ker C^{(4)}_+$, i.e. $k_0 = (a_0,0)^\top$, in which case $\mathfrak{m}^{(6)}_{x_0} = 0$. For the seventh order form, with $k_0 = (a,0)^\top$, one finds that h_6 is free provided h_1 satisfies

$$C^{(4)}_+X_0h_1 = -\frac{5}{6}C^{(4)}_+B_+S_+k_0, \tag{A.15}$$

in which case

$$\mathfrak{m}^{(5)}_{x_0}(\mathbb{E}^u_+(\cdot,0),\mathbb{S}_+(0))(w_0) = -\frac{5}{6}\left\langle S_+B_+C^{(4)}_+B_+S_+k_0, k_0\right\rangle = -\frac{5}{6}\left(\frac{18\phi''(x_0)^2a_0^2}{2\sqrt{\beta}-\sigma_2}\right) < 0, \tag{A.16}$$

so that $n_-(\mathfrak{m}^{(5)}_{x_0}) + n_-(\mathfrak{m}^{(7)}_{x_0}) = 2$.

Case (iii): $\phi(x_0) = \phi'(x_0) = \phi''(x_0) = 0, \phi'''(x_0) \neq 0$. Now $C^{(i)}_+ = 0$ for $1 \le i \le 5$, and $\mathfrak{m}^{(k)}_{x_0} = 0$ for $1 \le k \le 6$. With $W_k = W_1 = \mathbb{E}^u_+(x_0,0) = \mathbb{S}_+(0)$ for $1 \le k \le 7$, we find that $\mathfrak{m}^{(7)}_{x_0}$ is given by (A.9), thus $n_-(\mathfrak{m}^{(7)}_{x_0}) = 1$. Next, using $W_8 = \ker \mathfrak{m}^{(7)}_{x_0} = \operatorname{span}\{\mathbf{s}_1\}$, one finds that h_7 is free provided $X_0h_0 = k_0 \in \ker C^{(6)}_+$, i.e. $k_0 = (a_0,0)^\top$, in which case $\mathfrak{m}^{(8)}_{x_0} = 0$. For the ninth order form, with $k_0 = (a,0)^\top$, one finds that h_8 is free provided h_1 satisfies

$$C^{(6)}_+X_0h_1 = -\frac{7}{8}C^{(6)}_+B_+S_+k_0, \tag{A.17}$$

in which case

$$\mathfrak{m}^{(5)}_{x_0}(\mathbb{E}^u_+(\cdot,0),\mathbb{S}_+(0))(w_0) = -\frac{7}{8}\left\langle S_+B_+C^{(6)}_+B_+S_+k_0, k_0\right\rangle = -\frac{7}{8}\left(\frac{60\phi'''(x_0)^2a_0^2}{2\sqrt{\beta}-\sigma_2}\right) < 0, \tag{A.18}$$

so that $n_-(\mathfrak{m}^{(7)}_{x_0}) + n_-(\mathfrak{m}^{(9)}_{x_0}) = 2$.

References

[ABK94] N.N. Akhmediev, A.V. Buryak, and M. Karlsson. Radiationless optical solitons with oscillating tails. *Opt. Commun.*, 110(5-6):540 – 544, 1994. 2

[AGJ90] J. Alexander, R. Gardner, and C. Jones. A topological invariant arising in the stability analysis of travelling waves. *J. Reine Angew. Math.*, 410:167–212, 1990. 3, 10, 17, 31, 32

[Alb92] J. P. Albert. Positivity properties and stability of solitary-wave solutions of model equations for long waves. *Comm. Partial Differential Equations*, 17(1-2):1–22, 1992. 40

[Arn67] V. I. Arnol'd. On a characteristic class entering into conditions of quantization. *Funkcional. Anal. i Priložen.*, 1:1–14, 1967. 10, 11, 18

[Arn85] V. I. Arnol'd. Sturm theorems and symplectic geometry. *Funktsional. Anal. i Prilozhen.*, 19(4):1–10, 95, 1985. 11, 14

[BA95] A. V. Buryak and N. N. Akhmediev. Stability criterion for stationary bound states of solitons with radiationless oscillating tails. *Physical Review E*, 51(4):3572 – 3578, 1995. Cited by: 59. 2

[BCJ+18] M. Beck, G. Cox, C. Jones, Y. Latushkin, K. McQuighan, and A. Sukhtayev. Instability of pulses in gradient reaction-diffusion systems: a symplectic approach. *Philos. Trans. Roy. Soc. A*, 376(2117):20170187, 20, 2018. 4, 17, 41

[BGBK21] R. I. Bandara, A. Giraldo, N. G. R. Broderick, and B. Krauskopf. Infinitely many multi-pulse solitons of different symmetry types in the nonlinear Schrödinger equation with quartic dispersion. *Phys. Rev. A*, 103(6):063514, 2021. 2, 5, 7

[BJ95] A. Bose and C. K. R. T. Jones. Stability of the in-phase travelling wave solution in a pair of coupled nerve fibers. *Indiana Univ. Math. J.*, 44(1):189–220, 1995. 4

[BJ22] M. Beck and J. Jaquette. Validated spectral stability via conjugate points. *SIAM J. Appl. Dyn. Syst.*, 21(1):366–404, 2022. 3, 41

[BJP24] M. Beck, J. Jaquette, and H. Pieper. The Maslov index, degenerate crossings, and the stability of pulse solutions to the Swift-Hohenberg equations. *arXiv preprint*, 2024. 4, 5, 41

[Bot56] R. Bott. On the iteration of closed geodesics and the Sturm intersection theory. *Comm. Pure Appl. Math.*, 9(2):171–206, 1956. 18

[BRdSHE17] A. Blanco-Redondo, C. Martijn de Sterke, C. Husko, and B. Eggleton. High-energy ultra-short pulses from pure-quartic solitons. In *2017 European Conference on Lasers and Electro-Optics and European Quantum Electronics Conference*, page EE_3_2. Optica Publishing Group, 2017. 2

[BRdSS+16] Andrea Blanco-Redondo, C. Martijn de Sterke, J.E. Sipe, Thomas F. Krauss, Benjamin J. Eggleton, and Chad Husko. Pure-quartic solitons. *Nature Communications*, 7(1):10427, 2016. 2

[CCLM23] G. Cox, M. Curran, Y. Latushkin, and R. Marangell. Hamiltonian spectral flows, the Maslov index, and the stability of standing waves in the nonlinear Schrödinger equation. *SIAM J. Math. Anal.*, 55(5):4998–5050, 2023. 5, 14, 38, 39

[CDB09a] F. Chardard, F. Dias, and T. J. Bridges. Computing the Maslov index of solitary waves. I. Hamiltonian systems on a four-dimensional phase space. *Phys. D*, 238(18):1841–1867, 2009. 4

[CDB09b] F. Chardard, F. Dias, and T .J. Bridges. On the Maslov index of multi-pulse homoclinic orbits. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 465(2109):2897–2910, 2009. 4

[CDB11] F. Chardard, F. Dias, and T .J. Bridges. Computing the Maslov index of solitary waves, Part 2: Phase space with dimension greater than four. *Phys. D*, 240(17):1334–1344, 2011. 4

[CH07] C.-N. Chen and X. Hu. Maslov index for homoclinic orbits of Hamiltonian systems. *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, 24(4):589–603, 2007. 18

[CH14] C.-N. Chen and X. Hu. Stability analysis for standing pulse solutions to FitzHugh-Nagumo equations. *Calc. Var. Partial Differential Equations*, 49(1-2):827–845, 2014. 4

[CJ18] P. Cornwell and C. K. R. T. Jones. On the existence and stability of fast traveling waves in a doubly diffusive FitzHugh-Nagumo system. *SIAM J. Appl. Dyn. Syst.*, 17(1):754–787, 2018. 4

[Cor19] P. Cornwell. Opening the Maslov box for traveling waves in skew-gradient systems: counting eigenvalues and proving (in)stability. *Indiana Univ. Math. J.*, 68(6):1801–1832, 2019. 4, 10, 18, 31, 41

[CT93] A R Champneys and J F Toland. Bifurcation of a plethora of multi-modal homoclinic orbits for autonomous hamiltonian systems. *Nonlinearity*, 6(5):665, sep 1993. 2

[DJ11] J. Deng and C. K. R. T. Jones. Multi-dimensional Morse index theorems and a symplectic view of elliptic boundary value problems. *Trans. Amer. Math. Soc.*, 363(3):1487–1508, 2011. 4, 5, 14

[Dui76] J. J. Duistermaat. On the Morse index in variational calculus. *Adv. Math.*, 21(2):173–195, 1976. 18

[Edw64] H. M. Edwards. A generalized Sturm theorem. *Ann. of Math. (2)*, 80:22–57, 1964. 18

[Fur04] K. Furutani. Fredholm-Lagrangian-Grassmannian and the Maslov index. *J. Geom. Phys.*, 51(3):269–331, 2004. 10, 15

[GPP04a] R. Giambò, P. Piccione, and A. Portaluri. Computation of the Maslov index and the spectral flow via partial signatures. *C. R. Math. Acad. Sci. Paris*, 338(5):397–402, 2004. 4, 10, 11, 13, 14

[GPP04b] R. Giambo, P. Piccione, and A. Portaluri. On the Maslov index of symplectic paths that are not transversal to the Maslov cycle. Semi-Riemannian index theorems in the degenerate case. *arXiv preprint arXiv:math/0306187*, 2004. 4, 10, 11, 13, 14, 15

[Gri88] M. Grillakis. Linearized instability for nonlinear Schrödinger and Klein-Gordon equations. *Comm. Pure Appl. Math.*, 41(6):747–774, 1988. 3, 7

[GSS87] M. Grillakis, J. Shatah, and W. Strauss. Stability theory of solitary waves in the presence of symmetry. I. *J. Funct. Anal.*, 74(1):160–197, 1987. 2

[GSS90] M. Grillakis, J. Shatah, and W. Strauss. Stability theory of solitary waves in the presence of symmetry, ii. *J. Funct. Anal.*, 94(2):308–348, 1990. 2, 4

[HLS17] P. Howard, Y. Latushkin, and A. Sukhtayev. The Maslov index for Lagrangian pairs on $\mathbb{R}^{2n}$. *J. Math. Anal. Appl.*, 451(2):794 – 821, 2017. 10

[HLS18] P. Howard, Y. Latushkin, and A. Sukhtayev. The Maslov and Morse indices for system Schrödinger operators on $\mathbb{R}$. *Indiana Univ. Math. J.*, 67(5):1765–1815, 2018. 4, 17, 18, 41

[How21] P. Howard. Hörmander's index and oscillation theory. *J. Math. Anal. Appl.*, 500(1):1–38, 2021. 4, 5, 20

[How23] Peter Howard. The Maslov index and spectral counts for linear Hamiltonian systems on $\mathbb{R}$. *J. Dynam. Differential Equations*, 35(3):1947–1991, 2023. 4, 5, 20, 41

[Jon88] C. K. R. T. Jones. Instability of standing waves for nonlinear Schrödinger-type equations. *Ergodic Theory Dynam. Systems*, 8(Charles Conley Memorial Issue):119–138, 1988. 3, 4, 7

[Kar96] V. I. Karpman. Stabilization of soliton instabilities by higher-order dispersion: Fourth-order nonlinear schrödinger-type equations. *Phys. Rev. E*, 53:R1336–R1339, Feb 1996. 2

[Kat80] T. Kato. *Perturbation Theory for Linear Operators*. Springer-Verlag, Berlin, 1980. 31, 34

[KH94] M. Karlsson and A. Höök. Soliton-like pulses governed by fourth order dispersion in optical fibers. *Optics Communications*, 104(4):303–307, 1994. 2, 5

[KKS04] T. Kapitula, P. G. Kevrekidis, and B. Sandstede. Counting eigenvalues via the Krein signature in infinite-dimensional Hamiltonian systems. *Phys. D*, 195(3-4):263–282, 2004. 3

[KP13] T. Kapitula and K. Promislow. *Spectral and Dynamical Stability of Nonlinear Waves (Vol. 185)*. Springer, New York, 2013. 3, 4, 7, 9

[KS97] V. I. Karpman and A. G. Shagalov. Solitons and their stability in high dispersive systems. I. Fourth-order nonlinear Schrödinger-type equations with power-law nonlinearities. *Phys. Lett. A*, 228(1-2):59–65, 1997. 2, 7

[KS00] V. I. Karpman and A. G. Shagalov. Stability of solitons described by nonlinear Schrödinger-type equations with higher-order dispersion. *Phys. D*, 144(1-2):194–210, 2000. 2, 7

[NP15] F. Natali and A. Pastor. The fourth-order dispersive nonlinear Schrödinger equation: orbital stability of a standing wave. *SIAM J. Appl. Dyn. Syst.*, 14(3):1326–1347, 2015. 2, 8, 39, 40

[PA21] Ross Parker and Alejandro Aceves. Multi-pulse solitary waves in a fourth-order nonlinear Schrödinger equation. *Phys. D*, 422:132890, 2021. 2, 3

[Pel11] D. E. Pelinovsky. *Localization in Periodic Potentials: From Schrödinger Operators to the Gross–Pitaevskii Equation*. Cambridge University Press, New York, 2011. 3, 7

[PSS97] D. Peterhof, B. Sandstede, and A. Scheel. Exponential dichotomies for solitary-wave solutions of semilinear elliptic equations on infinite cylinders. *J. Differential Equations*, 140(2):266–308, 1997. 10

[PT08] P. Piccione and D. V. Tausk. *A student's guide to symplectic spaces, Grassmannians and Maslov index*. Publicações Matemáticas do IMPA. [IMPA Mathematical Publications]. Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, 2008. 4, 8, 11, 12, 13, 15

[Ree22] E. L. Rees. Graphical Discussion of the Roots of a Quartic Equation. *Amer. Math. Monthly*, 29(2):51–55, 1922. 9

[RS93] J. Robbin and D. Salamon. The Maslov index for paths. *Topology*, 32(4):827–844, 1993. 4, 10, 11, 13, 14, 15, 16

[San02] Björn Sandstede. Stability of travelling waves. In *Handbook of dynamical systems, Vol. 2*, pages 983–1055. North-Holland, Amsterdam, 2002. 9

[SS00] Björn Sandstede and Arnd Scheel. Absolute and convective instabilities of waves on unbounded and large bounded domains. *Phys. D*, 145(3-4):233–277, 2000. 41

[TABRdS18] Kevin K. K. Tam, T. J. Alexander, Andrea Blanco-Redondo, and C. Martijn de Sterke. Solitary wave solutions in nonlinear media with quartic and quadratic dispersion—implications for high-power lasers. In *Frontiers in Optics / Laser Science*, page JW4A.78. Optica Publishing Group, 2018. 2

[TABRdS19] Kevin K. K. Tam, Tristram J. Alexander, Andrea Blanco-Redondo, and C. Martijn de Sterke. Stationary and dynamical properties of pure-quartic solitons. *Opt. Lett.*, 44(13):3306–3309, Jul 2019. 2, 5

[TABRdS20] Kevin K. K. Tam, Tristram J. Alexander, Andrea Blanco-Redondo, and C. Martijn de Sterke. Generalized dispersion kerr solitons. *Phys. Rev. A*, 101:043822, Apr 2020. 2, 5

[VK73] N. G. Vakhitov and A. A. Kolokolov. Stationary solutions of the wave equation in a medium with nonlinearity saturation. *Radiophysics and Quantum Electronics*, 16:783–789, 1973. 3, 7

[Wei87] J. Weidmann. *Spectral theory of ordinary differential operators*, volume 1258 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1987. 6