A MIXED FINITE ELEMENT METHOD FOR A CLASS OF FOURTH-ORDER STOCHASTIC EVOLUTION EQUATIONS WITH MULTIPLICATIVE NOISE

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Abstract. We develop a fully discrete, semi-implicit mixed finite element method for approximating solutions to a class of fourth-order stochastic partial differential equations (SPDEs) with non-globally Lipschitz and non-monotone nonlinearities, perturbed by spatially smooth multiplicative Gaussian noise. The proposed scheme is applicable to a range of physically relevant nonlinear models, including the stochastic Landau–Lifshitz–Baryakhtar (sLLBar) equation, the stochastic convective Cahn–Hilliard equation with mass source, and the stochastic regularised Landau–Lifshitz–Bloch (sLLB) equation, among others. To overcome the difficulties posed by the interplay between the nonlinearities and the stochastic forcing, we adopt a 'truncate-then-discretise' strategy: the nonlinear term is first truncated before discretising the resulting modified problem. We show that the strong solution to the truncated problem converges in probability to that of the original problem. A fully discrete numerical scheme is then proposed for the truncated system, and we establish both convergence in probability and strong convergence (with quantitative rates) for the two fields used in the mixed formulation.

Contents

1. Introduction

Motivated by physical applications, we consider the following fourth-order system of nonlinear SPDEs with non-monotone nonlinearities, perturbed by a spatially smooth multiplicative Gaussian noise:

$$\mathrm{d}\boldsymbol{u} = \big(\lambda_1 \boldsymbol{H} - \lambda_2 \Delta \boldsymbol{H} - \gamma \boldsymbol{u} \times \boldsymbol{H} + \mathcal{S}(\boldsymbol{u})\big)\,\mathrm{d}t + G(\boldsymbol{u})\,\mathrm{d}W(t) \quad \text{for } (t,\boldsymbol{x}) \in (0,T)\times\mathscr{D}, \tag{1.1a}$$

$$\boldsymbol{H} = \Delta \boldsymbol{u} + f(\boldsymbol{u}) \quad \text{for } (t,\boldsymbol{x}) \in (0,T)\times\mathscr{D}, \tag{1.1b}$$

$$\boldsymbol{u}(0,\boldsymbol{x}) = \boldsymbol{u}_0(\boldsymbol{x}) \quad \text{for } \boldsymbol{x} \in \mathscr{D}, \tag{1.1c}$$

$$\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} = \boldsymbol{0},\ \frac{\partial \boldsymbol{H}}{\partial \boldsymbol{n}} = \boldsymbol{0} \quad \text{for } (t,\boldsymbol{x}) \in (0,T)\times\partial\mathscr{D}, \tag{1.1d}$$

Date: March 10, 2025.

where $\mathscr{D} \subset \mathbb{R}^d$, $d \le 3$, is a bounded regular domain, and $\boldsymbol{u} : \Omega \times [0,T] \times \mathscr{D} \to \mathbb{R}^3$ is a vector-valued random variable. Here, W is a real-valued Wiener process on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\ge 0})$ with respect to the usual filtration, and $G(\boldsymbol{u})$ is a Lipschitz function of $\boldsymbol{u}$ satisfying certain assumptions (details are elaborated in Section 2.2). The forcing term $f(\boldsymbol{u}) := \kappa\mu\boldsymbol{u} - \kappa|\boldsymbol{u}|^2\boldsymbol{u}$ arises from the Ginzburg–Landau theory, which is the negative variational derivative of $V(\boldsymbol{u}) := \kappa(|\boldsymbol{u}|^2 - \mu)^2/4$, a double-well potential function. Define

$$\mathcal{S}(\boldsymbol{u}) := \mathcal{M}(\boldsymbol{u}) + \mathcal{C}(\boldsymbol{u}),$$

where $\mathcal{M}(\boldsymbol{u})$ is a mass source term with at most quadratic growth and $\mathcal{C}(\boldsymbol{u})$ is a convective term given by

$$\mathcal{C}(\boldsymbol{u}) := \beta_1(\boldsymbol{\nu}\cdot\nabla)\boldsymbol{u} + \beta_2\boldsymbol{u}\times(\boldsymbol{\nu}\cdot\nabla)\boldsymbol{u}, \tag{1.2}$$

where $\boldsymbol{\nu}$ is specified in (2.4). All numerical coefficients are non-negative.

The problem (1.1) describes various problems in physics. When λ_1, λ_2, and γ are positive, problem (1.1) is the *stochastic Landau–Lifshitz–Baryakhtar* (sLLBar) system with spin current [23, 40, 43], which can be seen as a Cahn–Hilliard-type regularisation of the *stochastic Landau–Lifshitz–Bloch* (sLLB) equation in micromagnetics [7, 18, 31, 39]. When $\gamma = 0$, (1.1) is the stochastic bi-flux reaction-diffusion system [5] if $\beta_2 = 0$, a stochastic population growth/dispersal model with long-range effects [12] if $\beta_1 = \beta_2 = 0$, the *Cahn–Hilliard-Cook* (CHC) equation [29] if $\lambda_1 = \beta_1 = \mathcal{M}(\boldsymbol{u}) = 0$ (and the noise is additive), the *stochastic convective Cahn–Hilliard equation with mass source* (sCHm) [36, 32] if $\lambda_1 = \beta_2 = 0$, and the *stochastic convective Allen–Cahn/Cahn–Hilliard* (sAC/CH) equation [1] if $\beta_2 = 0$.

The development of numerical methods for physically relevant SPDEs with non-globally Lipschitz and non-monotone nonlinearities perturbed by multiplicative noise is an active area of research (see e.g. [6, 15, 26, 27] and many others). As (1.1) is a fourth-order equation, a conforming finite element method to solve the equation directly would require C^1-elements, which can be computationally costly. Numerically treating the problem in mixed form allows us to work with C^0-conforming finite elements and use the mixed finite element method (see (3.1)), at the expense of introducing an auxiliary unknown and performing a more delicate analysis. To the best of our knowledge, no numerical scheme has been proposed for the problem (1.1) in its generality, not even for the sLLBar equation (with or without spin current), the sCHm equation, or the sAC/CH equation.

On a related note, several numerical schemes have been proposed in the literature for the CHC equation with *additive* noise, including a C^1-conforming semi-discrete scheme [11], a fully implicit scheme [20], and a fully explicit scheme combined with spectral Galerkin method [9] (see also [19] for gradient-type noise, where strong convergence of an implicit scheme in H^{-1} is shown). Note that even in this setting ($\gamma = \beta_2 = 0$), numerical schemes of implicit/explicit-type proposed here have not been analysed before. Furthermore, adding a mass source and a convective/precession term in (1.1) causes a nontrivial difficulty in the analysis due to the non-conservative mass and the loss of gradient flow structure, which is already encountered even in the deterministic case [30, 38]. On the other hand, setting $\lambda_2 = 0$ in (1.1) gives the sLLB equation. A C^1-conforming finite element scheme for a regularised version of the sLLB equation (which is simpler than our problem (1.1)) is proposed in [22]. As we can consider (1.1) to be a physically relevant regularisation of sLLB, our scheme provides a more practical method to approximate the solution to sLLB by taking λ_2 sufficiently small (in light of the convergence result for sLLBar to sLLB [23, Section 8]).

Regarding the error analysis, we describe here some difficulties at the discrete level that need to be overcome. To this end, let A denote the Neumann Laplacian, A_h the discrete Laplacian, and Π_h the orthogonal projection onto some finite element space $\mathbb{V}_h$. Firstly, loosely speaking, the mixed finite element method aims to approximate $\boldsymbol{u}$ and $\boldsymbol{H}$ simultaneously, using finite element functions which belong to $\mathbb{H}^1$ (but not $\mathbb{H}^2$). This already makes the analysis of the mixed finite element scheme more challenging than its C^1-conforming counterpart, even in the deterministic case. Furthermore, the presence of the finite element projection Π_h in front of the nonlinearities present in (1.1) destroys some dissipativity properties of the continuous problem. For instance, while there exists a $C > 0$ such that for any sufficiently regular function $\boldsymbol{v}$,

$$-\langle f(\boldsymbol{v}), \boldsymbol{v}\rangle \geq -C, \text{ and } -\langle \nabla f(\boldsymbol{v}), \nabla \boldsymbol{v}\rangle \geq -C \|\nabla \boldsymbol{v}\|_{\mathbb{L}^2}^2,$$

we notice that for $\boldsymbol{v}_h \in \mathbb{V}_h$,

$$-\langle \nabla \Pi_h f(\boldsymbol{v}_h), \nabla \boldsymbol{v}_h\rangle \ngeq -C \|\nabla \boldsymbol{v}_h\|_{\mathbb{L}^2}^2.$$

As such, moment bounds for $\|\boldsymbol{u}_h\|_{\mathbb{L}^2}$ and $\|\nabla \boldsymbol{u}_h\|_{\mathbb{L}^2}$, where $\boldsymbol{u}_h$ is the finite element approximation of $\boldsymbol{u}$, are difficult to attain.

Similar difficulties are encountered in [20] and [19] for a simpler model. In their case, however, these issues could be overcome by exploiting the fact that the (scalar-valued) Cahn–Hilliard–Cook equation is the H^{-1}-gradient flow of a Lyapunov functional together with the mass conservation property to derive a moment bound for the H^{-1} norm of the finite element solution. This bound could then be bootstrapped and used to derive moment bounds in stronger norms and strong convergence of the scheme in H^{-1} or L^2 norms. However, it is not clear how to adapt such arguments to our case since (1.1) is *not* a gradient flow in the presence of the cross product term, the forcing term $f(\boldsymbol{u})$, and the convective term $\mathcal{C}(\boldsymbol{u})$. Furthermore, these nonlinearities are non-monotone, thus the general results from [25, 34] do not apply and the analysis needs to proceed differently here. We also remark that the nonlinear term $\Delta f(\boldsymbol{u})$ is absent in the regularised sLLB model considered in [22] and C^1-elements are used there, thus the complications mentioned in the previous and this paragraphs do not appear in the aforementioned paper.

We outline the approach taken in this paper as follows. To overcome the difficulties mentioned before, while still using C^0-conforming elements, we adopt the idea from [37] in the deterministic case by first truncating the potential function so as to have at most quadratic growth at infinity. Such truncation is both physically reasonable and a common practice [8, 13, 16, 28, 42]. In doing so, the forcing function f is approximated by a globally Lipschitz C^2-smooth function f_R. We also truncate the mass source [32]. We show that the strong solution to the problem with truncated potential converges in probability to that of (1.1) as the truncation parameter $R \uparrow \infty$. Note that even after these modifications, the problem (1.1) is still a system of SPDEs with non-monotone and non-globally Lipschitz nonlinearities due to the cross product term $\boldsymbol{u} \times \boldsymbol{H}$ and the non-variational term $\mathcal{C}(\boldsymbol{u})$, thus the analysis is not straightforward.

A mixed finite element method is proposed to solve the truncated problem. In the absence of cross product terms (the case of sCHm or sAC/CH equations), our fully discrete scheme is of IMEX-type. Error analysis based on the variational approach to SPDEs is then performed for this scheme, where convergence in probability and strong convergence with a rate are proven. This is achieved by establishing stability estimates in strong norms and showing error estimates localised to a sample space with large probability, following the approach in [4, 10]. The main results are stated in Theorem 3.7, Theorem 3.8, and Theorem 3.9.

2. Preliminaries

2.1. **Notations.** We begin by defining some notations used in this paper. Let $\mathscr{D}$ be a convex Lipschitz domain or a domain with C^2-smooth boundary. The function space $\mathbb{L}^p := \mathbb{L}^p(\mathscr{D};\mathbb{R}^3)$ denotes the usual space of p-th integrable functions defined on $\mathscr{D}$ and taking values in $\mathbb{R}^3$, and $\mathbb{W}^{k,p} := \mathbb{W}^{k,p}(\mathscr{D};\mathbb{R}^3)$ denotes the usual Sobolev space of functions on $\mathscr{D} \subset \mathbb{R}^d$ taking values in $\mathbb{R}^3$. We write $\mathbb{H}^k := \mathbb{W}^{k,2}$. The partial derivative $\partial/\partial x_i$ will be written by ∂_i for short. The partial derivative of f with respect to time t will be denoted by ∂_t. The operator Δ denotes the Neumann Laplacian acting on $\mathbb{R}^3$-valued functions with domain

$$\mathrm{D}(\Delta) := \left\{ \boldsymbol{v} \in \mathbb{H}^2 : \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} = 0 \text{ on } \partial\mathscr{D} \right\}.$$

If X is a Banach space, $L^p(0,T;X)$ and $W^{k,p}(0,T;X)$ denote respectively the usual Lebesgue and Sobolev spaces of functions on $(0,T)$ taking values in X. The space $C([0,T];X)$ denotes the space of continuous functions on $[0,T]$ taking values in X. The space $L^p(\Omega;X)$ denotes the space of X-valued random variable with finite p-th moment, where $(\Omega,\mathcal{F},\mathbb{P})$ is a probability space.

Throughout this paper, we denote the scalar product in a Hilbert space H by $\langle\cdot,\cdot\rangle_H$ and its corresponding norm by $\|\cdot\|_H$. The expectation of a random variable Y will be denoted by $\mathbb{E}[Y]$. We will not distinguish between the scalar product of $\mathbb{L}^2$ vector-valued functions taking values in $\mathbb{R}^3$ and the scalar product of $\mathbb{L}^2$ matrix-valued functions taking values in $\mathbb{R}^{3\times 3}$, and denote them by $\langle\cdot,\cdot\rangle$.

In various estimates, the constant C in the estimate denotes a generic constant which takes different values at different occurrences. If the dependence of C on some variables, e.g. R and T, is highlighted, we will write $C_{R,T}$.

2.2. **Assumptions.** Let $\varphi_R\in C_c^2(\mathbb{R}^3)$ be a C^2-smooth bump function whose support lies in the closed ball $B_{2R}(\mathbf{0})$, such that

$$\varphi_R(\boldsymbol{x})=\begin{cases}1, & \text{if } |\boldsymbol{x}|\le R\\ 0, & \text{if } |\boldsymbol{x}|\ge 2R.\end{cases}\tag{2.1}$$

The functions $f_R:\mathbb{R}^3\to\mathbb{R}^3$ and $\mathcal{M}_R:\mathbb{R}^3\to\mathbb{R}^3$ are defined by

$$f_R(\boldsymbol{u}):=\varphi_R(\boldsymbol{u})f(\boldsymbol{u}),\tag{2.2}$$

$$\mathcal{M}_R(\boldsymbol{u}):=\varphi_R(\boldsymbol{u})\mathcal{M}(\boldsymbol{u}),\tag{2.3}$$

For equation (1.1), we set $\lambda_1=\lambda_2=\kappa=\mu=1$ for simplicity. We further assume the following:

(1) The map $G:\mathbb{R}^3\to\mathbb{R}^3$ is Lipschitz continuous with Lipschitz constant C_G. Moreover, there exists a constant $C>0$ such that

$$\|\nabla G(\boldsymbol{v})-\nabla G(\boldsymbol{w})\|_{\mathbb{L}^2}\le C\,\|\nabla\boldsymbol{v}-\nabla\boldsymbol{w}\|_{\mathbb{L}^2},\quad \forall\boldsymbol{v},\boldsymbol{w}\in\mathbb{H}^1,$$
$$\|G(\boldsymbol{v})\|_{\mathbb{H}^2}\le C\big(1+\|\boldsymbol{v}\|_{\mathbb{H}^2}\big),\quad \forall\boldsymbol{v}\in\mathbb{H}^2.$$

(2) The spin current vector field $\boldsymbol{\nu}\in L^\infty\big(\mathbb{R}^+;\mathbb{L}^\infty(\mathscr{D};\mathbb{R}^d)\big)$ is given. For simplicity, set

$$\|\boldsymbol{\nu}\|_{L^\infty(\mathbb{R}^+;\mathbb{L}^\infty(\mathscr{D};\mathbb{R}^d))}=1.\tag{2.4}$$

We remark that our results are also valid for noise of the form $G(\boldsymbol{u})\,\mathrm{d}\boldsymbol{W}$, where $\boldsymbol{W}$ is an $\mathbb{H}^2$-valued Q-Wiener process of the form $\boldsymbol{W}(t)=\sum_{k=1}^\infty\sqrt{q_k}\boldsymbol{e}_kW_k(t)$ and $\{W_k\}$ is a family of real-valued independent Brownian motions. Here, $\{\boldsymbol{e}_k\}$ is an orthonormal basis of $\mathbb{H}^2$ such that $Q\boldsymbol{e}_k=q_k\boldsymbol{e}_k$ and $\sum_{j=1}^\infty q_j<\infty$. In this case, we assume $G:\mathbb{H}^2\to\mathcal{L}(\mathbb{H}^2)$ is Lipschitz continuous with sublinear growth. The assumptions here cover the case where $G(\boldsymbol{u})$ is the identity operator (additive noise) or the noise term given in [23] for the sLLBar equation. We consider just a single real-valued Brownian motion in this paper for simplicity of presentation.

2.3. **Existence, uniqueness, and regularity of solution.** The existence, uniqueness, and regularity of the (probabilistically and analytically) strong solution to the problem (1.1) are studied in [23]. We summarise the relevant results here. First, define a self-adjoint operator

$$A:=\Delta^2-\Delta,\quad \text{with } \mathrm{D}(A)=\mathrm{D}(\Delta^2).$$

The following theorem is shown in [23].

Theorem 2.1. Let $\boldsymbol{u}_0\in\mathrm{D}(A^{\frac12})$ and $T>0$ be given. Let $\beta\in[\frac12,1)$ and $\alpha\in(0,1-\beta)$. There exists a unique solution $\boldsymbol{u}$ of the problem (1.1) with regularity

$$\boldsymbol{u}\in L^p\big(\Omega;C^\alpha(0,T;\mathrm{D}(A^\beta))\big)\cap L^p\big(\Omega;L^2(0,T;\mathrm{D}(A))\big).$$

for any $p\ge 1$.

Consider now the problem (1.1) with truncated nonlinearities, where we set all numerical coefficients to be 1; namely for each $R > 0$, $(\boldsymbol{u}_R, \boldsymbol{H}_R)$ satisfies

$$\mathrm{d}\boldsymbol{u}_R = \big(\boldsymbol{H}_R - \Delta \boldsymbol{H}_R - \boldsymbol{u}_R \times \boldsymbol{H}_R + \mathcal{P}(\boldsymbol{u}_R)\big)\,\mathrm{d}t + G(\boldsymbol{u}_R)\,\mathrm{d}W(t) \quad \text{for } (t,\boldsymbol{x}) \in (0,T)\times\mathscr{D}, \tag{2.5a}$$

$$\boldsymbol{H}_R = \Delta \boldsymbol{u}_R + f_R(\boldsymbol{u}_R) \quad \text{for } (t,\boldsymbol{x}) \in (0,T)\times\mathscr{D}, \tag{2.5b}$$

$$\boldsymbol{u}_R(0,\boldsymbol{x}) = \boldsymbol{u}_0(\boldsymbol{x}) \quad \text{for } \boldsymbol{x} \in \mathscr{D}, \tag{2.5c}$$

$$\frac{\partial \boldsymbol{u}_R}{\partial \boldsymbol{n}} = \boldsymbol{0},\ \frac{\partial \boldsymbol{H}_R}{\partial \boldsymbol{n}} = \boldsymbol{0} \quad \text{for } (t,\boldsymbol{x}) \in (0,T)\times\partial\mathscr{D}, \tag{2.5d}$$

i.e. the problem (1.1) with $f(\boldsymbol{u})$ replaced by $f_R(\boldsymbol{u}_R)$, and

$$\mathcal{P}(\boldsymbol{u}_R) := \mathcal{C}(\boldsymbol{u}_R) + \mathcal{M}_R(\boldsymbol{u}_R). \tag{2.6}$$

A variational formulation for the problem (2.5) can be written as follows: For every $t \in [0,T]$ and $\mathbb{P}$-a.s., $(\boldsymbol{u}_R, \boldsymbol{H}_R)$ solves

$$\begin{aligned} \langle \boldsymbol{u}_R(t), \boldsymbol{\chi}\rangle &= \langle \boldsymbol{u}_0, \boldsymbol{\chi}\rangle + \int_0^t \langle \boldsymbol{H}_R(s), \boldsymbol{\chi}\rangle\,\mathrm{d}s + \int_0^t \langle \nabla \boldsymbol{H}_R(s), \nabla\boldsymbol{\chi}\rangle\,\mathrm{d}s \\ &\quad - \int_0^t \langle \boldsymbol{u}_R(s)\times \boldsymbol{H}(s), \boldsymbol{\chi}\rangle\,\mathrm{d}s + \int_0^t \langle \mathcal{P}(\boldsymbol{u}_R(s)), \boldsymbol{\chi}\rangle\,\mathrm{d}s \\ &\quad + \int_0^t \langle G(\boldsymbol{u}_R(s)), \boldsymbol{\chi}\rangle\,\mathrm{d}W(s), \\ \langle \boldsymbol{H}_R(t), \boldsymbol{\phi}\rangle &= -\langle \nabla \boldsymbol{u}_R(t), \nabla\boldsymbol{\phi}\rangle + \langle f_R(\boldsymbol{u}_R(t)), \boldsymbol{\phi}\rangle, \end{aligned} \tag{2.7}$$

for all $\boldsymbol{\chi}, \boldsymbol{\phi} \in \mathbb{H}^1$. This variational formulation will be used for the analysis of the numerical method proposed in Section 3. By similar argument as in [23, Theorem 2.5], we have the following result.

Proposition 2.2. Let $\boldsymbol{u}_0 \in \mathrm{D}(A^{\frac{1}{2}})$ and $T > 0$ be given. Let $\beta \in [\frac{1}{2}, 1)$ and $\alpha \in (0, 1-\beta)$. For each $R > 0$, there exists a unique pathwise solution $\boldsymbol{u}_R$ of the problem (2.5) with regularity

$$\boldsymbol{u}_R \in L^p\big(\Omega; C^\alpha(0,T;\mathrm{D}(A^\beta))\big) \cap L^p\big(\Omega; L^2(0,T;\mathrm{D}(A))\big).$$

for any $p \geq 1$.

Proposition 2.3. Let $\boldsymbol{u}_R$ and $\boldsymbol{u}$ be the solution to the problems (1.1) and (2.5), respectively. Then $\boldsymbol{u}_R(t) \to \boldsymbol{u}(t)$ a.s. on $[0,T]$ as $R \uparrow \infty$. Furthermore, for any $\varepsilon > 0$ and $p \geq 1$,

$$\mathbb{P}\left[\sup_{t\in[0,T]} \|\boldsymbol{u}_R(t) - \boldsymbol{u}(t)\|_{\mathbb{H}^2} > \varepsilon\right] \leq C_p R^{-p},$$

where C_p is a constant depending on T, p, and $\mathscr{D}$.

Proof. For each $R > 0$, let

$$\begin{aligned} \tau_R &:= \inf\{t \geq 0 : \|\boldsymbol{u}_R(t)\|_{\mathbb{H}^2} \geq R\} \wedge T, \\ \sigma_R &:= \inf\{t \geq 0 : \|\boldsymbol{u}(t)\|_{\mathbb{H}^2} \geq R\} \wedge T. \end{aligned}$$

Then $\boldsymbol{u}_R(t) = \boldsymbol{u}(t)$ a.s. for all $t \leq \tau_R$, and $\tau_R \uparrow T$ as $R \uparrow \infty$, which implies $\boldsymbol{u}_R(t) \to \boldsymbol{u}(t)$ a.s. on $[0,T]$ as $R \uparrow \infty$. Furthermore, for any $\varepsilon > 0$,

$$\begin{aligned} \mathbb{P}\left[\sup_{t\in[0,T]} \|\boldsymbol{u}_R(t) - \boldsymbol{u}(t)\|_{\mathbb{H}^2} > \varepsilon\right] &\leq \mathbb{P}\left[\sup_{t\in[0,\tau_R]} \|\boldsymbol{u}_R(t) - \boldsymbol{u}(t)\|_{\mathbb{H}^2} > \varepsilon\right] + \mathbb{P}\left[\tau_R < T\right] \\ &= \mathbb{P}\left[\{\tau_R < T\} \cap \{\sigma_R = T\}\right] + \mathbb{P}\left[\{\tau_R < T\}\cap\{\sigma_R < T\}\right] \\ &\leq \frac{1}{R^p}\mathbb{E}\left[\|\boldsymbol{u}\|^p_{L^\infty(\mathbb{H}^2)}\right], \end{aligned}$$

as required. □

As such, we can approximate the solution to the problem (1.1) using (2.5) by taking R sufficiently large. From this point onwards, we will focus on the numerical approximation of (2.5) and write $\boldsymbol{u}$ in place of $\boldsymbol{u}_R$.

2.4. **Finite element approximation.** Let $\mathcal{T}_h$ be a quasi-uniform triangulation of $\mathscr{D} \subset \mathbb{R}^d$ with maximal mesh-size h, and let $\mathbb{V}_h \subset \mathbb{W}^{1,\infty}$ be the Lagrange finite element space

$$\mathbb{V}_h := \{\boldsymbol{\phi} \in C(\overline{\mathscr{D}}; \mathbb{R}^3) : \boldsymbol{\phi}|_K \in \mathbb{P}_1(K; \mathbb{R}^3),\ \forall K \in \mathcal{T}_h\},$$

where $\mathbb{P}_1(K; \mathbb{R}^3)$ denotes the space of linear polynomials on K taking values in $\mathbb{R}^3$. Let $T > 0$ be fixed and k be the time-step size. Furthermore, let $\boldsymbol{u}_h^n$ and $\boldsymbol{H}_h^n$, respectively, be the approximation in $\mathbb{V}_h$ of $\boldsymbol{u}_R(t)$ and $\boldsymbol{H}_R(t)$ at time $t = t_n := nk$, where $n = 0, 1, 2, \dots, N$ and $N = \lfloor T/k \rfloor$.

We begin by defining several operators which will be used in the analysis. Firstly, there exists an orthogonal projection operator $\Pi_h : \mathbb{L}^2 \to \mathbb{V}_h$ such that

$$\langle \Pi_h \boldsymbol{v} - \boldsymbol{v}, \boldsymbol{\chi} \rangle = 0, \quad \forall \boldsymbol{\chi} \in \mathbb{V}_h. \tag{2.8}$$

The operator Π_h is stable [3, 14, 17] in $\mathbb{L}^p$ and $\mathbb{W}^{1,p}$ for $p \in (1, \infty)$: there exists a constant C independent of $\boldsymbol{v}$ such that

$$\|\Pi_h \boldsymbol{v}\|_{\mathbb{L}^p} \le C \|\boldsymbol{v}\|_{\mathbb{L}^p}, \quad \forall \boldsymbol{v} \in \mathbb{L}^p, \tag{2.9}$$

$$\|\nabla \Pi_h \boldsymbol{v}\|_{\mathbb{L}^p} \le C \|\nabla \boldsymbol{v}\|_{\mathbb{L}^p}, \quad \forall \boldsymbol{v} \in \mathbb{W}^{1,p}. \tag{2.10}$$

Moreover, it has the following approximation property:

$$\|\boldsymbol{v} - \Pi_h \boldsymbol{v}\|_{\mathbb{L}^p} + h \|\nabla(\boldsymbol{v} - \Pi_h \boldsymbol{v})\|_{\mathbb{L}^p} \le Ch^2 \|\boldsymbol{v}\|_{\mathbb{W}^{2,p}}. \tag{2.11}$$

We mainly use (2.9), (2.10), and (2.11) for $p = 2$.

Secondly, define the Ritz projection $\mathcal{R}_h : \mathbb{H}^1 \to \mathbb{V}_h$ by

$$\langle \nabla \mathcal{R}_h \boldsymbol{v} - \nabla \boldsymbol{v}, \nabla \boldsymbol{\chi} \rangle = 0, \quad \forall \boldsymbol{\chi} \in \mathbb{V}_h, \tag{2.12}$$

such that $\langle \mathcal{R}_h \boldsymbol{v} - \boldsymbol{v}, \boldsymbol{1} \rangle = 0$. The stability and approximation properties of the Ritz projection [33, 35] are assumed to hold. In particular, for $p \in (1, \infty)$,

$$\|\boldsymbol{v} - \mathcal{R}_h \boldsymbol{v}\|_{\mathbb{L}^p} + h \|\nabla(\boldsymbol{v} - \mathcal{R}_h \boldsymbol{v})\|_{\mathbb{L}^p} \le Ch^2 \|\boldsymbol{v}\|_{\mathbb{W}^{2,p}}. \tag{2.13}$$

Finally, the discrete Laplacian operator $\Delta_h : \mathbb{V}_h \to \mathbb{V}_h$ is defined by

$$\langle \Delta_h \boldsymbol{v}_h, \boldsymbol{\chi} \rangle = -\langle \nabla \boldsymbol{v}_h, \nabla \boldsymbol{\chi} \rangle, \quad \forall \boldsymbol{v}_h, \boldsymbol{\chi} \in \mathbb{V}_h. \tag{2.14}$$

Consequently, for any $p, q \in [1, \infty]$ such that $1/p + 1/q = 1$, by Hölder's inequality we have

$$\|\nabla \boldsymbol{v}_h\|_{\mathbb{L}^2}^2 \le \|\boldsymbol{v}_h\|_{\mathbb{L}^p} \|\Delta_h \boldsymbol{v}_h\|_{\mathbb{L}^q}, \tag{2.15}$$

$$\|\Delta_h \boldsymbol{v}_h\|_{\mathbb{L}^2}^2 \le \|\nabla \boldsymbol{v}_h\|_{\mathbb{L}^p} \|\nabla \Delta_h \boldsymbol{v}_h\|_{\mathbb{L}^q}. \tag{2.16}$$

2.5. **Identities and inequalities.** Some identities and inequalities that are frequently used in the analysis are collected in this section. Recall that $f(\boldsymbol{v}) = \boldsymbol{v} - |\boldsymbol{v}|^2 \boldsymbol{v}$, where $\boldsymbol{v} : \mathscr{D} \to \mathbb{R}^3$. For φ_R and f_R defined in (2.1) and (2.2), respectively, we have the following identities:

$$\nabla f(\boldsymbol{v}) = \nabla \boldsymbol{v} - 2\boldsymbol{v}(\boldsymbol{v} \cdot \nabla \boldsymbol{v}) - |\boldsymbol{v}|^2 \nabla \boldsymbol{v}, \tag{2.17}$$

$$\Delta f(\boldsymbol{v}) = \Delta \boldsymbol{v} - 2|\nabla \boldsymbol{v}|^2 \boldsymbol{v} - 2(\boldsymbol{v} \cdot \Delta \boldsymbol{v})\boldsymbol{v} - 4\nabla \boldsymbol{v}\ (\boldsymbol{v} \cdot \nabla \boldsymbol{v})^\top - |\boldsymbol{v}|^2 \Delta \boldsymbol{v}, \tag{2.18}$$

$$\nabla f_R(\boldsymbol{v}) = \nabla\big[\varphi_R(\boldsymbol{v})\big]\big(f(\boldsymbol{v})\big)^\top + \varphi_R(\boldsymbol{v}) \nabla\big[f(\boldsymbol{v})\big], \tag{2.19}$$

$$\Delta f_R(\boldsymbol{v}) = \varphi_R(\boldsymbol{v}) \Delta\big[f(\boldsymbol{v})\big] + f(\boldsymbol{v}) \Delta\big[\varphi_R(\boldsymbol{v})\big] + 2\nabla\big[f(\boldsymbol{v})\big] \cdot \nabla\big[\varphi_R(\boldsymbol{v})\big]^\top. \tag{2.20}$$

$$\Delta \varphi_R(\boldsymbol{v}) = \big(D\varphi_R(\boldsymbol{v})\big)\Delta \boldsymbol{v} + \sum_{i=1}^{d} \big(\partial_i \boldsymbol{v}\big)^\top \big(D^2 \varphi_R(\boldsymbol{v})\big)\big(\partial_i \boldsymbol{v}\big). \tag{2.21}$$

where $D\varphi_R(\boldsymbol{v})$ and $D^2\varphi_R(\boldsymbol{v})$ denotes, respectively, the Jacobian and the Hessian of φ_R evaluated at $\boldsymbol{v}$.

Lemma 2.4. Let $\mathscr{D} \subset \mathbb{R}^d$ be an open bounded domain with Lipschitz boundary and $\epsilon > 0$ be given. Then there exists a positive constant C such that the following inequalities hold:

(i) For any $\boldsymbol{v} \in \mathbb{H}^1$ and $p \in (2,6)$,
$$\|\boldsymbol{v}\|_{\mathbb{L}^p} \le C_\epsilon \|\boldsymbol{v}\|_{\mathbb{L}^2}^2 + \epsilon \|\nabla \boldsymbol{v}\|_{\mathbb{L}^2}^2 . \tag{2.22}$$

(ii) For any $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{H}^s$, where $s > d/2$,
$$\|\boldsymbol{v} \odot \boldsymbol{w}\|_{\mathbb{H}^s} \le C \|\boldsymbol{v}\|_{\mathbb{H}^s} \|\boldsymbol{w}\|_{\mathbb{H}^s} , \tag{2.23}$$
Here $\odot$ denotes either the dot product or cross product.

(iii) Let $\mathscr{D}$ be a convex polygonal or polyhedral domain with globally quasi-uniform triangulation. Let Δ_h be the discrete Laplacian operator defined in (2.14). For any $\boldsymbol{v}_h \in \mathbb{V}_h$,
$$\|\boldsymbol{v}_h\|_{\mathbb{L}^\infty} \le C \|\boldsymbol{v}_h\|_{\mathbb{L}^2}^{1-\frac{d}{4}} \left(\|\boldsymbol{v}_h\|_{\mathbb{L}^2}^{\frac{d}{4}} + \|\Delta_h \boldsymbol{v}_h\|_{\mathbb{L}^2}^{\frac{d}{4}} \right), \tag{2.24}$$
$$\|\nabla \boldsymbol{v}_h\|_{\mathbb{L}^6} \le C \|\Delta_h \boldsymbol{v}_h\|_{\mathbb{L}^2} . \tag{2.25}$$

Proof. Inequality (2.22) follows from the Gagliardo–Nirenberg inequalities. Inequality (2.23) is shown in [41, Lemma 2.2]. The estimates (2.24) and (2.25) are shown in [24, Appendix A]). ☐

Lemma 2.5. Let φ_R and f_R be the maps defined in (2.1) and (2.2), respectively. Let $p,q \in [1,\infty]$. Then there exists a positive constant C_R, depending on R, such that the following inequalities hold:

(1) For any $\boldsymbol{v} : \mathscr{D} \to \mathbb{R}^3$,
$$\|\nabla f_R(\boldsymbol{v})\|_{\mathbb{L}^p} \le C_R \|\nabla \boldsymbol{v}\|_{\mathbb{L}^p} , \tag{2.26}$$
$$\|\Delta f_R(\boldsymbol{v})\|_{\mathbb{L}^p} \le C_R \left(\|\nabla \boldsymbol{v}\|_{\mathbb{L}^{2p}}^2 + \|\Delta \boldsymbol{v}\|_{\mathbb{L}^p} \right). \tag{2.27}$$

(2) Suppose that $1/p + 1/q = 1/2$. For any $\boldsymbol{v} : \mathscr{D} \to \mathbb{R}^3$ and $\boldsymbol{w} : \mathscr{D} \to \mathbb{R}^3$,
$$\begin{aligned} \|\nabla f_R(\boldsymbol{v}) - \nabla f_R(\boldsymbol{w})\|_{\mathbb{L}^2} &\le C_R \left(1 + \|\boldsymbol{v}\|_{\mathbb{L}^\infty}^3\right) \|\nabla \boldsymbol{v} - \nabla \boldsymbol{w}\|_{\mathbb{L}^2} \\ &\quad + C_R \left(1 + \|\boldsymbol{v}\|_{\mathbb{L}^\infty}^3\right) \|\nabla \boldsymbol{v}\|_{\mathbb{L}^p} \|\boldsymbol{v} - \boldsymbol{w}\|_{\mathbb{L}^q} . \end{aligned} \tag{2.28}$$

Proof. Firstly, by (2.17) and (2.19) it is clear that we have
$$|\nabla f_R(\boldsymbol{v})| \le |D\varphi_R(\boldsymbol{v})| \, |\nabla \boldsymbol{v}| \, |f(\boldsymbol{v})| + |\varphi_R(\boldsymbol{v})| \, |\boldsymbol{v}|^2 \, |\nabla \boldsymbol{v}| \le C_R \, |\nabla \boldsymbol{v}| ,$$
which implies (2.26). Similarly, noting (2.18), (2.20), and (2.21) we have
$$\begin{aligned} |\Delta f_R(\boldsymbol{v})| &\le |\varphi_R(\boldsymbol{v})| \left(|\Delta \boldsymbol{v}| + 6 \, |\nabla \boldsymbol{v}|^2 \, |\boldsymbol{v}| + 3 \, |\boldsymbol{v}|^2 \, |\Delta \boldsymbol{v}| \right) \\ &\quad + \left(|\boldsymbol{v}| + |\boldsymbol{v}|^3 \right) \left(|D\varphi_R(\boldsymbol{v})| \, |\Delta \boldsymbol{v}| + d \, |\nabla \boldsymbol{v}|^2 \left| D^2 \varphi_R(\boldsymbol{v}) \right| \right) \\ &\quad + |D\varphi_R(\boldsymbol{v})| \, |\nabla \boldsymbol{v}| \left(|\nabla \boldsymbol{v}| + 3 \, |\boldsymbol{v}|^2 \, |\nabla \boldsymbol{v}| \right) \\ &\le C_R \left(|\nabla \boldsymbol{v}|^2 + |\Delta \boldsymbol{v}| \right), \end{aligned}$$
which implies (2.27). Next, using (2.19) again, we have
$$\begin{aligned} |\nabla f_R(\boldsymbol{v}) - \nabla f_R(\boldsymbol{w})| &\le \left| \nabla \left[\varphi_R(\boldsymbol{v}) - \varphi_R(\boldsymbol{w}) \right] \right| \, |f(\boldsymbol{v})| + \left| \nabla \left[\varphi_R(\boldsymbol{w}) \right] \right| \, |f(\boldsymbol{v}) - f(\boldsymbol{w})| \\ &\quad + |\varphi_R(\boldsymbol{v}) - \varphi_R(\boldsymbol{w})| \left| \nabla \left[f(\boldsymbol{v}) \right] \right| + |\varphi_R(\boldsymbol{w})| \left| \nabla \left[f(\boldsymbol{v}) - f(\boldsymbol{w}) \right] \right| . \end{aligned} \tag{2.29}$$
We now estimate each term on the right-hand side. Firstly,
$$\left| \nabla \left[\varphi_R(\boldsymbol{v}) - \varphi_R(\boldsymbol{w}) \right] \right| \, |f(\boldsymbol{v})| \le |D\varphi_R(\boldsymbol{w})| \, |\nabla \boldsymbol{v} - \nabla \boldsymbol{w}| \, |f(\boldsymbol{v})| + |D\varphi_R(\boldsymbol{w}) - D\varphi_R(\boldsymbol{v})| \, |\nabla \boldsymbol{v}| \, |f(\boldsymbol{v})|$$

$$\leq C_R \left(1 + |\boldsymbol{v}|^3\right) |\nabla \boldsymbol{v} - \nabla \boldsymbol{w}| + C_R \left(1 + |\boldsymbol{v}|^3\right) |\nabla \boldsymbol{v}| \, |\boldsymbol{v} - \boldsymbol{w}| \, .$$

Similarly, we have

$$\begin{aligned}\left|\nabla\big[\varphi_R(\boldsymbol{w})\big]\right| |f(\boldsymbol{v}) - f(\boldsymbol{w})| &\leq |D\varphi_R(\boldsymbol{w})| \, |\nabla \boldsymbol{w} - \nabla \boldsymbol{v}| \, |f(\boldsymbol{v}) - f(\boldsymbol{w})| + |D\varphi_R(\boldsymbol{w})| \, |\nabla \boldsymbol{v}| \, |f(\boldsymbol{v}) - f(\boldsymbol{w})| \\ &\leq C_R \left(1 + |\boldsymbol{v}|^3\right) |\nabla \boldsymbol{v} - \nabla \boldsymbol{w}| + C_R \left(1 + |\boldsymbol{v}|^2\right) |\nabla \boldsymbol{v}| \, |\boldsymbol{v} - \boldsymbol{w}| \, ,\end{aligned}$$

and

$$|\varphi_R(\boldsymbol{v}) - \varphi_R(\boldsymbol{w})| \left|\nabla\big[f(\boldsymbol{v})\big]\right| \leq C_R \left(1 + |\boldsymbol{v}|^2\right) |\nabla \boldsymbol{v}| \, |\boldsymbol{v} - \boldsymbol{w}| \, .$$

Finally, for the last term, we note that

$$\begin{aligned}\nabla f(\boldsymbol{v}) - \nabla f(\boldsymbol{w}) = (\nabla \boldsymbol{v} - \nabla \boldsymbol{w}) &- 2\Big(\boldsymbol{v}((\boldsymbol{v} - \boldsymbol{w}) \cdot \nabla \boldsymbol{v}) + (\boldsymbol{v} - \boldsymbol{w})(\boldsymbol{w} \cdot \nabla \boldsymbol{v}) + \boldsymbol{w}(\boldsymbol{w} \cdot (\nabla \boldsymbol{v} - \nabla \boldsymbol{w}))\Big) \\ &- \Big(((\boldsymbol{v} - \boldsymbol{w}) \cdot (\boldsymbol{v} + \boldsymbol{w}))\nabla \boldsymbol{v} + |\boldsymbol{w}|^2 (\nabla \boldsymbol{v} - \nabla \boldsymbol{w})\Big) \, .\end{aligned}$$

This implies

$$|\varphi_R(\boldsymbol{w})| \left|\nabla\big[f(\boldsymbol{v}) - f(\boldsymbol{w})\big]\right| \leq C_R \, |\nabla \boldsymbol{v} - \nabla \boldsymbol{w}| + C_R \, (1 + |\boldsymbol{v}|) \, |\nabla \boldsymbol{v}| \, |\boldsymbol{v} - \boldsymbol{w}| \, .$$

Thus, continuing from (2.29) we obtain

$$|\nabla f_R(\boldsymbol{v}) - \nabla f_R(\boldsymbol{w})| \leq C_R \left(1 + |\boldsymbol{v}|^3\right) |\nabla \boldsymbol{v} - \nabla \boldsymbol{w}| + C_R \left(1 + |\boldsymbol{v}|^3\right) |\nabla \boldsymbol{v}| \, |\boldsymbol{v} - \boldsymbol{w}| \, ,$$

from which (2.28) follows by Hölder's inequality. ☐

Lemma 2.6. Let $\mathcal{C}$ be the map defined in (1.2) and $\boldsymbol{\nu}$ be given. For each $\epsilon > 0$, there exists a positive constant C such that for any $\boldsymbol{v} \in \mathbb{H}^1$ and $\boldsymbol{w} \in \mathbb{L}^\infty$,

$$\left| \left\langle \mathcal{C}(\boldsymbol{v}), \boldsymbol{v} \right\rangle_{\mathbb{L}^2} \right| \leq C \left(1 + \|\boldsymbol{\nu}\|_{\mathbb{L}^2(\mathscr{D};\mathbb{R}^d)}^2\right) \|\boldsymbol{v}\|_{\mathbb{L}^2}^2 + \epsilon \, \|\nabla \boldsymbol{v}\|_{\mathbb{L}^2}^2 \, , \tag{2.30}$$

$$\left| \left\langle \mathcal{C}(\boldsymbol{v}), \boldsymbol{w} \right\rangle_{\mathbb{L}^2} \right| \leq C \left(1 + \|\boldsymbol{\nu}\|_{\mathbb{L}^\infty(\mathscr{D};\mathbb{R}^d)}^2\right) \left(1 + \|\boldsymbol{w}\|_{\mathbb{L}^\infty}^2\right) \|\boldsymbol{v}\|_{\mathbb{L}^2}^2 + \epsilon \, \|\nabla \boldsymbol{v}\|_{\mathbb{L}^2}^2 \, . \tag{2.31}$$

Now, let $p, q, r \in [1, \infty]$ be such that $1/p + 1/q + 1/r = 1$. For each $\epsilon > 0$, there exists a positive constant C such that for any $\boldsymbol{v} \in \mathbb{W}^{1,q} \cap L^p$ and $\boldsymbol{w} \in \mathbb{L}^r$,

$$\left| \left\langle \mathcal{C}(\boldsymbol{v}), \boldsymbol{w} \right\rangle_{\mathbb{L}^2} \right| \leq C \, \|\boldsymbol{\nu}\|_{\mathbb{L}^\infty(\mathscr{D};\mathbb{R}^d)}^2 \left(1 + \|\boldsymbol{v}\|_{\mathbb{L}^p}^2\right) \|\nabla \boldsymbol{v}\|_{\mathbb{L}^q}^2 + \epsilon \, \|\boldsymbol{w}\|_{\mathbb{L}^r}^2 \, . \tag{2.32}$$

Proof. Inequalities (2.30) and (2.31) follow directly from Young's inequality and the fact that $\mathcal{M}_R$ is Lipschitz. To show (2.32), we apply Hölder's and Young's inequalities to obtain

$$\begin{aligned}\left| \left\langle \mathcal{C}(\boldsymbol{v}), \boldsymbol{w} \right\rangle_{\mathbb{L}^2} \right| &\leq C \, \|\boldsymbol{\nu}\|_{\mathbb{L}^p(\mathscr{D};\mathbb{R}^d)} \, \|\nabla \boldsymbol{v}\|_{\mathbb{L}^q} \, \|\boldsymbol{w}\|_{\mathbb{L}^r} + \|\boldsymbol{\nu}\|_{\mathbb{L}^\infty(\mathscr{D};\mathbb{R}^d)} \, \|\boldsymbol{v}\|_{\mathbb{L}^p} \, \|\nabla \boldsymbol{v}\|_{\mathbb{L}^q} \, \|\boldsymbol{w}\|_{\mathbb{L}^r} \\ &\leq C \, \|\boldsymbol{\nu}\|_{\mathbb{L}^\infty(\mathscr{D};\mathbb{R}^d)}^2 \left(1 + \|\boldsymbol{v}\|_{\mathbb{L}^p}^2\right) \|\nabla \boldsymbol{v}\|_{\mathbb{L}^q}^2 + \epsilon \, \|\boldsymbol{w}\|_{\mathbb{L}^r}^2 \, ,\end{aligned}$$

as required. This completes the proof of the lemma. ☐

3. A fully-discrete mixed finite element method

In this section, we propose a mixed finite element method for (1.1) with partially implicit Euler time-stepping. We start with $\boldsymbol{u}_h^0 = \Pi_h \boldsymbol{u}(0) \in \mathbb{V}_h$. Let $t_n \in [0, T]$, where $n \in \{1, 2, \dots, N\}$ and $N = \lfloor T/k \rfloor$, given $\boldsymbol{u}_h^{n-1} \in \mathbb{V}_h$, we find $\mathcal{F}_{t_n}$-adapted and $\mathbb{V}_h \times \mathbb{V}_h$-valued random variables $\{(\boldsymbol{u}_h^n, \boldsymbol{H}_h^n)\}$ satisfying $\mathbb{P}$-a.s.,

$$\begin{cases} \left\langle \boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}, \boldsymbol{\chi}_h \right\rangle = k \left\langle \boldsymbol{H}_h^n, \boldsymbol{\chi}_h \right\rangle + k \left\langle \nabla \boldsymbol{H}_h^n, \nabla \boldsymbol{\chi}_h \right\rangle - k\gamma \left\langle \boldsymbol{u}_h^n \times \boldsymbol{H}_h^n, \boldsymbol{\chi}_h \right\rangle + k \left\langle \mathcal{C}(\boldsymbol{u}_h^n), \boldsymbol{\chi}_h \right\rangle \\ \qquad + k \left\langle \mathcal{M}_R(\boldsymbol{u}_h^{n-1}), \boldsymbol{\chi}_h \right\rangle + \left\langle G(\boldsymbol{u}_h^{n-1}), \boldsymbol{\chi}_h \right\rangle \overline{\Delta} W^n, \\ \left\langle \boldsymbol{H}_h^n, \boldsymbol{\phi}_h \right\rangle = -\left\langle \nabla \boldsymbol{u}_h^n, \nabla \boldsymbol{\phi}_h \right\rangle + \left\langle f_R(\boldsymbol{u}_h^{n-1}), \boldsymbol{\phi}_h \right\rangle , \end{cases} \tag{3.1}$$

for all $\boldsymbol{\chi}_h, \boldsymbol{\phi}_h \in \mathbb{V}_h$. Here, $\overline{\Delta} W^n := W(t_n) - W(t_{n-1}) \sim \mathcal{N}(0,k)$. In particular, when $\gamma = \beta_2 = 0$, this is a fully-discrete IMEX-type scheme for the sCHm or the sAC/CH equations.

Lemma 3.1. There exists a sequence $\{(\boldsymbol{u}_h^n, \boldsymbol{H}_h^n)\}$ of $\mathbb{V}_h \times \mathbb{V}_h$-valued random variables which solves (3.1).

Proof. Fix $\omega \in \Omega$. We aim to use induction and a form of Brouwer's fixed point theorem to show the existence of a sequence $\{\boldsymbol{u}_h^n(\omega)\}_{n=1}^N$ solving (3.1). Suppose that $\boldsymbol{u}_h^0(\omega), \boldsymbol{u}_h^1(\omega), \dots, \boldsymbol{u}_h^{n-1}(\omega)$ are given. Consider a continuous map $\mathcal{G}_n^\omega : \mathbb{V}_h \to \mathbb{V}_h$ defined by

$$\begin{aligned}\mathcal{G}_n^\omega(\boldsymbol{v}) &= \boldsymbol{v} - \boldsymbol{u}_h^{n-1}(\omega) - k\Delta_h \boldsymbol{v} - k\Pi_h f_R\big(\boldsymbol{u}_h^{n-1}(\omega)\big) + k\Delta_h^2 \boldsymbol{v} + k\Delta_h \Pi_h f_R\big(\boldsymbol{u}_h^{n-1}(\omega)\big) \\ &\quad + k\boldsymbol{v} \times \big(\Delta_h \boldsymbol{v} + \Pi_h f_R(\boldsymbol{u}_h^{n-1}(\omega))\big) - k\Pi_h \mathcal{C}(\boldsymbol{v}) - k\Pi_h \mathcal{M}_R(\boldsymbol{u}_h^{n-1}(\omega)) \\ &\quad - \Pi_h G(\boldsymbol{u}_h^{n-1}(\omega))\overline{\Delta} W^n(\omega).\end{aligned}$$

For all $\boldsymbol{v} \in \mathbb{V}_h$, by Young's inequality, Lipschitz continuity of G, f_R, and $\mathcal{M}_R$, and (2.26) we have

$$\begin{aligned}\langle \mathcal{G}_n^\omega(\boldsymbol{v}), \boldsymbol{v}\rangle &\geq \frac{1}{2}\|\boldsymbol{v}\|_{\mathbb{L}^2}^2 - \frac{1}{2}\left\|\boldsymbol{u}_h^{n-1}(\omega)\right\|_{\mathbb{L}^2}^2 + k\|\nabla \boldsymbol{v}\|_{\mathbb{L}^2}^2 + k\|\Delta_h \boldsymbol{v}\|_{\mathbb{L}^2}^2 - k\left\langle \Pi_h f_R(\boldsymbol{u}_h^{n-1}(\omega)), \boldsymbol{v}\right\rangle \\ &\quad - k\left\langle \nabla \Pi_h f_R(\boldsymbol{u}_h^{n-1}(\omega)), \nabla \boldsymbol{v}\right\rangle - k\left\langle (\boldsymbol{\nu}\cdot\nabla)\boldsymbol{v}, \boldsymbol{v}\right\rangle - k\left\langle \mathcal{M}_R(\boldsymbol{v}), \boldsymbol{v}\right\rangle \\ &\quad - \left\langle G(\boldsymbol{u}_h^{n-1}(\omega))\overline{\Delta} W^n(\omega), \boldsymbol{v}\right\rangle \\ &\geq \frac{1}{2}\|\boldsymbol{v}\|_{\mathbb{L}^2}^2 - \frac{1}{2}\left\|\boldsymbol{u}_h^{n-1}(\omega)\right\|_{\mathbb{L}^2}^2 + k\|\nabla \boldsymbol{v}\|_{\mathbb{L}^2}^2 + k\|\Delta_h \boldsymbol{v}\|_{\mathbb{L}^2}^2 - 2C_R k\left\|\boldsymbol{u}_h^{n-1}(\omega)\right\|_{\mathbb{L}^2}^2 - \frac{k}{8}\|\boldsymbol{v}\|_{\mathbb{L}^2}^2 \\ &\quad - \frac{C_R k}{4}\left\|\boldsymbol{u}_h^{n-1}(\omega)\right\|_{\mathbb{L}^2}^2 - k\|\Delta_h \boldsymbol{v}\|_{\mathbb{L}^2}^2 - k\|\nabla \boldsymbol{v}\|_{\mathbb{L}^2}^2 - \frac{k}{4}\|\boldsymbol{v}\|_{\mathbb{L}^2}^2 - C_R k\|\boldsymbol{v}\|_{\mathbb{L}^2}^2 \\ &\quad - 2C_G\left\|\boldsymbol{u}_h^{n-1}(\omega)\right\|_{\mathbb{L}^2}^2\left|\overline{\Delta} W^n(\omega)\right|^2 - \frac{k}{8}\|\boldsymbol{v}\|_{\mathbb{L}^2}^2 \\ &= \frac{1}{2}(1 - k - 2C_R k)\|\boldsymbol{v}\|_{\mathbb{L}^2}^2 - \frac{1}{4}\left(9C_R k + 8C_G\left|\overline{\Delta} W^n(\omega)\right|^2\right)\left\|\boldsymbol{u}_h^{n-1}(\omega)\right\|_{\mathbb{L}^2}^2.\end{aligned}$$

Now, suppose that $k < (1+2C_R)^{-1}$, and set $\beta := 1 - k - 2C_R k > 0$. Let

$$\mathcal{B}_n := \left\{\boldsymbol{\varphi} \in \mathbb{V}_h : \|\boldsymbol{\varphi}\|_{\mathbb{L}^2}^2 = 4\beta^{-1}\Lambda_n(\omega)\right\},$$

where

$$\Lambda_n(\omega) := \frac{1}{4}\left(9C_R k + 8C_G\left|\overline{\Delta} W^n(\omega)\right|^2\right)\left\|\boldsymbol{u}_h^{n-1}(\omega)\right\|_{\mathbb{L}^2}^2 < \infty.$$

Then we have $\langle \mathcal{G}_n^\omega(\boldsymbol{v}), \boldsymbol{v}\rangle \geq 0$ for all $\boldsymbol{v} \in \mathcal{B}_n(\omega)$. Brouwer's fixed point theorem [21, Corollary VI.1.1] implies the existence of $\boldsymbol{u}_h^n(\omega)$ such that $\mathcal{G}_n^\omega\big(\boldsymbol{u}_h^n(\omega)\big) = 0$, thus also of $\boldsymbol{H}_h^n(\omega) = \Delta_h \boldsymbol{u}_h^n(\omega) + \Pi_h f_R\big(\boldsymbol{u}_h^{n-1}(\omega)\big)$. The $\mathcal{F}_{t_n}$-measurability of the map $\boldsymbol{u}_h^n : \Omega \to \mathbb{V}_h$, thus also of $\boldsymbol{H}_h^n$, follows from the same argument as in [2, Theorem 2.2]. □

We establish some stability estimates in the following lemmas.

Lemma 3.2. Let $p \in [1,\infty)$ be a natural number. Suppose that $\{(\boldsymbol{u}_h^n, \boldsymbol{H}_h^n)\}$ satisfies (3.1). There exists a positive constant C such that

$$\begin{aligned}&\mathbb{E}\left[\max_{l\leq n}\left\|\boldsymbol{u}_h^l\right\|_{\mathbb{L}^2}^{2^p}\right] + \mathbb{E}\left[\sum_{j=1}^n \left\|\boldsymbol{u}_h^j - \boldsymbol{u}_h^{j-1}\right\|_{\mathbb{L}^2}^2 \left\|\boldsymbol{u}_h^j\right\|_{\mathbb{L}^2}^{2^p-2}\right] \\ &\quad + \mathbb{E}\left[\sum_{j=1}^n k\left(\left\|\nabla \boldsymbol{u}_h^j\right\|_{\mathbb{L}^2}^2 + \left\|\Delta_h \boldsymbol{u}_h^j\right\|_{\mathbb{L}^2}^2\right)\left\|\boldsymbol{u}_h^j\right\|_{\mathbb{L}^2}^{2^p-2}\right]\end{aligned}$$

$$+\mathbb{E}\left[\left(k\sum_{j=1}^{n}\left\|\nabla \boldsymbol{u}_h^j\right\|_{\mathbb{L}^2}^2\right)^{2^{p-1}}\right]+\mathbb{E}\left[\left(k\sum_{j=1}^{n}\left\|\Delta_h \boldsymbol{u}_h^j\right\|_{\mathbb{L}^2}^2\right)^{2^{p-1}}\right]\leq C. \tag{3.2}$$

where C depends on T, p, R, and $\|\boldsymbol{u}_0\|_{\mathbb{L}^2}$, but is independent of n, h, and k.

Proof. We begin the proof by showing the inequality for $p=1$. Setting $\boldsymbol{\chi}_h=\boldsymbol{u}_h^n$, we have

$$\begin{aligned}\frac{1}{2}\left(\|\boldsymbol{u}_h^n\|_{\mathbb{L}^2}^2-\left\|\boldsymbol{u}_h^{n-1}\right\|_{\mathbb{L}^2}^2\right)+\frac{1}{2}\left\|\boldsymbol{u}_h^n-\boldsymbol{u}_h^{n-1}\right\|_{\mathbb{L}^2}^2&=k\left\langle \boldsymbol{H}_h^n,\boldsymbol{u}_h^n\right\rangle+k\left\langle \nabla\boldsymbol{H}_h^n,\nabla\boldsymbol{u}_h^n\right\rangle+k\left\langle \mathcal{C}(\boldsymbol{u}_h^n),\boldsymbol{u}_h^n\right\rangle\\&\quad+k\left\langle \mathcal{M}_R(\boldsymbol{u}_h^{n-1}),\boldsymbol{u}_h^n\right\rangle+\left\langle G(\boldsymbol{u}_h^{n-1}),\boldsymbol{u}_h^n\right\rangle\overline{\Delta}W^n.\end{aligned} \tag{3.3}$$

Successively taking $\boldsymbol{\phi}_h=\boldsymbol{u}_h^n$ and $\boldsymbol{\phi}_h=-\Delta_h\boldsymbol{u}_h^n$, then substituting the results to (3.3), we obtain

$$\begin{aligned}&\frac{1}{2}\left(\|\boldsymbol{u}_h^n\|_{\mathbb{L}^2}^2-\left\|\boldsymbol{u}_h^{n-1}\right\|_{\mathbb{L}^2}^2\right)+\frac{1}{2}\left\|\boldsymbol{u}_h^n-\boldsymbol{u}_h^{n-1}\right\|_{\mathbb{L}^2}^2+k\left\|\nabla\boldsymbol{u}_h^n\right\|_{\mathbb{L}^2}^2+k\left\|\Delta_h\boldsymbol{u}_h^n\right\|_{\mathbb{L}^2}^2\\&=k\left\langle f_R(\boldsymbol{u}_h^{n-1}),\boldsymbol{u}_h^n\right\rangle-k\left\langle f_R(\boldsymbol{u}_h^{n-1}),\Delta_h\boldsymbol{u}_h^n\right\rangle+k\left\langle \mathcal{C}(\boldsymbol{u}_h^n),\boldsymbol{u}_h^n\right\rangle+k\left\langle \mathcal{M}_R(\boldsymbol{u}_h^{n-1}),\boldsymbol{u}_h^n\right\rangle\\&\quad+\left\langle G(\boldsymbol{u}_h^{n-1}),\boldsymbol{u}_h^{n-1}\right\rangle\overline{\Delta}W^n+\left\langle G(\boldsymbol{u}_h^{n-1}),\boldsymbol{u}_h^n-\boldsymbol{u}_h^{n-1}\right\rangle\overline{\Delta}W^n\\&=:J_1+J_2+\cdots+J_6.\end{aligned} \tag{3.4}$$

We will estimate each term on the last line. By Lipschitz continuity of f_R, G, and $\mathcal{M}_R$, (2.32), and Young's inequality we have

$$\begin{aligned}|J_1|&\leq C_R k\left\|\boldsymbol{u}_h^{n-1}\right\|_{\mathbb{L}^2}^2+\frac{k}{2}\left\|\boldsymbol{u}_h^n\right\|_{\mathbb{L}^2}^2,\\|J_2|&\leq C_R k\left\|\boldsymbol{u}_h^{n-1}\right\|_{\mathbb{L}^2}^2+\frac{k}{2}\left\|\Delta_h\boldsymbol{u}_h^n\right\|_{\mathbb{L}^2}^2,\\|J_3|+|J_4|&\leq C_R k\left\|\boldsymbol{u}_h^{n-1}\right\|_{\mathbb{L}^2}^2+\frac{1}{8}\left\|\boldsymbol{u}_h^n-\boldsymbol{u}_h^{n-1}\right\|_{\mathbb{L}^2}^2+\frac{k}{2}\left\|\nabla\boldsymbol{u}_h^n\right\|_{\mathbb{L}^2}^2,\end{aligned}$$

We aim to estimate the moments of the last two terms. To this end, note that by Young's inequality,

$$|J_6|\leq\frac{1}{4}\left\|\boldsymbol{u}_h^n-\boldsymbol{u}_h^{n-1}\right\|_{\mathbb{L}^2}^2+\left\|G(\boldsymbol{u}_h^{n-1})\right\|_{\mathbb{L}^2}^2\left|\overline{\Delta}W^n\right|^2. \tag{3.5}$$

By the tower property of the conditional expectation and the independence of the Wiener increment, we infer that

$$\begin{aligned}\mathbb{E}\left[\max_{l\leq n}\sum_{j=1}^{l}\left\|G(\boldsymbol{u}_h^{j-1})\right\|_{\mathbb{L}^2}^2\left|\overline{\Delta}W^j\right|^2\right]&=\sum_{j=1}^{n}\mathbb{E}\left[\mathbb{E}\Big[\left\|G(\boldsymbol{u}_h^{j-1})\right\|_{\mathbb{L}^2}^2\left|\overline{\Delta}W^j\right|^2\Big|\mathcal{F}_{t_{j-1}}\Big]\right]\\&=\sum_{j=1}^{n}\mathbb{E}\left[\left\|G(\boldsymbol{u}_h^{j-1})\right\|_{\mathbb{L}^2}^2\mathbb{E}\Big[\left|\overline{\Delta}W^j\right|^2\Big|\mathcal{F}_{t_{j-1}}\Big]\right]\\&\leq Ck\sum_{j=1}^{n}\mathbb{E}\left[1+\left\|\boldsymbol{u}_h^{j-1}\right\|_{\mathbb{L}^2}^2\right].\end{aligned} \tag{3.6}$$

Noting the assumptions on G, we also have by the Burkholder–Davis–Gundy inequality,

$$\mathbb{E}\left[\max_{l\leq n}\sum_{j=1}^{l}\left\langle G(\boldsymbol{u}_h^{j-1}),\boldsymbol{u}_h^{j-1}\right\rangle\overline{\Delta}W^j\right]\leq C\mathbb{E}\left[\left(k\sum_{j=1}^{n}\left(1+\left\|\boldsymbol{u}_h^{j-1}\right\|_{\mathbb{L}^2}^2\right)\left\|\boldsymbol{u}_h^{j-1}\right\|_{\mathbb{L}^2}^2\right)^{\frac{1}{2}}\right]$$

$$\begin{aligned}
&\leq C\mathbb{E}\left[\max_{j\leq n}\left(1+\left\|\boldsymbol{u}_h^{j-1}\right\|_{\mathbb{L}^2}^2\right)^{\frac{1}{2}}\left(k\sum_{j=1}^n\left\|\boldsymbol{u}_h^{j-1}\right\|_{\mathbb{L}^2}^2\right)^{\frac{1}{2}}\right]\\
&\leq C+\frac{1}{4}\mathbb{E}\left[\max_{l\leq n}\left\|\boldsymbol{u}_h^{l}\right\|_{\mathbb{L}^2}^2\right]+C\mathbb{E}\left[\sum_{j=1}^n k\left\|\boldsymbol{u}_h^{j-1}\right\|_{\mathbb{L}^2}^2\right]. \qquad (3.7)
\end{aligned}$$

As such, summing (3.4) over $j\in\{1,2,\dots,l\}$, taking the maximum over l, applying the expected value, and absorbing the second term in (3.7) to the right-hand side, we infer from (3.5), (3.6), and (3.7) that

$$\begin{aligned}
&\mathbb{E}\left[\max_{l\leq n}\left\|\boldsymbol{u}_h^{l}\right\|_{\mathbb{L}^2}^2\right]+\mathbb{E}\left[\sum_{j=1}^n\left\|\boldsymbol{u}_h^{j}-\boldsymbol{u}_h^{j-1}\right\|_{\mathbb{L}^2}^2\right]+\mathbb{E}\left[\sum_{j=1}^n k\left(\|\nabla\boldsymbol{u}_h^n\|_{\mathbb{L}^2}^2+\|\Delta_h\boldsymbol{u}_h^n\|_{\mathbb{L}^2}^2\right)\right]\\
&\quad\leq\|\boldsymbol{u}_h^0\|_{\mathbb{L}^2}^2+C\left(1+k\sum_{j=1}^n\mathbb{E}\left[\max_{l\leq j}\left\|\boldsymbol{u}_h^{l-1}\right\|_{\mathbb{L}^2}^2\right]\right),
\end{aligned}$$

where C depends on T and R. The first inequality then follows by the discrete Gronwall lemma.

Next, we aim to prove the second inequality. We will show the case $p=2$ in detail. Multiplying (3.4) by $\|\boldsymbol{u}_h^n\|_{\mathbb{L}^2}^2$, we obtain

$$\begin{aligned}
&\frac{1}{4}\left(\|\boldsymbol{u}_h^n\|_{\mathbb{L}^2}^4-\|\boldsymbol{u}_h^{n-1}\|_{\mathbb{L}^2}^4+\left(\|\boldsymbol{u}_h^n\|_{\mathbb{L}^2}^2-\|\boldsymbol{u}_h^{n-1}\|_{\mathbb{L}^2}^2\right)^2\right)+\frac{1}{2}\|\boldsymbol{u}_h^n-\boldsymbol{u}_h^{n-1}\|_{\mathbb{L}^2}^2\|\boldsymbol{u}_h^n\|_{\mathbb{L}^2}^2\\
&+k\|\nabla\boldsymbol{u}_h^n\|_{\mathbb{L}^2}^2\|\boldsymbol{u}_h^n\|_{\mathbb{L}^2}^2+k\|\Delta_h\boldsymbol{u}_h^n\|_{\mathbb{L}^2}^2\|\boldsymbol{u}_h^n\|_{\mathbb{L}^2}^2\\
&=k\left\langle f_R(\boldsymbol{u}_h^{n-1}),\boldsymbol{u}_h^n\right\rangle\|\boldsymbol{u}_h^n\|_{\mathbb{L}^2}^2-k\left\langle f_R(\boldsymbol{u}_h^{n-1}),\Delta_h\boldsymbol{u}_h^n\right\rangle\|\boldsymbol{u}_h^n\|_{\mathbb{L}^2}^2+k\left\langle\mathcal{C}(\boldsymbol{u}_h^n),\boldsymbol{u}_h^n\right\rangle\|\boldsymbol{u}_h^n\|_{\mathbb{L}^2}^2\\
&\quad+k\left\langle\mathcal{M}_R(\boldsymbol{u}_h^{n-1}),\boldsymbol{u}_h^n\right\rangle\|\boldsymbol{u}_h^n\|_{\mathbb{L}^2}^2+\left\langle G(\boldsymbol{u}_h^{n-1})\overline{\Delta}W^n,\boldsymbol{u}_h^{n-1}\right\rangle\|\boldsymbol{u}_h^n\|_{\mathbb{L}^2}^2\\
&\quad+\left\langle G(\boldsymbol{u}_h^{n-1})\overline{\Delta}W^n,\boldsymbol{u}_h^n-\boldsymbol{u}_h^{n-1}\right\rangle\|\boldsymbol{u}_h^n\|_{\mathbb{L}^2}^2\\
&=I_1+I_2+\cdots+I_6. \qquad (3.8)
\end{aligned}$$

From the corresponding estimates for J_1 to J_4 in (3.4), the first four terms can be estimated as:

$$\begin{aligned}
|I_1|&\leq\frac{k}{16}\left\|\boldsymbol{u}_h^n-\boldsymbol{u}_h^{n-1}\right\|_{\mathbb{L}^2}^2\|\boldsymbol{u}_h^n\|_{\mathbb{L}^2}^2+Ck\|\boldsymbol{u}_h^n\|_{\mathbb{L}^2}^4,\\
|I_2|&\leq\frac{k}{16}\|\Delta_h\boldsymbol{u}_h^n\|_{\mathbb{L}^2}^2\|\boldsymbol{u}_h^n\|_{\mathbb{L}^2}^2+Ck\|\boldsymbol{u}_h^n\|_{\mathbb{L}^2}^4+Ck\left\|\boldsymbol{u}_h^{n-1}\right\|_{\mathbb{L}^2}^4,\\
|I_3|+|I_4|&\leq\frac{k}{16}\|\nabla\boldsymbol{u}_h^n\|_{\mathbb{L}^2}^2\|\boldsymbol{u}_h^n\|_{\mathbb{L}^2}^2+\frac{k}{16}\left\|\boldsymbol{u}_h^n-\boldsymbol{u}_h^{n-1}\right\|_{\mathbb{L}^2}^2\|\boldsymbol{u}_h^n\|_{\mathbb{L}^2}^2+Ck\|\boldsymbol{u}_h^n\|_{\mathbb{L}^2}^4.
\end{aligned}$$

For the term I_5, by Young's inequality we have

$$\begin{aligned}
I_5&=\left\langle G(\boldsymbol{u}_h^{n-1})\overline{\Delta}W^n,\boldsymbol{u}_h^{n-1}\right\rangle\left[\left(\|\boldsymbol{u}_h^n\|_{\mathbb{L}^2}^2-\left\|\boldsymbol{u}_h^{n-1}\right\|_{\mathbb{L}^2}^2\right)+\left\|\boldsymbol{u}_h^{n-1}\right\|_{\mathbb{L}^2}^2\right]\\
&\leq\frac{1}{16}\left(\|\boldsymbol{u}_h^n\|_{\mathbb{L}^2}^2-\left\|\boldsymbol{u}_h^{n-1}\right\|_{\mathbb{L}^2}^2\right)^2+C\left\|\boldsymbol{u}_h^{n-1}\right\|_{\mathbb{L}^2}^4\left|\overline{\Delta}W^n\right|^2+\left\langle G(\boldsymbol{u}_h^{n-1})\overline{\Delta}W^n,\boldsymbol{u}_h^{n-1}\right\rangle\left\|\boldsymbol{u}_h^{n-1}\right\|_{\mathbb{L}^2}^2. \qquad (3.9)
\end{aligned}$$

Similarly, for the term I_6 we infer that

$$\begin{aligned}
I_6&\leq C\left\|\boldsymbol{u}_h^{n-1}\right\|_{\mathbb{L}^2}^2|\overline{\Delta}W^n|^2\|\boldsymbol{u}_h^n\|_{\mathbb{L}^2}^2+\frac{1}{16}\left\|\boldsymbol{u}_h^n-\boldsymbol{u}_h^{n-1}\right\|_{\mathbb{L}^2}^2\|\boldsymbol{u}_h^n\|_{\mathbb{L}^2}^2\\
&\leq\frac{1}{16}\left(\|\boldsymbol{u}_h^n\|_{\mathbb{L}^2}^2-\left\|\boldsymbol{u}_h^{n-1}\right\|_{\mathbb{L}^2}^2\right)^2+C\left\|\boldsymbol{u}_h^{n-1}\right\|_{\mathbb{L}^2}^4\left(|\overline{\Delta}W^n|^4+|\overline{\Delta}W^n|^2\right)\\
&\quad+\frac{1}{16}\left\|\boldsymbol{u}_h^n-\boldsymbol{u}_h^{n-1}\right\|_{\mathbb{L}^2}^2\|\boldsymbol{u}_h^n\|_{\mathbb{L}^2}^2. \qquad (3.10)
\end{aligned}$$

Altogether, for sufficiently small k, we obtain

$$\begin{aligned}
&\frac{1}{4}\left(\|\boldsymbol{u}_h^n\|_{\mathbb{L}^2}^4 - \left\|\boldsymbol{u}_h^{n-1}\right\|_{\mathbb{L}^2}^4\right) + \frac{1}{8}\left(\|\boldsymbol{u}_h^n\|_{\mathbb{L}^2}^2 - \left\|\boldsymbol{u}_h^{n-1}\right\|_{\mathbb{L}^2}^2\right)^2 + \frac{1}{4}\left\|\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}\right\|_{\mathbb{L}^2}^2 \|\boldsymbol{u}_h^n\|_{\mathbb{L}^2}^2 \\
&+ \frac{k}{4}\|\nabla \boldsymbol{u}_h^n\|_{\mathbb{L}^2}^2 \|\boldsymbol{u}_h^n\|_{\mathbb{L}^2}^2 + \frac{k}{4}\|\Delta_h \boldsymbol{u}_h^n\|_{\mathbb{L}^2}^2 \|\boldsymbol{u}_h^n\|_{\mathbb{L}^2}^2 \\
&\le Ck\|\boldsymbol{u}_h^n\|_{\mathbb{L}^2}^4 + Ck\left\|\boldsymbol{u}_h^{n-1}\right\|_{\mathbb{L}^2}^4 + C\left\|\boldsymbol{u}_h^{n-1}\right\|_{\mathbb{L}^2}^4 \left(|\overline{\Delta} W^n|^4 + |\overline{\Delta} W^n|^2\right) \\
&\quad + \left\langle G(\boldsymbol{u}_h^{n-1})\overline{\Delta} W^n, \boldsymbol{u}_h^{n-1}\right\rangle \left\|\boldsymbol{u}_h^{n-1}\right\|_{\mathbb{L}^2}^2. \qquad (3.11)
\end{aligned}$$

We need to estimate the moment of the right-hand side of the last inequality. To this end, note that by the Burkholder–Davis–Gundy inequality, similarly to (3.7) we have

$$\begin{aligned}
\mathbb{E}\left[\max_{l\le n}\sum_{j=1}^{l}\left\langle G(\boldsymbol{u}_h^{j-1})\overline{\Delta} W^j, \boldsymbol{u}_h^{j-1}\right\rangle \left\|\boldsymbol{u}_h^{j-1}\right\|_{\mathbb{L}^2}^2\right] &\le C\mathbb{E}\left[\left(k\sum_{j=1}^{n}\left(1+\left\|\boldsymbol{u}_h^{j-1}\right\|_{\mathbb{L}^2}^2\right)\left\|\boldsymbol{u}_h^{j-1}\right\|_{\mathbb{L}^2}^6\right)^{\frac{1}{2}}\right] \\
&\le C + \frac{1}{4}\mathbb{E}\left[\max_{l\le n}\left\|\boldsymbol{u}_h^l\right\|_{\mathbb{L}^2}^4\right] + C\mathbb{E}\left[\sum_{j=1}^{n} k\left\|\boldsymbol{u}_h^{j-1}\right\|_{\mathbb{L}^2}^4\right]. \qquad (3.12)
\end{aligned}$$

With this estimate, we can continue from (3.11). Summing (3.11) over $j \in \{1, 2, \ldots, l\}$, taking the maximum over l, applying the expected value as before, we deduce the required inequality for $p = 2$ by the discrete Gronwall lemma, except for the last two terms on the left-hand side. For general $p \ge 2$, the inductive step is as follows: once we obtain inequality of the form

$$\begin{aligned}
&\frac{1}{2^p}\left(\|\boldsymbol{u}_h^n\|_{\mathbb{L}^2}^{2^p} - \left\|\boldsymbol{u}_h^{n-1}\right\|_{\mathbb{L}^2}^{2^p}\right) + \frac{1}{2^{p+1}}\left(\|\boldsymbol{u}_h^n\|_{\mathbb{L}^2}^{2^{p-1}} - \left\|\boldsymbol{u}_h^{n-1}\right\|_{\mathbb{L}^2}^{2^{p-1}}\right)^2 + \frac{1}{2^p}\left\|\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}\right\|_{\mathbb{L}^2}^2 \|\boldsymbol{u}_h^n\|_{\mathbb{L}^2}^{2^p-2} \\
&+ k\|\nabla \boldsymbol{u}_h^n\|_{\mathbb{L}^2}^2 \|\boldsymbol{u}_h^n\|_{\mathbb{L}^2}^{2^p-2} + k\|\Delta_h \boldsymbol{u}_h^n\|_{\mathbb{L}^2}^2 \|\boldsymbol{u}_h^n\|_{\mathbb{L}^2}^{2^p-2} \\
&\le Ck\|\boldsymbol{u}_h^n\|_{\mathbb{L}^2}^{2^p} + Ck\left\|\boldsymbol{u}_h^{n-1}\right\|_{\mathbb{L}^2}^{2^p} + C\left\|\boldsymbol{u}_h^{n-1}\right\|_{\mathbb{L}^2}^{2^p}\left(|\overline{\Delta} W^n|^{2^p} + |\overline{\Delta} W^n|^{2^{p-1}}\right) \\
&\quad + \left\langle G(\boldsymbol{u}_h^{n-1})\overline{\Delta} W^n, \boldsymbol{u}_h^{n-1}\right\rangle \left\|\boldsymbol{u}_h^{n-1}\right\|_{\mathbb{L}^2}^{2^p-2}, \qquad (3.13)
\end{aligned}$$

we multiply it by $\|\boldsymbol{u}_h^n\|_{\mathbb{L}^2}^{2^p}$. Note that the above inequality for $p = 2$ is (3.11). In this manner, we obtain

$$\begin{aligned}
&\frac{1}{2^{p+1}}\left(\|\boldsymbol{u}_h^n\|_{\mathbb{L}^2}^{2^{p+1}} - \left\|\boldsymbol{u}_h^{n-1}\right\|_{\mathbb{L}^2}^{2^{p+1}}\right) + \frac{1}{2^{p+2}}\left(\|\boldsymbol{u}_h^n\|_{\mathbb{L}^2}^{2^p} - \left\|\boldsymbol{u}_h^{n-1}\right\|_{\mathbb{L}^2}^{2^p}\right)^2 + \frac{1}{2^{p+1}}\left\|\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}\right\|_{\mathbb{L}^2}^2 \|\boldsymbol{u}_h^n\|_{\mathbb{L}^2}^{2^{p+1}-2} \\
&+ k\|\nabla \boldsymbol{u}_h^n\|_{\mathbb{L}^2}^2 \|\boldsymbol{u}_h^n\|_{\mathbb{L}^2}^{2^{p+1}-2} + k\|\Delta_h \boldsymbol{u}_h^n\|_{\mathbb{L}^2}^2 \|\boldsymbol{u}_h^n\|_{\mathbb{L}^2}^{2^{p+1}-2} \\
&\le Ck\|\boldsymbol{u}_h^n\|_{\mathbb{L}^2}^{2^{p+1}} + Ck\left\|\boldsymbol{u}_h^{n-1}\right\|_{\mathbb{L}^2}^{2^{p+1}} + C\left\|\boldsymbol{u}_h^{n-1}\right\|_{\mathbb{L}^2}^{2^p}\left(|\overline{\Delta} W^n|^{2^p} + |\overline{\Delta} W^n|^{2^{p-1}}\right)\|\boldsymbol{u}_h^n\|_{\mathbb{L}^2}^{2^p} \\
&\quad + \left\langle G(\boldsymbol{u}_h^{n-1})\overline{\Delta} W^n, \boldsymbol{u}_h^{n-1}\right\rangle \left\|\boldsymbol{u}_h^{n-1}\right\|_{\mathbb{L}^2}^{2^{p+1}-2} \\
&=: Ck\|\boldsymbol{u}_h^n\|_{\mathbb{L}^2}^{2^{p+1}} + Ck\left\|\boldsymbol{u}_h^{n-1}\right\|_{\mathbb{L}^2}^{2^{p+1}} + S_1 + S_2.
\end{aligned}$$

For the term S_1, we add and subtract $\left\|\boldsymbol{u}_h^{n-1}\right\|_{\mathbb{L}^2}^{2^p}$, and apply Young's inequality to obtain

$$S_1 \le \frac{1}{2^{p+3}}\left(\|\boldsymbol{u}_h^n\|_{\mathbb{L}^2}^{2^{p-1}} - \left\|\boldsymbol{u}_h^{n-1}\right\|_{\mathbb{L}^2}^{2^{p-1}}\right)^2 + C\left\|\boldsymbol{u}_h^{n-1}\right\|_{\mathbb{L}^2}^{2^{p+1}}\left(|\overline{\Delta} W^n|^{2^{p+1}} + |\overline{\Delta} W^n|^{2^p}\right),$$

and thus after rearranging, we obtain inequality of the form (3.13) with p replaced by $p+1$. Now, for the term S_2, we can estimate its moment by the same argument as in (3.12):

$$\begin{aligned}
&\mathbb{E}\left[\max_{l\leq n}\sum_{j=1}^{l}\left\langle G(\boldsymbol{u}_h^{j-1})\overline{\Delta}W^j, \boldsymbol{u}_h^{j-1}\right\rangle \left\|\boldsymbol{u}_h^{j-1}\right\|_{\mathbb{L}^2}^{2^{p+1}-2}\right] \\
&\leq C\mathbb{E}\left[\left(k\sum_{j=1}^{n}\left(1+\left\|\boldsymbol{u}_h^{j-1}\right\|_{\mathbb{L}^2}^{4}\right)\left\|\boldsymbol{u}_h^{j-1}\right\|_{\mathbb{L}^2}^{2^{p+2}-4}\right)^{\frac{1}{2}}\right] \\
&\leq C+\frac{1}{2^{p+2}}\mathbb{E}\left[\max_{l\leq n}\left\|\boldsymbol{u}_h^{l}\right\|_{\mathbb{L}^2}^{2^{p+1}}\right]+C\mathbb{E}\left[\sum_{j=1}^{n}k\left\|\boldsymbol{u}_h^{j-1}\right\|_{\mathbb{L}^2}^{2^{p+1}}\right].
\end{aligned}$$

Summing (3.13) over $j\in\{1,2,\dots,l\}$, taking the maximum over l and the expected value, we obtain (3.2) for general p, except for the last two terms on the left-hand side. In particular, we have shown for any $q\geq 1$,

$$\mathbb{E}\left[\max_{l\leq n}\left\|\boldsymbol{u}_h^{l}\right\|_{\mathbb{L}^2}^{q}\right]\leq C, \tag{3.14}$$

Finally, we sum (3.4) over $j\in\{1,2,\dots,n\}$ and raise it to the 2^{p-1}-th power. Noting (3.5) and applying similar argument as before yield

$$\begin{aligned}
&\|\boldsymbol{u}_h^n\|_{\mathbb{L}^2}^{2^p}+\left(k\sum_{j=1}^{n}\left\|\nabla\boldsymbol{u}_h^{j}\right\|_{\mathbb{L}^2}^{2}\right)^{2^{p-1}}+\left(k\sum_{j=1}^{n}\left\|\Delta_h\boldsymbol{u}_h^{j}\right\|_{\mathbb{L}^2}^{2}\right)^{2^{p-1}} \\
&\leq C\left(k\sum_{j=1}^{n}\left\|\boldsymbol{u}_h^{j-1}\right\|_{\mathbb{L}^2}^{2}\right)^{2^{p-1}}+\left(\sum_{j=1}^{n}\left\|G(\boldsymbol{u}_h^{j-1})\overline{\Delta}W^j\right\|_{\mathbb{L}^2}^{2}\right)^{2^{p-1}}+\left(\sum_{j=1}^{n}\left\langle G(\boldsymbol{u}_h^{j-1}),\boldsymbol{u}_h^{j-1}\right\rangle\overline{\Delta}W^n\right)^{2^{p-1}} \\
&=: R_1+R_2+R_3.
\end{aligned}\tag{3.15}$$

The expected value of R_1 is clearly bounded by (3.14). For R_2, we have by Jensen's inequality, (3.14), and the same argument as in (3.6),

$$\mathbb{E}[R_2]\leq Cn^{2^{p-1}-1}k^{2^{p-1}}\sum_{j=1}^{n}\mathbb{E}\left[1+\left\|\boldsymbol{u}_h^{j-1}\right\|_{\mathbb{L}^2}^{2^p}\right]\leq C.$$

For the term R_3, by the Burkholder–Davis–Gundy and the Jensen inequalities, noting the assumption on G, we obtain

$$\begin{aligned}
\mathbb{E}[R_3]&\leq C_p\,\mathbb{E}\left[\left(\sum_{j=1}^{n}k\left\|G(\boldsymbol{u}_h^{j-1})\right\|_{\mathbb{L}^2}^{2}\left\|\boldsymbol{u}_h^{j-1}\right\|_{\mathbb{L}^2}^{2}\right)^{2^{p-2}}\right] \\
&\leq C_p\,\mathbb{E}\left[\max_{j\leq n}\left\|\boldsymbol{u}_h^{j-1}\right\|_{\mathbb{L}^2}^{2^{p-1}}\left(k\sum_{j=1}^{n}\left(1+\left\|\boldsymbol{u}_h^{j-1}\right\|_{\mathbb{L}^2}^{2}\right)\right)^{2^{p-2}}\right] \\
&\leq\frac{1}{4}\mathbb{E}\left[\max_{j\leq n}\left\|\boldsymbol{u}_h^{j-1}\right\|_{\mathbb{L}^2}^{2^p}\right]+Ck^{2^{p-1}}\mathbb{E}\left[\left(\sum_{j=1}^{n}\left(1+\left\|\boldsymbol{u}_h^{j-1}\right\|_{\mathbb{L}^2}^{2}\right)\right)^{2^{p-1}}\right]
\end{aligned}$$

$$\leq \frac{1}{4}\mathbb{E}\left[\max_{j\leq n}\left\|\boldsymbol{u}_h^{j-1}\right\|_{\mathbb{L}^2}^{2p}\right] + CT + C_T\mathbb{E}\left[\max_{j\leq n}\left\|\boldsymbol{u}_h^{j-1}\right\|_{\mathbb{L}^2}^{2p}\right] \leq C.$$

This completes the proof of inequality (3.2). □

Lemma 3.3. Suppose that $\{(\boldsymbol{u}_h^n, \boldsymbol{H}_h^n)\}$ satisfies (3.1). There exists a positive constant C such that

$$\mathbb{E}\left[\max_{l\leq n}\left\|\boldsymbol{u}_h^l\right\|_{\mathbb{H}^1}^2\right] + \mathbb{E}\left[\sum_{j=1}^n \left\|\boldsymbol{u}_h^j - \boldsymbol{u}_h^{j-1}\right\|_{\mathbb{H}^1}^2\right] + \mathbb{E}\left[\sum_{j=1}^n k\left\|\boldsymbol{H}_h^j\right\|_{\mathbb{H}^1}^2\right] \leq C,$$

where C depends on T, R, and $\|\boldsymbol{u}_0\|_{\mathbb{H}^1}$, but is independent of n, h, and k.

Proof. We set $\boldsymbol{\chi}_h = \boldsymbol{H}_h^n$ and $\boldsymbol{\phi}_h = \boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}$ in (3.1) to obtain

$$\begin{aligned}\left\langle \boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}, \boldsymbol{H}_h^n\right\rangle &= k\left\|\boldsymbol{H}_h^n\right\|_{\mathbb{L}^2}^2 + k\left\|\nabla\boldsymbol{H}_h^n\right\|_{\mathbb{L}^2}^2 + k\left\langle \mathcal{C}(\boldsymbol{u}_h^n), \boldsymbol{H}_h^n\right\rangle + k\left\langle \mathcal{M}_R(\boldsymbol{u}_h^{n-1}), \boldsymbol{H}_h^n\right\rangle \\ &\quad + \left\langle G(\boldsymbol{u}_h^{n-1}), \boldsymbol{H}_h^n\right\rangle \overline{\Delta} W^n, \qquad (3.16)\\ \left\langle \boldsymbol{H}_h^n, \boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}\right\rangle &= -\frac{1}{2}\left(\left\|\nabla\boldsymbol{u}_h^n\right\|_{\mathbb{L}^2}^2 - \left\|\nabla\boldsymbol{u}_h^{n-1}\right\|_{\mathbb{L}^2}^2\right) - \frac{1}{2}\left\|\nabla\boldsymbol{u}_h^n - \nabla\boldsymbol{u}_h^{n-1}\right\|_{\mathbb{L}^2}^2 \\ &\quad + \left\langle f_R(\boldsymbol{u}_h^{n-1}), \boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}\right\rangle. \qquad (3.17)\end{aligned}$$

Next, putting $\boldsymbol{\phi}_h = \Pi_h G(\boldsymbol{u}_h^{n-1})$ yields

$$\begin{aligned}\left\langle G(\boldsymbol{u}_h^{n-1}), \boldsymbol{H}_h^n\right\rangle \overline{\Delta} W^n &= \left\langle \nabla\Pi_h G(\boldsymbol{u}_h^{n-1}), \nabla\boldsymbol{u}_h^{n-1}\right\rangle \overline{\Delta} W^n - \left\langle \nabla\Pi_h G(\boldsymbol{u}_h^{n-1}), \nabla\boldsymbol{u}_h^n - \nabla\boldsymbol{u}_h^{n-1}\right\rangle \overline{\Delta} W^n \\ &\quad + \left\langle \Pi_h G(\boldsymbol{u}_h^{n-1}), f_R(\boldsymbol{u}_h^{n-1})\right\rangle \overline{\Delta} W^n. \qquad (3.18)\end{aligned}$$

Furthermore, taking $\boldsymbol{\chi}_h = \Pi_h f_R(\boldsymbol{u}_h^{n-1})$ gives

$$\begin{aligned}\left\langle f_R(\boldsymbol{u}_h^{n-1}), \boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}\right\rangle &= k\left\langle \boldsymbol{H}_h^n, f_R(\boldsymbol{u}_h^{n-1})\right\rangle + k\left\langle \nabla\boldsymbol{H}_h^n, \nabla\Pi_h f_R(\boldsymbol{u}_h^{n-1})\right\rangle \\ &\quad - k\left\langle \boldsymbol{u}_h^n \times \boldsymbol{H}_h^n, \Pi_h f_R(\boldsymbol{u}_h^{n-1})\right\rangle + k\left\langle \mathcal{C}(\boldsymbol{u}_h^n), \Pi_h f_R(\boldsymbol{u}_h^{n-1})\right\rangle \\ &\quad + k\left\langle \mathcal{M}_R(\boldsymbol{u}_h^{n-1}), \Pi_h f_R(\boldsymbol{u}_h^{n-1})\right\rangle + \left\langle G(\boldsymbol{u}_h^{n-1}), \Pi_h f_R(\boldsymbol{u}_h^{n-1})\right\rangle \overline{\Delta} W^n. \quad (3.19)\end{aligned}$$

Subtracting (3.17) from (3.16), then adding the resulting expression with (3.18) and (3.19), we obtain

$$\begin{aligned}&\frac{1}{2}\left(\left\|\nabla\boldsymbol{u}_h^n\right\|_{\mathbb{L}^2}^2 - \left\|\nabla\boldsymbol{u}_h^{n-1}\right\|_{\mathbb{L}^2}^2\right) + \frac{1}{2}\left\|\nabla\boldsymbol{u}_h^n - \nabla\boldsymbol{u}_h^{n-1}\right\|_{\mathbb{L}^2}^2 + k\left\|\boldsymbol{H}_h^n\right\|_{\mathbb{L}^2}^2 + k\left\|\nabla\boldsymbol{H}_h^n\right\|_{\mathbb{L}^2}^2 \\ &= -k\left\langle \mathcal{C}(\boldsymbol{u}_h^n), \boldsymbol{H}_h^n\right\rangle - k\left\langle \mathcal{M}_R(\boldsymbol{u}_h^{n-1}), \boldsymbol{H}_h^n\right\rangle + k\left\langle \boldsymbol{H}_h^n, f_R(\boldsymbol{u}_h^{n-1})\right\rangle + k\left\langle \nabla\boldsymbol{H}_h^n, \nabla\Pi_h f_R(\boldsymbol{u}_h^{n-1})\right\rangle \\ &\quad - k\left\langle \boldsymbol{u}_h^n \times \boldsymbol{H}_h^n, \Pi_h f_R(\boldsymbol{u}_h^{n-1})\right\rangle + k\left\langle \mathcal{C}(\boldsymbol{u}_h^n), \Pi_h f_R(\boldsymbol{u}_h^{n-1})\right\rangle + k\left\langle \mathcal{M}_R(\boldsymbol{u}_h^{n-1}), \Pi_h f_R(\boldsymbol{u}_h^{n-1})\right\rangle \\ &\quad - \left\langle \nabla\Pi_h G(\boldsymbol{u}_h^{n-1}), \nabla\boldsymbol{u}_h^{n-1}\right\rangle \overline{\Delta} W^n - \left\langle \nabla\Pi_h G(\boldsymbol{u}_h^{n-1}), \nabla\boldsymbol{u}_h^n - \nabla\boldsymbol{u}_h^{n-1}\right\rangle \overline{\Delta} W^n \\ &=: I_1 + I_2 + \cdots + I_9. \qquad (3.20)\end{aligned}$$

We need to bound each term on the last line. For the first two terms, by (2.32), the Gagliardo–Nirenberg and Young inequalities, it is clear that

$$|I_1| + |I_2| \leq Ck\left\|\boldsymbol{u}_h^{n-1}\right\|_{\mathbb{L}^2}^2 + Ck\left(1 + \left\|\boldsymbol{u}_h^n\right\|_{\mathbb{L}^2}^2\right)\left\|\nabla\boldsymbol{u}_h^n\right\|_{\mathbb{L}^4}^2 + \frac{k}{8}\left\|\boldsymbol{H}_h^n\right\|_{\mathbb{L}^2}^2 + \frac{k}{8}\left\|\nabla\boldsymbol{H}_h^n\right\|_{\mathbb{L}^2}^2. \qquad (3.21)$$

For the terms I_3 and I_4, by the assumptions on f_R, Young's inequality and (2.26) we have

$$|I_3| + |I_4| \leq C_R k\left\|\boldsymbol{u}_h^{n-1}\right\|_{\mathbb{L}^2}^2 + C_R k\left\|\nabla\boldsymbol{u}_h^{n-1}\right\|_{\mathbb{L}^2}^2 + \frac{k}{4}\left\|\boldsymbol{H}_h^n\right\|_{\mathbb{L}^2}^2 + \frac{k}{4}\left\|\nabla\boldsymbol{H}_h^n\right\|_{\mathbb{L}^2}^2. \qquad (3.22)$$

For the term I_5, by Young's inequality, the Sobolev embedding $\mathbb{H}^1 \hookrightarrow \mathbb{L}^4$, and the stability of Π_h, we infer

$$|I_5| \leq C_R k\left\|\boldsymbol{u}_h^n\right\|_{\mathbb{L}^4}\left\|\boldsymbol{H}_h^n\right\|_{\mathbb{L}^4}\left\|\Pi_h f_R(\boldsymbol{u}_h^{n-1})\right\|_{\mathbb{L}^2}$$

$$\leq Ck \|\boldsymbol{u}_h^n\|_{\mathbb{L}^4}^2 \left\|\boldsymbol{u}_h^{n-1}\right\|_{\mathbb{L}^2}^2 + \frac{k}{4} \|\boldsymbol{H}_h^n\|_{\mathbb{L}^2}^2 + \frac{k}{4} \|\nabla \boldsymbol{H}_h^n\|_{\mathbb{L}^2}^2 . \tag{3.23}$$

For the terms I_6 and I_7, we again apply (2.32) and the Lipschitz continuity of $\mathcal{M}_R$ to obtain

$$\begin{aligned}|I_6| + |I_7| &\leq Ck \left\|\boldsymbol{u}_h^{n-1}\right\|_{\mathbb{L}^2}^2 + Ck \left(1 + \left\|\boldsymbol{u}_h^{n-1}\right\|_{\mathbb{L}^\infty}^2\right) \|\boldsymbol{u}_h^n\|_{\mathbb{L}^2}^2 + Ck \|\nabla \boldsymbol{u}_h^n\|_{\mathbb{L}^2}^2 \\ &\leq Ck \left\|\boldsymbol{u}_h^{n-1}\right\|_{\mathbb{L}^2}^2 + Ck \left(1 + \left\|\boldsymbol{u}_h^{n-1}\right\|_{\mathbb{L}^2}^2 + \left\|\Delta_h \boldsymbol{u}_h^{n-1}\right\|_{\mathbb{L}^2}^2\right) \|\boldsymbol{u}_h^n\|_{\mathbb{L}^2}^2 + Ck \|\nabla \boldsymbol{u}_h^n\|_{\mathbb{L}^2}^2 ,\end{aligned} \tag{3.24}$$

where in the last step we also used (2.24) and Young's inequality. For the term I_9, we have

$$\begin{aligned}|I_9| &\leq \frac{1}{4} \left\|\nabla \boldsymbol{u}_h^n - \nabla \boldsymbol{u}_h^{n-1}\right\|_{\mathbb{L}^2}^2 + 4 \left\|\nabla \Pi_h G(\boldsymbol{u}_h^{n-1})\right\|_{\mathbb{L}^2}^2 \left|\overline{\Delta} W^n\right|^2 \\ &\leq \frac{1}{4} \left\|\nabla \boldsymbol{u}_h^n - \nabla \boldsymbol{u}_h^{n-1}\right\|_{\mathbb{L}^2}^2 + C \left\|\boldsymbol{u}_h^{n-1}\right\|_{\mathbb{H}^1}^2 \left|\overline{\Delta} W^n\right|^2 ,\end{aligned} \tag{3.25}$$

where in the last step we used the $\mathbb{H}^1$-stability of the $\mathbb{L}^2$-projection [3] and the definition of G. The term I_8 in (3.20) remains as is for now.

We now collect all the estimates (3.21), (3.22), (3.23), (3.24), (3.25), and continue from (3.20), taking care to absorb appropriate terms to the left-hand side. Summing over $j \in \{1, 2, \dots, l\}$, taking the maximum over l, and applying the expected value, we obtain

$$\begin{aligned}&\mathbb{E}\left[\max_{l\leq n} \left\|\nabla \boldsymbol{u}_h^l\right\|_{\mathbb{L}^2}^2\right] + \mathbb{E}\left[\sum_{j=1}^n \left\|\nabla \boldsymbol{u}_h^j - \nabla \boldsymbol{u}_h^{j-1}\right\|_{\mathbb{L}^2}^2\right] + \mathbb{E}\left[\sum_{j=1}^n k \left\|\boldsymbol{H}_h^j\right\|_{\mathbb{H}^1}^2\right] \\ &\quad\leq \|\boldsymbol{u}_h^0\|_{\mathbb{H}^1}^2 + C\,\mathbb{E}\left[\max_{l\leq n} \left\|\boldsymbol{u}_h^l\right\|_{\mathbb{L}^2}^2\right] + C\,\mathbb{E}\left[\sum_{j=1}^n k \left(1 + \left\|\boldsymbol{u}_h^j\right\|_{\mathbb{L}^2}^2\right) \left\|\nabla \boldsymbol{u}_h^j\right\|_{\mathbb{L}^4}^2\right] \\ &\quad+ C\,\mathbb{E}\left[\sum_{j=1}^n k \left\|\boldsymbol{u}_h^j\right\|_{\mathbb{L}^4}^2 \left\|\boldsymbol{u}_h^{j-1}\right\|_{\mathbb{L}^2}^2\right] + C\,\mathbb{E}\left[\sum_{j=1}^n k \left(1 + \left\|\boldsymbol{u}_h^{j-1}\right\|_{\mathbb{L}^2}^2 + \left\|\Delta_h \boldsymbol{u}_h^{j-1}\right\|_{\mathbb{L}^2}^2\right) \left\|\boldsymbol{u}_h^j\right\|_{\mathbb{L}^2}^2\right] \\ &\quad+ \mathbb{E}\left[\max_{l\leq n} \sum_{j=1}^l \left\langle G(\boldsymbol{u}_h^{j-1}), \Delta_h \boldsymbol{u}_h^{j-1}\right\rangle \overline{\Delta} W^j\right] + C\,\mathbb{E}\left[\sum_{j=1}^n \left\|\boldsymbol{u}_h^{j-1}\right\|_{\mathbb{H}^1}^2 \left|\overline{\Delta} W^j\right|^2\right] \\ &\quad=: J_1 + J_2 + \dots + J_7.\end{aligned} \tag{3.26}$$

It remains to estimate the last five terms on the right-hand side. Firstly, the term J_3 is bounded by (2.25) and Lemma 3.2. Next, we have

$$J_4 \leq C\,\mathbb{E}\left[\left(\max_{l\leq n} \left\|\boldsymbol{u}_h^{l-1}\right\|_{\mathbb{L}^2}^2\right)\left(\sum_{j=1}^n k \left\|\boldsymbol{u}_h^j\right\|_{\mathbb{L}^4}^2\right)\right] \leq C + C\,\mathbb{E}\left[\left(\sum_{j=1}^n k \left\|\boldsymbol{u}_h^j\right\|_{\mathbb{H}^1}^2\right)^2\right] \leq C.$$

The term J_5 can be similarly bounded. For the term J_6, by the Burkholder–Davis–Gundy inequality and the $\mathbb{H}^1$-stability of Π_h, we obtain

$$\begin{aligned}\mathbb{E}\left[\max_{l\leq n} \sum_{j=1}^l \left\langle G(\boldsymbol{u}_h^{j-1}), \Delta_h \boldsymbol{u}_h^{j-1}\right\rangle \overline{\Delta} W^j\right] &\leq C\mathbb{E}\left[\left(k \sum_{j=1}^n \left(1 + \left\|\boldsymbol{u}_h^{j-1}\right\|_{\mathbb{L}^2}^2\right) \left\|\Delta_h \boldsymbol{u}_h^{j-1}\right\|_{\mathbb{L}^2}^2\right)^{\frac{1}{2}}\right] \\ &\leq C\mathbb{E}\left[\max_{j\leq n} \left(1 + \left\|\boldsymbol{u}_h^{j-1}\right\|_{\mathbb{L}^2}^2\right)^{\frac{1}{2}} \left(k \sum_{j=1}^n \left\|\Delta_h \boldsymbol{u}_h^{j-1}\right\|_{\mathbb{L}^2}^2\right)^{\frac{1}{2}}\right]\end{aligned}$$

$$\leq C + C\mathbb{E}\left[\sum_{j=1}^{n} k \left\|\Delta_h \boldsymbol{u}_h^{j-1}\right\|_{\mathbb{L}^2}^2\right] \leq C,$$

where in the last step we used Lemma 3.2. Similarly for the last term, we infer from the independence of the Wiener increment that

$$J_7 \leq C\mathbb{E}\left[\sum_{j=1}^{n} k \left\|\boldsymbol{u}_h^{j-1}\right\|_{\mathbb{H}^1}^2\right] \leq C.$$

Substituting these estimates back into (3.26), we deduce the required inequality. □

In the following, we assume that $\beta_2 = 0$ in (1.2). This estimate will be used only in Theorem 3.9.

Lemma 3.4. Suppose that $\beta_2 = 0$ in (1.2) and let $p \in [1, \infty)$ be a natural number. There exists a positive constant C such that

$$\mathbb{E}\left[\max_{l\leq n} \left\|\boldsymbol{u}_h^l\right\|_{\mathbb{H}^1}^{2^p}\right] + \mathbb{E}\left[\sum_{j=1}^{n} k \left\|\boldsymbol{H}_h^j\right\|_{\mathbb{H}^1}^2 \left\|\nabla \boldsymbol{u}_h^j\right\|_{\mathbb{L}^2}^{2^{p-1}}\right] + \mathbb{E}\left[\left(k\sum_{j=1}^{n} \left\|\boldsymbol{H}_h^j\right\|_{\mathbb{H}^1}^2\right)^{2^{p-1}}\right] \leq C, \tag{3.27}$$

where C depends on T, R, and $\|\boldsymbol{u}_0\|_{\mathbb{H}^1}$, but is independent of n, h, and k.

Proof. As before, we prove the case $p = 2$ in detail. Similarly to the proof of Lemma 3.2, we multiply (3.20) by $\|\nabla \boldsymbol{u}_n\|_{\mathbb{L}^2}^2$ to obtain

$$\begin{aligned}
&\frac{1}{4}\left(\|\nabla \boldsymbol{u}_h^n\|_{\mathbb{L}^2}^4 - \left\|\nabla \boldsymbol{u}_h^{n-1}\right\|_{\mathbb{L}^2}^4 + \left(\|\nabla \boldsymbol{u}_h^n\|_{\mathbb{L}^2}^2 - \left\|\nabla \boldsymbol{u}_h^{n-1}\right\|_{\mathbb{L}^2}^2\right)^2\right) + \frac{1}{2}\left\|\nabla \boldsymbol{u}_h^n - \nabla \boldsymbol{u}_h^{n-1}\right\|_{\mathbb{L}^2}^2 \|\nabla \boldsymbol{u}_h^n\|_{\mathbb{L}^2}^2 \\
&\quad + k\|\boldsymbol{H}_h^n\|_{\mathbb{L}^2}^2 \|\nabla \boldsymbol{u}_h^n\|_{\mathbb{L}^2}^2 + k\|\nabla \boldsymbol{H}_h^n\|_{\mathbb{L}^2}^2 \|\nabla \boldsymbol{u}_h^n\|_{\mathbb{L}^2}^2 \\
&= -k\langle \mathcal{C}(\boldsymbol{u}_h^n), \boldsymbol{H}_h^n\rangle \|\nabla \boldsymbol{u}_h^n\|_{\mathbb{L}^2}^2 - k\left\langle \mathcal{M}_R(\boldsymbol{u}_h^{n-1}), \boldsymbol{H}_h^n\right\rangle \|\nabla \boldsymbol{u}_h^n\|_{\mathbb{L}^2}^2 + k\left\langle \boldsymbol{H}_h^n, f_R(\boldsymbol{u}_h^{n-1})\right\rangle \|\nabla \boldsymbol{u}_h^n\|_{\mathbb{L}^2}^2 \\
&\quad + k\left\langle \nabla \boldsymbol{H}_h^n, \nabla \Pi_h f_R(\boldsymbol{u}_h^{n-1})\right\rangle \|\nabla \boldsymbol{u}_h^n\|_{\mathbb{L}^2}^2 - k\left\langle \boldsymbol{u}_h^n \times \boldsymbol{H}_h^n, \Pi_h f_R(\boldsymbol{u}_h^{n-1})\right\rangle \|\nabla \boldsymbol{u}_h^n\|_{\mathbb{L}^2}^2 \\
&\quad + k\left\langle \mathcal{C}(\boldsymbol{u}_h^n), \Pi_h f_R(\boldsymbol{u}_h^{n-1})\right\rangle \|\nabla \boldsymbol{u}_h^n\|_{\mathbb{L}^2}^2 + k\left\langle \mathcal{M}_R(\boldsymbol{u}_h^{n-1}), \Pi_h f_R(\boldsymbol{u}_h^{n-1})\right\rangle \|\nabla \boldsymbol{u}_h^n\|_{\mathbb{L}^2}^2 \\
&\quad - \left\langle \nabla \Pi_h G(\boldsymbol{u}_h^{n-1}), \nabla \boldsymbol{u}_h^{n-1}\right\rangle \overline{\Delta} W^n \|\nabla \boldsymbol{u}_h^n\|_{\mathbb{L}^2}^2 - \left\langle \nabla \Pi_h G(\boldsymbol{u}_h^{n-1}), \nabla \boldsymbol{u}_h^n - \nabla \boldsymbol{u}_h^{n-1}\right\rangle \overline{\Delta} W^n \|\nabla \boldsymbol{u}_h^n\|_{\mathbb{L}^2}^2 \\
&=: I_1 + I_2 + \cdots + I_9.
\end{aligned} \tag{3.28}$$

Noting $\beta_2 = 0$, we can estimate the first five terms following the corresponding bounds in (3.21) to (3.24):

$$\begin{aligned}
|I_1| &\leq Ck\|\boldsymbol{u}_h^n\|_{\mathbb{L}^2}^2 \|\nabla \boldsymbol{u}_h^n\|_{\mathbb{L}^2}^2 + Ck\|\nabla \boldsymbol{u}_h^n\|_{\mathbb{L}^2}^4 + \frac{k}{16}\|\boldsymbol{H}_h^n\|_{\mathbb{L}^2}^2 \|\nabla \boldsymbol{u}_h^n\|_{\mathbb{L}^2}^2, \\
|I_2| &\leq Ck\left\|\boldsymbol{u}_h^{n-1}\right\|_{\mathbb{L}^2}^4 + Ck\|\boldsymbol{u}_h^n\|_{\mathbb{L}^2}^4 + Ck\|\nabla \boldsymbol{u}_h^n\|_{\mathbb{L}^2}^4 + \frac{k}{16}\|\boldsymbol{H}_h^n\|_{\mathbb{L}^2}^2 \|\nabla \boldsymbol{u}_h^n\|_{\mathbb{L}^2}^2, \\
|I_3| &\leq Ck\left\|\boldsymbol{u}_h^{n-1}\right\|_{\mathbb{L}^2}^4 + \frac{k}{16}\|\nabla \boldsymbol{u}_h^n\|_{\mathbb{L}^2}^4 + \frac{k}{16}\|\boldsymbol{H}_h^n\|_{\mathbb{L}^2}^2 \|\nabla \boldsymbol{u}_h^n\|_{\mathbb{L}^2}^2, \\
|I_4| &\leq Ck\left\|\nabla \boldsymbol{u}_h^{n-1}\right\|_{\mathbb{L}^2}^4 + \frac{k}{16}\|\nabla \boldsymbol{u}_h^n\|_{\mathbb{L}^2}^4 + \frac{k}{16}\|\nabla \boldsymbol{H}_h^n\|_{\mathbb{L}^2}^2 \|\nabla \boldsymbol{u}_h^n\|_{\mathbb{L}^2}^2, \\
|I_5| &\leq Ck\|\boldsymbol{u}_h^n\|_{\mathbb{L}^2}^2 \left\|\boldsymbol{u}_h^{n-1}\right\|_{\mathbb{L}^2}^2 \left(\|\nabla \boldsymbol{u}_h^n\|_{\mathbb{L}^2}^2 + \|\Delta_h \boldsymbol{u}_h^n\|_{\mathbb{L}^2}^2\right) + \frac{k}{16}\|\boldsymbol{H}_h^n\|_{\mathbb{H}^1}^2 \|\nabla \boldsymbol{u}_h^n\|_{\mathbb{L}^2}^2, \\
|I_6| + |I_7| &\leq Ck\|\boldsymbol{u}_h^n\|_{\mathbb{L}^2}^2 \left\|\boldsymbol{u}_h^{n-1}\right\|_{\mathbb{L}^2}^2 \left(\|\nabla \boldsymbol{u}_h^n\|_{\mathbb{L}^2}^2 + \|\Delta_h \boldsymbol{u}_h^n\|_{\mathbb{L}^2}^2\right) + Ck\left\|\boldsymbol{u}_h^{n-1}\right\|_{\mathbb{L}^2}^4 + Ck\|\nabla \boldsymbol{u}_h^n\|_{\mathbb{L}^2}^4,
\end{aligned}$$

where for the terms I_5 and I_6 we also used (2.15). The expected values of the terms I_8 and I_9 can be estimated as (3.9), (3.10), and (3.12). Substituting these estimates into (3.28), summing, taking the maximum and the expected value, and applying the results of Lemma 3.2 as done previously shows (3.27) for the first two terms. To establish the inequality for the last term, we sum (3.28) over $j \in \{1, 2, \dots, n\}$, square the result, and follow the same argument as in (3.15). The general case follows by induction as in the proof of Lemma 3.2. We omit further details for brevity. □

To facilitate the proof of the error estimate, we decompose the error of the numerical method at time t_n, $n = 0, 1, \dots, N$, as:

$$\boldsymbol{u}(t_n) - \boldsymbol{u}_h^n = (\boldsymbol{u}(t_n) - \mathcal{R}_h \boldsymbol{u}(t_n)) + (\mathcal{R}_h \boldsymbol{u}(t_n) - \boldsymbol{u}_h^n) =: \boldsymbol{\rho}^n + \boldsymbol{\theta}^n, \tag{3.29}$$

$$\boldsymbol{H}(t_n) - \boldsymbol{H}_h^n = (\boldsymbol{H}(t_n) - \mathcal{R}_h \boldsymbol{H}(t_n)) + (\mathcal{R}_h \boldsymbol{H}(t_n) - \boldsymbol{H}_h^n) =: \boldsymbol{\eta}^n + \boldsymbol{\xi}^n. \tag{3.30}$$

As such by the definition of the Ritz projection (2.12),

$$\langle \nabla \boldsymbol{\rho}^n, \nabla \boldsymbol{\chi}_h \rangle = \langle \nabla \boldsymbol{\eta}^n, \nabla \boldsymbol{\chi}_h \rangle = 0, \quad \forall \boldsymbol{\chi}_h \in \mathbb{V}_h. \tag{3.31}$$

Furthermore, define a sequence of subsets of Ω which depend on κ and m:

$$\Omega_{\kappa,m} := \left\{ \omega \in \Omega : \max_{t \le t_m \wedge T} \|\boldsymbol{u}(t)\|_{\mathbb{H}^2}^2 + \max_{t \le t_m \wedge T} \|\boldsymbol{H}(t)\|_{\mathbb{L}^2}^2 + \max_{n \le m} \|\boldsymbol{u}_h^n\|_{\mathbb{H}^1}^2 \le \kappa \right\}, \tag{3.32}$$

where $\kappa > 0$ is to be specified. It is clear that for any $\kappa > 0$ and $m \in \mathbb{N}$, we have $\Omega_{\kappa,m} \supset \Omega_{\kappa,m+1}$. Thus, for any time-discrete random variable $\boldsymbol{v}^n$,

$$\begin{aligned}
&\mathbb{E}\left[\max_{m \le n} \sum_{\ell=1}^{m} \mathbb{1}_{\Omega_{\kappa,\ell-1}} \left\langle \boldsymbol{v}^\ell - \boldsymbol{v}^{\ell-1}, \boldsymbol{v}^\ell \right\rangle \right] \\
&= \frac{1}{2} \mathbb{E}\left[\max_{m \le n} \left(\mathbb{1}_{\Omega_{\kappa,m-1}} \|\boldsymbol{v}^m\|_{\mathbb{L}^2}^2 - \mathbb{1}_{\Omega_{\kappa,0}} \left\|\boldsymbol{v}^0\right\|_{\mathbb{L}^2}^2 + \sum_{\ell=2}^{m} \left(\mathbb{1}_{\Omega_{\kappa,m-2}} - \mathbb{1}_{\Omega_{\kappa,m-1}} \right) \left\|\boldsymbol{v}^{m-1}\right\|_{\mathbb{L}^2}^2 \right) \right] \\
&\quad + \frac{1}{2} \sum_{\ell=1}^{n} \mathbb{E}\left[\mathbb{1}_{\Omega_{\kappa,\ell-1}} \left\| \boldsymbol{v}^\ell - \boldsymbol{v}^{\ell-1} \right\|_{\mathbb{L}^2}^2 \right] \\
&\ge \frac{1}{2} \mathbb{E}\left[\max_{m \le n} \left(\mathbb{1}_{\Omega_{\kappa,m-1}} \|\boldsymbol{v}^m\|_{\mathbb{L}^2}^2 \right) - \mathbb{1}_{\Omega_{\kappa,0}} \left\|\boldsymbol{v}^0\right\|_{\mathbb{L}^2}^2 \right] + \frac{1}{2} \sum_{\ell=1}^{n} \mathbb{E}\left[\mathbb{1}_{\Omega_{\kappa,\ell-1}} \left\| \boldsymbol{v}^\ell - \boldsymbol{v}^{\ell-1} \right\|_{\mathbb{L}^2}^2 \right].
\end{aligned} \tag{3.33}$$

We are now ready to prove an auxiliary error estimate. Similar to the assumptions in the following proposition, a technical mesh constraint condition $h = O(k)$ is also implicitly assumed in [22].

Proposition 3.5. Let $(\boldsymbol{u}, \boldsymbol{H})$ be the solution of (2.7) with initial data $\boldsymbol{u}_0 \in \mathbb{H}^2$, and let $(\boldsymbol{u}_h^n, \boldsymbol{H}_h^n)$ be the solution to (3.1). Let $\Omega_{\kappa,m}$ be as defined in (3.32). Let $\boldsymbol{\theta}^n$ and $\boldsymbol{\xi}^n$ be as defined in (3.29) and (3.30), respectively. Assume that $h = O(k)$. Then for $n \in \{1, 2, \dots, N\}$, we have

$$\mathbb{E}\left[\max_{m \le n} \left(\mathbb{1}_{\Omega_{\kappa,m-1}} \|\boldsymbol{\theta}^m\|_{\mathbb{L}^2}^2 \right) \right] + k \sum_{\ell=1}^{n} \mathbb{E}\left[\mathbb{1}_{\Omega_{\kappa,\ell-1}} \left(\left\| \nabla \boldsymbol{\theta}^\ell \right\|_{\mathbb{L}^2}^2 + \left\| \boldsymbol{\xi}^\ell \right\|_{\mathbb{L}^2}^2 \right) \right] \le e^{C_{R,T} \kappa^2} \left(h^2 + k^{\frac{1}{2} - \delta} \right),$$

where $C_{R,T}$ is a constant depending on R and T, but is independent of h and k.

Proof. Subtracting (2.7) from (3.1), rewriting the indices, and noting (3.29) and (3.30), we have for any $\boldsymbol{\chi}_h, \boldsymbol{\phi}_h \in \mathbb{V}_h$,

$$\begin{aligned}
\left\langle \boldsymbol{\theta}^\ell - \boldsymbol{\theta}^{\ell-1}, \boldsymbol{\chi}_h \right\rangle &= -\left\langle \boldsymbol{\rho}^\ell - \boldsymbol{\rho}^{\ell-1}, \boldsymbol{\chi}_h \right\rangle + \int_{t_{\ell-1}}^{t_\ell} \left\langle \boldsymbol{H}(s) - \boldsymbol{H}_h^\ell, \boldsymbol{\chi}_h \right\rangle \mathrm{d}s + \int_{t_{\ell-1}}^{t_\ell} \left\langle \nabla \boldsymbol{H}(s) - \nabla \boldsymbol{H}_h^\ell, \nabla \boldsymbol{\chi}_h \right\rangle \mathrm{d}s \\
&\quad - \int_{t_{\ell-1}}^{t_\ell} \left\langle \left(\boldsymbol{u}(s) - \boldsymbol{u}_h^\ell \right) \times \boldsymbol{H}(s), \boldsymbol{\chi}_h \right\rangle \mathrm{d}s - \int_{t_{\ell-1}}^{t_\ell} \left\langle \boldsymbol{u}_h^\ell \times \left(\boldsymbol{H}(s) - \boldsymbol{H}_h^\ell \right), \boldsymbol{\chi}_h \right\rangle \mathrm{d}s \\
&\quad + \int_{t_{\ell-1}}^{t_\ell} \left\langle \boldsymbol{\nu} \cdot \nabla \left(\boldsymbol{u}(s) - \boldsymbol{u}_h^\ell \right), \boldsymbol{\chi}_h \right\rangle \mathrm{d}s + \int_{t_{\ell-1}}^{t_\ell} \left\langle \left(\boldsymbol{u}(s) - \boldsymbol{u}_h^\ell \right) \times (\boldsymbol{\nu} \cdot \nabla) \boldsymbol{u}_h^\ell, \boldsymbol{\chi}_h \right\rangle \mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&+\int_{t_{\ell-1}}^{t_\ell} \left\langle \boldsymbol{u}(s)\times(\boldsymbol{\nu}\cdot\nabla)\big(\boldsymbol{u}(s)-\boldsymbol{u}_h^\ell\big), \boldsymbol{\chi}_h \right\rangle \mathrm{d}s + \int_{t_{\ell-1}}^{t_\ell} \left\langle \mathcal{M}_R(\boldsymbol{u}(s)) - \mathcal{M}_R(\boldsymbol{u}_h^{\ell-1}), \boldsymbol{\chi}_h \right\rangle \mathrm{d}s \\
&+\int_{t_{\ell-1}}^{t_\ell} \left\langle G(\boldsymbol{u}(s)) - G(\boldsymbol{u}_h^{\ell-1}), \boldsymbol{\chi}_h \right\rangle \mathrm{d}W(s),
\end{aligned} \tag{3.34}
$$

$$
\left\langle \boldsymbol{\xi}^\ell, \boldsymbol{\phi}_h \right\rangle = -\left\langle \boldsymbol{\eta}^\ell, \boldsymbol{\phi}_h \right\rangle - \left\langle \nabla\boldsymbol{\theta}^\ell + \nabla\boldsymbol{\rho}^\ell, \nabla\boldsymbol{\phi}_h \right\rangle + \left\langle f_R(\boldsymbol{u}(t_\ell)) - f_R(\boldsymbol{u}_h^{\ell-1}), \boldsymbol{\phi}_h \right\rangle. \tag{3.35}
$$

We now put $\boldsymbol{\chi}_h = \boldsymbol{\theta}^\ell$ in (3.34) and $\boldsymbol{\phi}_h = k\boldsymbol{\theta}^\ell$ in (3.35), then multiply the resulting equations by $\mathbb{1}_{\Omega_{\kappa,\ell-1}}$, where the set $\Omega_{\kappa,n}$ was defined in (3.32). We then sum the resulting expression over $\ell \in \{1,2,\dots,m\}$, take the maximum over $m \le n$, and apply the expectation value. Noting (3.29), (3.30), (3.31), and (3.33), we obtain

$$
\begin{aligned}
&\frac{1}{2}\mathbb{E}\left[\max_{m\le n}\left(\mathbb{1}_{\Omega_{\kappa,m-1}}\|\boldsymbol{\theta}^m\|_{\mathbb{L}^2}^2\right)\right] + \frac{1}{2}\sum_{\ell=1}^n \mathbb{E}\left[\mathbb{1}_{\Omega_{\kappa,\ell-1}}\left\|\boldsymbol{\theta}^\ell - \boldsymbol{\theta}^{\ell-1}\right\|_{\mathbb{L}^2}^2\right] \\
&\le \frac{1}{2}\mathbb{E}\left[\left\|\boldsymbol{\theta}^0\right\|_{\mathbb{L}^2}^2\right] - \mathbb{E}\left[\max_{m\le n}\sum_{\ell=1}^m \mathbb{1}_{\Omega_{\kappa,\ell-1}}\left\langle \boldsymbol{\rho}^\ell - \boldsymbol{\rho}^{\ell-1}, \boldsymbol{\theta}^\ell \right\rangle\right] \\
&\quad + \mathbb{E}\left[\max_{m\le n}\sum_{\ell=1}^m \mathbb{1}_{\Omega_{\kappa,\ell-1}}\int_{t_{\ell-1}}^{t_\ell}\left\langle \boldsymbol{\eta}^\ell + \boldsymbol{\xi}^\ell + \boldsymbol{H}(s) - \boldsymbol{H}(t_\ell), \boldsymbol{\theta}^\ell \right\rangle \mathrm{d}s\right] \\
&\quad + \mathbb{E}\left[\max_{m\le n}\sum_{\ell=1}^m \mathbb{1}_{\Omega_{\kappa,\ell-1}}\int_{t_{\ell-1}}^{t_\ell}\left\langle \nabla\boldsymbol{\xi}^\ell + \nabla\boldsymbol{H}(s) - \nabla\boldsymbol{H}(t_\ell), \nabla\boldsymbol{\theta}^\ell \right\rangle \mathrm{d}s\right] \\
&\quad - \mathbb{E}\left[\max_{m\le n}\sum_{\ell=1}^m \mathbb{1}_{\Omega_{\kappa,\ell-1}}\int_{t_{\ell-1}}^{t_\ell}\left\langle (\boldsymbol{\rho}^\ell + \boldsymbol{\theta}^\ell + \boldsymbol{u}(s) - \boldsymbol{u}(t_\ell)) \times \boldsymbol{H}(s), \boldsymbol{\theta}^\ell \right\rangle \mathrm{d}s\right] \\
&\quad - \mathbb{E}\left[\max_{m\le n}\sum_{\ell=1}^m \mathbb{1}_{\Omega_{\kappa,\ell-1}}\int_{t_{\ell-1}}^{t_\ell}\left\langle \boldsymbol{u}_h^\ell \times (\boldsymbol{\eta}^\ell + \boldsymbol{\xi}^\ell + \boldsymbol{H}(s) - \boldsymbol{H}(t_\ell)), \boldsymbol{\theta}^\ell \right\rangle \mathrm{d}s\right] \\
&\quad + \mathbb{E}\left[\max_{m\le n}\sum_{\ell=1}^m \mathbb{1}_{\Omega_{\kappa,\ell-1}}\int_{t_{\ell-1}}^{t_\ell}\left\langle \boldsymbol{\nu}\cdot\nabla(\boldsymbol{\rho}^\ell + \boldsymbol{\theta}^\ell + \boldsymbol{u}(s) - \boldsymbol{u}(t_\ell)), \boldsymbol{\theta}^\ell \right\rangle \mathrm{d}s\right] \\
&\quad + \mathbb{E}\left[\max_{m\le n}\sum_{\ell=1}^m \mathbb{1}_{\Omega_{\kappa,\ell-1}}\int_{t_{\ell-1}}^{t_\ell}\left\langle (\boldsymbol{\rho}^\ell + \boldsymbol{\theta}^\ell + \boldsymbol{u}(s) - \boldsymbol{u}(t_\ell)) \times (\boldsymbol{\nu}\cdot\nabla)\boldsymbol{u}_h^\ell, \boldsymbol{\theta}^\ell \right\rangle \mathrm{d}s\right] \\
&\quad + \mathbb{E}\left[\max_{m\le n}\sum_{\ell=1}^m \mathbb{1}_{\Omega_{\kappa,\ell-1}}\int_{t_{\ell-1}}^{t_\ell}\left\langle \boldsymbol{u}(s) \times (\boldsymbol{\nu}\cdot\nabla)(\boldsymbol{\rho}^\ell + \boldsymbol{\theta}^\ell + \boldsymbol{u}(s) - \boldsymbol{u}(t_\ell)), \boldsymbol{\theta}^\ell \right\rangle \mathrm{d}s\right] \\
&\quad + \mathbb{E}\left[\max_{m\le n}\sum_{\ell=1}^m \mathbb{1}_{\Omega_{\kappa,\ell-1}}\int_{t_{\ell-1}}^{t_\ell}\left\langle \mathcal{M}_R(\boldsymbol{u}(s)) - \mathcal{M}_R(\boldsymbol{u}_h^{\ell-1}), \boldsymbol{\theta}^\ell \right\rangle \mathrm{d}s\right] \\
&\quad + \mathbb{E}\left[\max_{m\le n}\sum_{\ell=1}^m \mathbb{1}_{\Omega_{\kappa,\ell-1}}\int_{t_{\ell-1}}^{t_\ell}\left\langle G(\boldsymbol{u}(s)) - G(\boldsymbol{u}_h^{\ell-1}), \boldsymbol{\theta}^\ell \right\rangle \mathrm{d}W(s)\right],
\end{aligned} \tag{3.36}
$$

and

$$
\begin{aligned}
k\mathbb{E}\left[\max_{m\le n}\sum_{\ell=1}^m \mathbb{1}_{\Omega_{\kappa,\ell-1}}\left\langle \boldsymbol{\xi}^\ell, \boldsymbol{\theta}^\ell \right\rangle\right] &= -k\mathbb{E}\left[\max_{m\le n}\sum_{\ell=1}^m \mathbb{1}_{\Omega_{\kappa,\ell-1}}\left\|\nabla\boldsymbol{\theta}^\ell\right\|_{\mathbb{L}^2}^2\right] - k\mathbb{E}\left[\max_{m\le n}\sum_{\ell=1}^m \mathbb{1}_{\Omega_{\kappa,\ell-1}}\left\langle \boldsymbol{\eta}^\ell, \boldsymbol{\theta}^\ell \right\rangle\right] \\
&\quad + k\mathbb{E}\left[\max_{m\le n}\sum_{\ell=1}^m \mathbb{1}_{\Omega_{\kappa,\ell-1}}\left\langle f_R(\boldsymbol{u}(t_\ell)) - f_R(\boldsymbol{u}_h^{\ell-1}), \boldsymbol{\theta}^\ell \right\rangle\right].
\end{aligned} \tag{3.37}
$$

Next, we put $\boldsymbol{\phi}_h = k\boldsymbol{\xi}^n$ to obtain

$$k\mathbb{E}\left[\max_{m\le n}\sum_{\ell=1}^{m}\mathbb{1}_{\Omega_{\kappa,\ell-1}}\left\|\boldsymbol{\xi}^\ell\right\|_{\mathbb{L}^2}^2\right] = -k\mathbb{E}\left[\max_{m\le n}\sum_{\ell=1}^{m}\mathbb{1}_{\Omega_{\kappa,\ell-1}}\left\langle\boldsymbol{\eta}^\ell,\boldsymbol{\xi}^\ell\right\rangle\right] - k\mathbb{E}\left[\max_{m\le n}\sum_{\ell=1}^{m}\mathbb{1}_{\Omega_{\kappa,\ell-1}}\left\langle\nabla\boldsymbol{\theta}^\ell,\nabla\boldsymbol{\xi}^\ell\right\rangle\right] + k\mathbb{E}\left[\max_{m\le n}\sum_{\ell=1}^{m}\mathbb{1}_{\Omega_{\kappa,\ell-1}}\left\langle f_R(\boldsymbol{u}(t_\ell)) - f_R(\boldsymbol{u}_h^{\ell-1}),\boldsymbol{\xi}^\ell\right\rangle\right]. \tag{3.38}$$

Adding (3.36), (3.37), and (3.38), we have

$$\begin{aligned}
&\frac{1}{2}\mathbb{E}\left[\max_{m\le n}\left(\mathbb{1}_{\Omega_{\kappa,m-1}}\|\boldsymbol{\theta}^m\|_{\mathbb{L}^2}^2\right)\right] + \frac{1}{2}\sum_{\ell=1}^{n}\mathbb{E}\left[\mathbb{1}_{\Omega_{\kappa,\ell-1}}\left\|\boldsymbol{\theta}^\ell-\boldsymbol{\theta}^{\ell-1}\right\|_{\mathbb{L}^2}^2\right] \\
&\quad + k\mathbb{E}\left[\sum_{\ell=1}^{n}\mathbb{1}_{\Omega_{\kappa,\ell-1}}\left\|\nabla\boldsymbol{\theta}^\ell\right\|_{\mathbb{L}^2}^2\right] + k\mathbb{E}\left[\sum_{\ell=1}^{n}\mathbb{1}_{\Omega_{\kappa,\ell-1}}\left\|\boldsymbol{\xi}^\ell\right\|_{\mathbb{L}^2}^2\right] \\
&\le \frac{1}{2}\mathbb{E}\left[\left\|\boldsymbol{\theta}^0\right\|_{\mathbb{L}^2}^2\right] - \mathbb{E}\left[\max_{m\le n}\sum_{\ell=1}^{m}\mathbb{1}_{\Omega_{\kappa,\ell-1}}\left\langle\boldsymbol{\rho}^\ell-\boldsymbol{\rho}^{\ell-1},\boldsymbol{\theta}^\ell\right\rangle\right] \\
&\quad + \mathbb{E}\left[\max_{m\le n}\sum_{\ell=1}^{m}\mathbb{1}_{\Omega_{\kappa,\ell-1}}\int_{t_{\ell-1}}^{t_\ell}\left\langle\boldsymbol{\eta}^\ell+\boldsymbol{\xi}^\ell+\boldsymbol{H}(s)-\boldsymbol{H}(t_\ell),\boldsymbol{\theta}^\ell\right\rangle\mathrm{d}s\right] \\
&\quad + \mathbb{E}\left[\max_{m\le n}\sum_{\ell=1}^{m}\mathbb{1}_{\Omega_{\kappa,\ell-1}}\int_{t_{\ell-1}}^{t_\ell}\left\langle\nabla\boldsymbol{H}(s)-\nabla\boldsymbol{H}(t_\ell),\nabla\boldsymbol{\theta}^\ell\right\rangle\mathrm{d}s\right] \\
&\quad - \mathbb{E}\left[\max_{m\le n}\sum_{\ell=1}^{m}\mathbb{1}_{\Omega_{\kappa,\ell-1}}\int_{t_{\ell-1}}^{t_\ell}\left\langle(\boldsymbol{\rho}^\ell+\boldsymbol{\theta}^\ell+\boldsymbol{u}(s)-\boldsymbol{u}(t_\ell))\times\boldsymbol{H}(s),\boldsymbol{\theta}^\ell\right\rangle\mathrm{d}s\right] \\
&\quad - \mathbb{E}\left[\max_{m\le n}\sum_{\ell=1}^{m}\mathbb{1}_{\Omega_{\kappa,\ell-1}}\int_{t_{\ell-1}}^{t_\ell}\left\langle\boldsymbol{u}_h^\ell\times(\boldsymbol{\eta}^\ell+\boldsymbol{\xi}^\ell+\boldsymbol{H}(s)-\boldsymbol{H}(t_\ell)),\boldsymbol{\theta}^\ell\right\rangle\mathrm{d}s\right] \\
&\quad + \mathbb{E}\left[\max_{m\le n}\sum_{\ell=1}^{m}\mathbb{1}_{\Omega_{\kappa,\ell-1}}\int_{t_{\ell-1}}^{t_\ell}\left\langle\boldsymbol{\nu}\cdot\nabla(\boldsymbol{\rho}^\ell+\boldsymbol{\theta}^\ell+\boldsymbol{u}(s)-\boldsymbol{u}(t_\ell)),\boldsymbol{\theta}^\ell\right\rangle\mathrm{d}s\right] \\
&\quad + \mathbb{E}\left[\max_{m\le n}\sum_{\ell=1}^{m}\mathbb{1}_{\Omega_{\kappa,\ell-1}}\int_{t_{\ell-1}}^{t_\ell}\left\langle(\boldsymbol{\rho}^\ell+\boldsymbol{\theta}^\ell+\boldsymbol{u}(s)-\boldsymbol{u}(t_\ell))\times(\boldsymbol{\nu}\cdot\nabla)\boldsymbol{u}_h^\ell,\boldsymbol{\theta}^\ell\right\rangle\mathrm{d}s\right] \\
&\quad + \mathbb{E}\left[\max_{m\le n}\sum_{\ell=1}^{m}\mathbb{1}_{\Omega_{\kappa,\ell-1}}\int_{t_{\ell-1}}^{t_\ell}\left\langle\boldsymbol{u}(s)\times(\boldsymbol{\nu}\cdot\nabla)(\boldsymbol{\rho}^\ell+\boldsymbol{\theta}^\ell+\boldsymbol{u}(s)-\boldsymbol{u}(t_\ell)),\boldsymbol{\theta}^\ell\right\rangle\mathrm{d}s\right] \\
&\quad + \mathbb{E}\left[\max_{m\le n}\sum_{\ell=1}^{m}\mathbb{1}_{\Omega_{\kappa,\ell-1}}\int_{t_{\ell-1}}^{t_\ell}\left\langle\mathcal{M}_R(\boldsymbol{u}(s))-\mathcal{M}_R(\boldsymbol{u}_h^{\ell-1}),\boldsymbol{\theta}^\ell\right\rangle\mathrm{d}s\right] \\
&\quad + k\mathbb{E}\left[\max_{m\le n}\sum_{\ell=1}^{m}\mathbb{1}_{\Omega_{\kappa,\ell-1}}\left\langle f_R(\boldsymbol{u}(t_\ell))-f_R(\boldsymbol{u}_h^{\ell-1}),\boldsymbol{\theta}^\ell\right\rangle\right] \\
&\quad - k\mathbb{E}\left[\max_{m\le n}\sum_{\ell=1}^{m}\mathbb{1}_{\Omega_{\kappa,\ell-1}}\left\langle\boldsymbol{\eta}^\ell,\boldsymbol{\theta}^\ell\right\rangle\right] - k\mathbb{E}\left[\max_{m\le n}\sum_{\ell=1}^{m}\mathbb{1}_{\Omega_{\kappa,\ell-1}}\left\langle\boldsymbol{\eta}^\ell,\boldsymbol{\xi}^\ell\right\rangle\right] \\
&\quad + \mathbb{E}\left[\max_{m\le n}\sum_{\ell=1}^{m}\mathbb{1}_{\Omega_{\kappa,\ell-1}}\int_{t_{\ell-1}}^{t_\ell}\left\langle\mathcal{M}_R(\boldsymbol{u}(s))-\mathcal{M}_R(\boldsymbol{u}_h^{\ell-1}),\boldsymbol{\xi}^\ell\right\rangle\mathrm{d}s\right]
\end{aligned}$$

$$+\mathbb{E}\left[\max_{m\le n}\sum_{\ell=1}^{m}\mathbb{1}_{\Omega_{\kappa,\ell-1}}\int_{t_{\ell-1}}^{t_\ell}\left\langle G(\boldsymbol{u}(s))-G(\boldsymbol{u}_h^{\ell-1}),\boldsymbol{\theta}^\ell\right\rangle \mathrm{d}W(s)\right]$$

$$=: \frac{1}{2}\mathbb{E}\left[\|\boldsymbol{\theta}^0\|_{\mathbb{L}^2}^2\right]+I_1+I_2+\cdots+I_{14}. \tag{3.39}$$

We will estimate each term on the last line, noting the regularity of the solution in Proposition 2.2. Let $\epsilon>0$ be a constant to be determined later. For the first term, by (2.13) and Young's inequality,

$$\begin{aligned}|I_1| &\le Ck^{-1}\mathbb{E}\left[\sum_{\ell=1}^n\left\|\boldsymbol{\rho}^\ell-\boldsymbol{\rho}^{\ell-1}\right\|_{\mathbb{L}^2}^2\right]+Ck\mathbb{E}\left[\sum_{\ell=1}^n\mathbb{1}_{\Omega_{\kappa,\ell-1}}\left\|\boldsymbol{\theta}^\ell\right\|_{\mathbb{L}^2}^2\right]\\ &\le Cnh^4k^{-1}+Ck\mathbb{E}\left[\sum_{\ell=1}^n\mathbb{1}_{\Omega_{\kappa,\ell-1}}\left\|\boldsymbol{\theta}^\ell\right\|_{\mathbb{L}^2}^2\right].\end{aligned}$$

For the second term, by Young's inequality, (2.13), and Proposition 2.2,

$$\begin{aligned}|I_2| &\le Ch^4+\epsilon k\mathbb{E}\left[\sum_{\ell=1}^n\mathbb{1}_{\Omega_{\kappa,\ell-1}}\left\|\boldsymbol{\xi}^\ell\right\|_{\mathbb{L}^2}^2\right]+Ck\mathbb{E}\left[\sum_{\ell=1}^n\mathbb{1}_{\Omega_{\kappa,\ell-1}}\left\|\boldsymbol{\theta}^\ell\right\|_{\mathbb{L}^2}^2\right]\\ &\quad+C\mathbb{E}\left[\sum_{\ell=1}^n\mathbb{1}_{\Omega_{\kappa,\ell-1}}\int_{t_{\ell-1}}^{t_\ell}\|\boldsymbol{H}(s)-\boldsymbol{H}(t_\ell)\|_{\mathbb{L}^2}^2\,\mathrm{d}s\right]\\ &\le Ch^4+Ck^{1-\delta}+\epsilon k\mathbb{E}\left[\sum_{\ell=1}^n\mathbb{1}_{\Omega_{\kappa,\ell-1}}\left\|\boldsymbol{\xi}^\ell\right\|_{\mathbb{L}^2}^2\right]+Ck\mathbb{E}\left[\sum_{\ell=1}^n\mathbb{1}_{\Omega_{\kappa,\ell-1}}\left\|\boldsymbol{\theta}^\ell\right\|_{\mathbb{L}^2}^2\right].\end{aligned}$$

Similarly, for the terms I_3 and I_4 we have

$$\begin{aligned}|I_3| &\le \epsilon k\mathbb{E}\left[\sum_{\ell=1}^n\mathbb{1}_{\Omega_{\kappa,\ell-1}}\left\|\nabla\boldsymbol{\theta}^\ell\right\|_{\mathbb{L}^2}^2\right]+C\mathbb{E}\left[\sum_{\ell=1}^n\mathbb{1}_{\Omega_{\kappa,\ell-1}}\int_{t_{\ell-1}}^{t_\ell}\|\nabla\boldsymbol{H}(s)-\nabla\boldsymbol{H}(t_\ell)\|_{\mathbb{L}^2}^2\,\mathrm{d}s\right]\\ &\le \epsilon k\mathbb{E}\left[\sum_{\ell=1}^n\mathbb{1}_{\Omega_{\kappa,\ell-1}}\left\|\nabla\boldsymbol{\theta}^\ell\right\|_{\mathbb{L}^2}^2\right]+Ck^{\frac12-\delta},\end{aligned} \tag{3.40}$$

and noting (3.32),

$$\begin{aligned}|I_4| &\le Ch^4+Ck\mathbb{E}\left[\sum_{\ell=1}^n\mathbb{1}_{\Omega_{\kappa,\ell-1}}\left\|\boldsymbol{\theta}^\ell\right\|_{\mathbb{L}^2}^2\right]+C\mathbb{E}\left[\sum_{\ell=1}^n\mathbb{1}_{\Omega_{\kappa,\ell-1}}\int_{t_{\ell-1}}^{t_\ell}\|\boldsymbol{H}(s)\|_{\mathbb{L}^2}^2\,\|\boldsymbol{u}(s)-\boldsymbol{u}(t_\ell)\|_{\mathbb{L}^4}^2\,\mathrm{d}s\right]\\ &\le Ch^4+Ck\mathbb{E}\left[\sum_{\ell=1}^n\mathbb{1}_{\Omega_{\kappa,\ell-1}}\left\|\boldsymbol{\theta}^\ell\right\|_{\mathbb{L}^2}^2\right]+Ck\mathbb{E}\left[\sum_{\ell=1}^n\mathbb{1}_{\Omega_{\kappa,\ell-1}}\sup_{s\in[t_{\ell-1},t_\ell]}\|\boldsymbol{H}(s)\|_{\mathbb{L}^2}^2\,\|\boldsymbol{u}(s)-\boldsymbol{u}(t_\ell)\|_{\mathbb{L}^4}^2\right]\\ &\le Ch^4+Ck\mathbb{E}\left[\sum_{\ell=1}^n\mathbb{1}_{\Omega_{\kappa,\ell-1}}\left\|\boldsymbol{\theta}^\ell\right\|_{\mathbb{L}^2}^2\right]+C\kappa k^{1-\delta}.\end{aligned}$$

For the term I_5, by Hölder's and Young's inequalities and (2.22), for any $\epsilon>0$ we have

$$\begin{aligned}|I_5| &\le \mathbb{E}\left[\sum_{\ell=1}^n\mathbb{1}_{\Omega_{\kappa,\ell-1}}\int_{t_{\ell-1}}^{t_\ell}\left\|\boldsymbol{u}_h^\ell\right\|_{\mathbb{L}^4}\left(\left\|\boldsymbol{\eta}^\ell\right\|_{\mathbb{L}^2}+\left\|\boldsymbol{\xi}^\ell\right\|_{\mathbb{L}^2}+\|\boldsymbol{H}(s)-\boldsymbol{H}(t_\ell)\|_{\mathbb{L}^2}\right)\left\|\boldsymbol{\theta}^\ell\right\|_{\mathbb{L}^4}\mathrm{d}s\right]\\ &\le Ck\mathbb{E}\left[\sum_{\ell=1}^n\mathbb{1}_{\Omega_{\kappa,\ell-1}}\left\|\boldsymbol{u}_h^\ell\right\|_{\mathbb{L}^4}^2\left\|\boldsymbol{\theta}^\ell\right\|_{\mathbb{L}^4}^2\right]+Ch^4+\epsilon k\mathbb{E}\left[\sum_{\ell=1}^n\mathbb{1}_{\Omega_{\kappa,\ell-1}}\left\|\boldsymbol{\xi}^\ell\right\|_{\mathbb{L}^2}^2\right]\\ &\quad+\epsilon\mathbb{E}\left[\sum_{\ell=1}^n\mathbb{1}_{\Omega_{\kappa,\ell-1}}\int_{t_{\ell-1}}^{t_\ell}\|\boldsymbol{H}(s)-\boldsymbol{H}(t_\ell)\|_{\mathbb{L}^2}^2\,\mathrm{d}s\right]\end{aligned}$$

$$\le Ch^4 + Ck^{1-\delta} + C\kappa^2 k\mathbb{E}\left[\sum_{\ell=1}^{n} \mathbb{1}_{\Omega_{\kappa,\ell-1}} \left\|\boldsymbol{\theta}^\ell\right\|_{\mathbb{L}^2}^2\right] + \epsilon k\mathbb{E}\left[\sum_{\ell=1}^{n} \mathbb{1}_{\Omega_{\kappa,\ell-1}} \left\|\nabla\boldsymbol{\theta}^\ell\right\|_{\mathbb{L}^2}^2\right]$$
$$+ \epsilon k\mathbb{E}\left[\sum_{\ell=1}^{n} \mathbb{1}_{\Omega_{\kappa,\ell-1}} \left\|\boldsymbol{\xi}^\ell\right\|_{\mathbb{L}^2}^2\right].$$

For the terms I_6 to I_8, we note Lemma 3.2 and apply similar argument to obtain

$$\begin{aligned}
|I_6| &\le Ck\mathbb{E}\left[\sum_{\ell=1}^{n} \mathbb{1}_{\Omega_{\kappa,\ell-1}} \left\|\boldsymbol{\theta}^\ell\right\|_{\mathbb{L}^2}^2\right] + \epsilon h^2 + \epsilon k\mathbb{E}\left[\sum_{\ell=1}^{n} \mathbb{1}_{\Omega_{\kappa,\ell-1}} \left\|\nabla\boldsymbol{\theta}^\ell\right\|_{\mathbb{L}^2}^2\right] \\
&\quad + \epsilon\mathbb{E}\left[\sum_{\ell=1}^{n} \mathbb{1}_{\Omega_{\kappa,\ell-1}} \int_{t_{\ell-1}}^{t_\ell} \|\nabla\boldsymbol{u}(s) - \nabla\boldsymbol{u}(t_\ell)\|_{\mathbb{L}^2}^2 \,\mathrm{d}s\right] \\
&\le Ck\mathbb{E}\left[\sum_{\ell=1}^{n} \mathbb{1}_{\Omega_{\kappa,\ell-1}} \left\|\boldsymbol{\theta}^\ell\right\|_{\mathbb{L}^2}^2\right] + \epsilon k\mathbb{E}\left[\sum_{\ell=1}^{n} \mathbb{1}_{\Omega_{\kappa,\ell-1}} \left\|\nabla\boldsymbol{\theta}^\ell\right\|_{\mathbb{L}^2}^2\right] + Ch^2 + Ck^{1-\delta}, \\
|I_7| &\le \epsilon k\mathbb{E}\left[\sum_{\ell=1}^{n} \mathbb{1}_{\Omega_{\kappa,\ell-1}} \left\|\boldsymbol{\theta}^\ell\right\|_{\mathbb{L}^4}^2\right] + C\mathbb{E}\left[\max_{j\le n} \left\|\boldsymbol{\rho}^j\right\|_{\mathbb{L}^4}^4\right]^{\frac12} \mathbb{E}\left[\left(k\sum_{\ell=1}^{n} \left\|\nabla\boldsymbol{u}_h^\ell\right\|_{\mathbb{L}^2}^2\right)^2\right]^{\frac12} \\
&\quad + Ck\mathbb{E}\left[\sum_{\ell=1}^{n} \mathbb{1}_{\Omega_{\kappa,\ell-1}} \int_{t_{\ell-1}}^{t_\ell} \|\nabla\boldsymbol{u}_h^n\|_{\mathbb{L}^2}^2 \|\boldsymbol{u}(s) - \boldsymbol{u}(t_n)\|_{\mathbb{L}^\infty}^2 \,\mathrm{d}s\right] \\
&\le \epsilon k\mathbb{E}\left[\sum_{\ell=1}^{n} \mathbb{1}_{\Omega_{\kappa,\ell-1}} \left\|\boldsymbol{\theta}^\ell\right\|_{\mathbb{H}^1}^2\right] + Ch^4 + C\kappa k^{1-\delta}, \\
|I_8| &\le C\mathbb{E}\left[\sum_{\ell=1}^{n} \mathbb{1}_{\Omega_{\kappa,\ell-1}} \left\|\boldsymbol{\theta}^\ell\right\|_{\mathbb{L}^2}^2 \int_{t_{\ell-1}}^{t_\ell} \|\boldsymbol{u}(s)\|_{\mathbb{L}^\infty}^2 \,\mathrm{d}s\right] + \epsilon h^2\mathbb{E}\left[\|\boldsymbol{u}\|_{L^\infty(\mathbb{L}^\infty)}^2\right] \\
&\quad + \epsilon k\mathbb{E}\left[\sum_{\ell=1}^{n} \mathbb{1}_{\Omega_{\kappa,\ell-1}} \left\|\nabla\boldsymbol{\theta}^\ell\right\|_{\mathbb{L}^2}^2\right] + \epsilon\mathbb{E}\left[\mathbb{1}_{\Omega_{\kappa,n}} \int_{t_{n-1}}^{t_n} \|\nabla\boldsymbol{u}(s) - \nabla\boldsymbol{u}(t_n)\|_{\mathbb{L}^2}^2 \,\mathrm{d}s\right] \\
&\le C\kappa k\mathbb{E}\left[\sum_{\ell=1}^{n} \mathbb{1}_{\Omega_{\kappa,\ell-1}} \left\|\boldsymbol{\theta}^\ell\right\|_{\mathbb{L}^2}^2\right] + Ch^2 + \epsilon k^{1-\delta} + \epsilon k\mathbb{E}\left[\sum_{\ell=1}^{n} \mathbb{1}_{\Omega_{\kappa,\ell-1}} \left\|\nabla\boldsymbol{\theta}^\ell\right\|_{\mathbb{L}^2}^2\right].
\end{aligned}$$

For the terms I_9, I_{10}, and I_{13}, we apply the Lipschitz continuity assumption on $\mathcal{M}_R$ and f_R to obtain

$$|I_9| + |I_{10}| + |I_{13}| \le Ch^4 + Ck^{1-\delta} + Ck\mathbb{E}\left[\sum_{\ell=1}^{n} \mathbb{1}_{\Omega_{\kappa,\ell-1}} \left\|\boldsymbol{\theta}^\ell\right\|_{\mathbb{L}^2}^2\right] + \epsilon k\mathbb{E}\left[\sum_{\ell=1}^{n} \mathbb{1}_{\Omega_{\kappa,\ell-1}} \left\|\boldsymbol{\xi}^\ell\right\|_{\mathbb{L}^2}^2\right].$$

The terms I_{11} and I_{12} can be estimated easily as

$$|I_{11}| + |I_{12}| \le Ck\mathbb{E}\left[\sum_{\ell=1}^{n} \mathbb{1}_{\Omega_{\kappa,\ell-1}} \left\|\boldsymbol{\theta}^\ell\right\|_{\mathbb{L}^2}^2\right] + Ch^4 + \epsilon k\mathbb{E}\left[\sum_{\ell=1}^{n} \mathbb{1}_{\Omega_{\kappa,\ell-1}} \left\|\boldsymbol{\xi}^\ell\right\|_{\mathbb{L}^2}^2\right].$$

For the term I_{14}, we split the stochastic integral as

$$\begin{aligned}
I_{14} &= \mathbb{E}\left[\max_{m\le n} \sum_{\ell=1}^{m} \mathbb{1}_{\Omega_{\kappa,\ell-1}} \int_{t_{\ell-1}}^{t_\ell} \left\langle G(\boldsymbol{u}(s)) - G(\boldsymbol{u}_h^{\ell-1}), \boldsymbol{\theta}^{\ell-1}\right\rangle \mathrm{d}W(s)\right] \\
&\quad + \mathbb{E}\left[\max_{m\le n} \sum_{\ell=1}^{m} \mathbb{1}_{\Omega_{\kappa,\ell-1}} \int_{t_{\ell-1}}^{t_\ell} \left\langle G(\boldsymbol{u}(s)) - G(\boldsymbol{u}_h^{\ell-1}), \boldsymbol{\theta}^\ell - \boldsymbol{\theta}^{\ell-1}\right\rangle \mathrm{d}W(s)\right] =: I_{14a} + I_{14b}.
\end{aligned}$$

For the first term above, noting the assumptions on G and the Hölder continuity of $\boldsymbol{u}$, we apply the Burkholder–Davis–Gundy and the Young inequalities to obtain

$$\begin{aligned}
I_{14a} &\le C\mathbb{E}\left[\left(\sum_{\ell=1}^n \mathbb{1}_{\Omega_{\kappa,\ell-1}} \int_{t_{\ell-1}}^{t_\ell} \left\|G(\boldsymbol{u}(s)) - G(\boldsymbol{u}_h^{\ell-1})\right\|_{\mathbb{L}^2}^2 \left\|\boldsymbol{\theta}^{\ell-1}\right\|_{\mathbb{L}^2}^2 \,\mathrm{d}s\right)^{\frac{1}{2}}\right] \\
&\le C\mathbb{E}\left[\left(\max_{m\le n} \mathbb{1}_{\Omega_{\kappa,m-1}} \left\|\boldsymbol{\theta}^{m-1}\right\|_{\mathbb{L}^2}\right)\left(\sum_{\ell=1}^n \mathbb{1}_{\Omega_{\kappa,\ell-1}} \int_{t_{\ell-1}}^{t_\ell} \left\|G(\boldsymbol{u}(s)) - G(\boldsymbol{u}_h^{\ell-1})\right\|_{\mathbb{L}^2}^2\right)^{\frac{1}{2}}\right] \\
&\le \epsilon\mathbb{E}\left[\max_{m\le n} \mathbb{1}_{\Omega_{\kappa,m-1}} \left\|\boldsymbol{\theta}^{m}\right\|_{\mathbb{L}^2}^2\right] + Ch^4 + Ck^{1-\delta} + Ck\mathbb{E}\left[\sum_{\ell=1}^n \mathbb{1}_{\Omega_{\kappa,\ell-1}} \left\|\boldsymbol{\theta}^{\ell-1}\right\|_{\mathbb{L}^2}^2\right].
\end{aligned}$$

For the term I_{14b}, we use Young's inequality, Itô's isometry, the assumptions on G, and the Hölder continuity of $\boldsymbol{u}$ to obtain

$$\begin{aligned}
I_{14b} &\le C\mathbb{E}\left[\sum_{\ell=1}^n \left\|\mathbb{1}_{\Omega_{\kappa,\ell-1}} \int_{t_{\ell-1}}^{t_\ell} \left(G(\boldsymbol{u}(s)) - G(\boldsymbol{u}_h^{\ell-1})\right) \mathrm{d}W(s)\right\|_{\mathbb{L}^2}^2\right] + \epsilon\mathbb{E}\left[\sum_{\ell=1}^n \mathbb{1}_{\Omega_{\kappa,\ell-1}} \left\|\boldsymbol{\theta}^{\ell} - \boldsymbol{\theta}^{\ell-1}\right\|_{\mathbb{L}^2}^2\right] \\
&= C\mathbb{E}\left[\sum_{\ell=1}^n \mathbb{1}_{\Omega_{\kappa,\ell-1}} \int_{t_{\ell-1}}^{t_\ell} \left\|G(\boldsymbol{u}(s)) - G(\boldsymbol{u}_h^{\ell-1})\right\|_{\mathbb{L}^2}^2 \mathrm{d}s\right] + \epsilon\mathbb{E}\left[\sum_{\ell=1}^n \mathbb{1}_{\Omega_{\kappa,\ell-1}} \left\|\boldsymbol{\theta}^{\ell} - \boldsymbol{\theta}^{\ell-1}\right\|_{\mathbb{L}^2}^2\right] \\
&\le Ch^4 + Ck^{1-\delta} + Ck\mathbb{E}\left[\sum_{\ell=1}^n \mathbb{1}_{\Omega_{\kappa,\ell-1}} \left\|\boldsymbol{\theta}^{\ell-1}\right\|_{\mathbb{L}^2}^2\right] + \epsilon\mathbb{E}\left[\sum_{\ell=1}^n \mathbb{1}_{\Omega_{\kappa,\ell-1}} \left\|\boldsymbol{\theta}^{\ell} - \boldsymbol{\theta}^{\ell-1}\right\|_{\mathbb{L}^2}^2\right].
\end{aligned}$$

We substitute all the above estimates into (3.39), set $\epsilon = 1/16$, and rearrange the terms. Now, continuing from (3.39), noting the assumption $h = O(k)$ we obtain

$$\begin{aligned}
&\mathbb{E}\left[\max_{m\le n}\left(\mathbb{1}_{\Omega_{\kappa,m-1}} \left\|\boldsymbol{\theta}^{m}\right\|_{\mathbb{L}^2}^2\right)\right] + \sum_{\ell=1}^n \mathbb{E}\left[\mathbb{1}_{\Omega_{\kappa,\ell-1}} \left\|\boldsymbol{\theta}^{\ell} - \boldsymbol{\theta}^{\ell-1}\right\|_{\mathbb{L}^2}^2\right] \\
&\quad + k\mathbb{E}\left[\sum_{\ell=1}^n \mathbb{1}_{\Omega_{\kappa,\ell-1}} \left\|\nabla\boldsymbol{\theta}^{\ell}\right\|_{\mathbb{L}^2}^2\right] + k\mathbb{E}\left[\sum_{\ell=1}^n \mathbb{1}_{\Omega_{\kappa,\ell-1}} \left\|\boldsymbol{\xi}^{\ell}\right\|_{\mathbb{L}^2}^2\right] \\
&\le \mathbb{E}\left[\left\|\boldsymbol{\theta}^{0}\right\|_{\mathbb{L}^2}^2\right] + Ch^2 + C(1+\kappa)k^{\frac{1}{2}-\delta} + C(1+\kappa^2)k\sum_{\ell=1}^n \mathbb{E}\left[\mathbb{1}_{\Omega_{\kappa,\ell-1}} \left\|\boldsymbol{\theta}^{\ell}\right\|_{\mathbb{L}^2}^2\right]. \qquad (3.41)
\end{aligned}$$

By choosing $\boldsymbol{u}_h^0$ such that $\mathbb{E}\left[\left\|\boldsymbol{\theta}^{0}\right\|_{\mathbb{L}^2}^2\right] \le Ch^2$, say $\boldsymbol{u}_h^0 = \Pi_h \boldsymbol{u}_0$, we infer the required result for sufficiently small k by the discrete Gronwall lemma. □

We also deduce the following estimate in stronger norms.

Proposition 3.6. Assume that the hypotheses of Proposition 3.5 hold. Then for $n \in \{1, 2, \dots, N\}$, we have

$$\mathbb{E}\left[\max_{m\le n}\left(\mathbb{1}_{\Omega_{\kappa,m-1}} \left\|\nabla\boldsymbol{\theta}^{m}\right\|_{\mathbb{L}^2}^2\right)\right] + k\sum_{\ell=1}^n \mathbb{E}\left[\mathbb{1}_{\Omega_{\kappa,\ell-1}} \left(\left\|\nabla\Delta_h\boldsymbol{\theta}^{\ell}\right\|_{\mathbb{L}^2}^2 + \left\|\nabla\boldsymbol{\xi}^{\ell}\right\|_{\mathbb{L}^2}^2\right)\right] \le Ce^{C\kappa^2}\left(h^2 + k^{\frac{1}{2}-\delta}\right),$$

where C is a constant depending on R, T, and the coefficients of the equation, but is independent of h and k.

Proof. We put $\boldsymbol{\chi}_h = -\Delta_h\boldsymbol{\theta}^{\ell}$ in (3.34), then multiply the resulting equations by $\mathbb{1}_{\Omega_{\kappa,\ell-1}}$, sum the resulting expression over $\ell \in \{1, 2, \dots, m\}$, take the expectation value, and argue similarly as in (3.36) and (3.39) to obtain

$$
\begin{aligned}
&\frac{1}{2}\mathbb{E}\left[\max_{m\le n}\left(\mathbb{1}_{\Omega_{\kappa,m-1}}\|\nabla\boldsymbol{\theta}^m\|_{\mathbb{L}^2}^2\right)\right]+\frac{1}{2}\mathbb{E}\left[\sum_{\ell=1}^n \mathbb{1}_{\Omega_{\kappa,\ell-1}}\left\|\nabla\boldsymbol{\theta}^\ell-\nabla\boldsymbol{\theta}^{\ell-1}\right\|_{\mathbb{L}^2}^2\right]\\
&\le \frac{1}{2}\mathbb{E}\left[\left\|\nabla\boldsymbol{\theta}^0\right\|_{\mathbb{L}^2}^2\right]+\mathbb{E}\left[\sum_{\ell=1}^n \mathbb{1}_{\Omega_{\kappa,\ell-1}}\left\langle \boldsymbol{\rho}^\ell-\boldsymbol{\rho}^{\ell-1},\Delta_h\boldsymbol{\theta}^\ell\right\rangle\right]\\
&\quad-\mathbb{E}\left[\sum_{\ell=1}^n \mathbb{1}_{\Omega_{\kappa,\ell-1}}\int_{t_{\ell-1}}^{t_\ell}\left\langle \boldsymbol{\eta}^\ell+\boldsymbol{\xi}^\ell+\boldsymbol{H}(s)-\boldsymbol{H}(t_\ell),\Delta_h\boldsymbol{\theta}^\ell\right\rangle \mathrm{d}s\right]\\
&\quad-\mathbb{E}\left[\sum_{\ell=1}^n \mathbb{1}_{\Omega_{\kappa,\ell-1}}\int_{t_{\ell-1}}^{t_\ell}\left\langle \nabla\boldsymbol{\xi}^\ell+\nabla\boldsymbol{H}(s)-\nabla\boldsymbol{H}(t_\ell),\nabla\Delta_h\boldsymbol{\theta}^\ell\right\rangle \mathrm{d}s\right]\\
&\quad+\mathbb{E}\left[\sum_{\ell=1}^n \mathbb{1}_{\Omega_{\kappa,\ell-1}}\int_{t_{\ell-1}}^{t_\ell}\left\langle (\boldsymbol{\rho}^\ell+\boldsymbol{\theta}^\ell+\boldsymbol{u}(s)-\boldsymbol{u}(t_\ell))\times\boldsymbol{H}(s),\Delta_h\boldsymbol{\theta}^\ell\right\rangle \mathrm{d}s\right]\\
&\quad+\mathbb{E}\left[\sum_{\ell=1}^n \mathbb{1}_{\Omega_{\kappa,\ell-1}}\int_{t_{\ell-1}}^{t_\ell}\left\langle \boldsymbol{u}_h^\ell\times(\boldsymbol{\eta}^\ell+\boldsymbol{\xi}^\ell+\boldsymbol{H}(s)-\boldsymbol{H}(t_\ell)),\Delta_h\boldsymbol{\theta}^\ell\right\rangle \mathrm{d}s\right]\\
&\quad-\mathbb{E}\left[\sum_{\ell=1}^n \mathbb{1}_{\Omega_{\kappa,\ell-1}}\int_{t_{\ell-1}}^{t_\ell}\left\langle \boldsymbol{\nu}\cdot\nabla(\boldsymbol{\rho}^\ell+\boldsymbol{\theta}^\ell+\boldsymbol{u}(s)-\boldsymbol{u}(t_\ell)),\Delta_h\boldsymbol{\theta}^\ell\right\rangle \mathrm{d}s\right]\\
&\quad-\mathbb{E}\left[\sum_{\ell=1}^n \mathbb{1}_{\Omega_{\kappa,\ell-1}}\int_{t_{\ell-1}}^{t_\ell}\left\langle (\boldsymbol{\rho}^\ell+\boldsymbol{\theta}^\ell+\boldsymbol{u}(s)-\boldsymbol{u}(t_\ell))\times(\boldsymbol{\nu}\cdot\nabla)\boldsymbol{u}_h^\ell,\Delta_h\boldsymbol{\theta}^\ell\right\rangle \mathrm{d}s\right]\\
&\quad-\mathbb{E}\left[\sum_{\ell=1}^n \mathbb{1}_{\Omega_{\kappa,\ell-1}}\int_{t_{\ell-1}}^{t_\ell}\left\langle \boldsymbol{u}(s)\times(\boldsymbol{\nu}\cdot\nabla)(\boldsymbol{\rho}^\ell+\boldsymbol{\theta}^\ell+\boldsymbol{u}(s)-\boldsymbol{u}(t_\ell)),\Delta_h\boldsymbol{\theta}^\ell\right\rangle \mathrm{d}s\right]\\
&\quad-\mathbb{E}\left[\sum_{\ell=1}^n \mathbb{1}_{\Omega_{\kappa,\ell-1}}\int_{t_{\ell-1}}^{t_\ell}\left\langle \mathcal{M}_R(\boldsymbol{u}(s))-\mathcal{M}_R(\boldsymbol{u}_h^{\ell-1}),\Delta_h\boldsymbol{\theta}^\ell\right\rangle \mathrm{d}s\right]\\
&\quad-\mathbb{E}\left[\sum_{\ell=1}^n \mathbb{1}_{\Omega_{\kappa,\ell-1}}\int_{t_{\ell-1}}^{t_\ell}\left\langle G(\boldsymbol{u}(s))-G(\boldsymbol{u}_h^{\ell-1}),\Delta_h\boldsymbol{\theta}^\ell\right\rangle \mathrm{d}W(s)\right].
\end{aligned}
\tag{3.42}
$$

Similarly, we take $\boldsymbol{\phi}_h = k\Delta_h^2\boldsymbol{\theta}^\ell$ and rearrange the terms. Noting the definition of Δ_h in (2.14), we have

$$
\begin{aligned}
k\mathbb{E}\left[\sum_{\ell=1}^n \mathbb{1}_{\Omega_{\kappa,\ell-1}}\left\|\nabla\Delta_h\boldsymbol{\theta}^\ell\right\|_{\mathbb{L}^2}^2\right]&=k\mathbb{E}\left[\sum_{\ell=1}^n \mathbb{1}_{\Omega_{\kappa,\ell-1}}\left\langle \nabla\boldsymbol{\xi}^\ell,\nabla\Delta_h\boldsymbol{\theta}^\ell\right\rangle \mathrm{d}s\right]\\
&\quad-k\mathbb{E}\left[\sum_{\ell=1}^n \mathbb{1}_{\Omega_{\kappa,\ell-1}}\left\langle \nabla\Pi_h\boldsymbol{\eta}^\ell,\nabla\Delta_h\boldsymbol{\theta}^\ell\right\rangle\right]\\
&\quad+k\mathbb{E}\left[\sum_{\ell=1}^n \mathbb{1}_{\Omega_{\kappa,\ell-1}}\left\langle \nabla\Pi_h f_R(\boldsymbol{u}(t_\ell))-\nabla\Pi_h f_R(\boldsymbol{u}_h^{\ell-1}),\nabla\Delta_h\boldsymbol{\theta}^\ell\right\rangle\right].
\end{aligned}
\tag{3.43}
$$

Adding (3.42) and (3.43), upon rearranging the terms we obtain

$$
\frac{1}{2}\mathbb{E}\left[\max_{m\le n}\left(\mathbb{1}_{\Omega_{\kappa,m-1}}\|\nabla\boldsymbol{\theta}^m\|_{\mathbb{L}^2}^2\right)\right]+\frac{1}{2}\mathbb{E}\left[\sum_{\ell=1}^n \mathbb{1}_{\Omega_{\kappa,\ell-1}}\left\|\nabla\boldsymbol{\theta}^\ell-\nabla\boldsymbol{\theta}^{\ell-1}\right\|_{\mathbb{L}^2}^2\right]
$$

$$
\begin{aligned}
&+ k\mathbb{E}\left[\sum_{\ell=1}^n \mathbb{1}_{\Omega_{\kappa,\ell-1}} \left\|\Delta_h \boldsymbol{\theta}^\ell\right\|_{\mathbb{L}^2}^2\right] + k\mathbb{E}\left[\sum_{\ell=1}^n \mathbb{1}_{\Omega_{\kappa,\ell-1}} \left\|\nabla\Delta_h \boldsymbol{\theta}^\ell\right\|_{\mathbb{L}^2}^2\right] \\
&\le \frac{1}{2}\mathbb{E}\left[\|\nabla\boldsymbol{\theta}^0\|_{\mathbb{L}^2}^2\right] + k\mathbb{E}\left[\sum_{\ell=1}^n \mathbb{1}_{\Omega_{\kappa,\ell-1}} \left\|\Delta_h \boldsymbol{\theta}^\ell\right\|_{\mathbb{L}^2}^2\right] \\
&+ \mathbb{E}\left[\sum_{\ell=1}^n \mathbb{1}_{\Omega_{\kappa,\ell-1}} \left\langle \boldsymbol{\rho}^\ell - \boldsymbol{\rho}^{\ell-1}, \Delta_h\boldsymbol{\theta}^\ell\right\rangle\right] \\
&- \mathbb{E}\left[\sum_{\ell=1}^n \mathbb{1}_{\Omega_{\kappa,\ell-1}} \int_{t_{\ell-1}}^{t_\ell} \left\langle \boldsymbol{\eta}^\ell + \boldsymbol{\xi}^\ell + \boldsymbol{H}(s) - \boldsymbol{H}(t_\ell), \Delta_h\boldsymbol{\theta}^\ell\right\rangle \mathrm{d}s\right] \\
&- \mathbb{E}\left[\sum_{\ell=1}^n \mathbb{1}_{\Omega_{\kappa,\ell-1}} \int_{t_{\ell-1}}^{t_\ell} \left\langle \nabla\boldsymbol{H}(s) - \nabla\boldsymbol{H}(t_\ell), \nabla\Delta_h\boldsymbol{\theta}^\ell\right\rangle \mathrm{d}s\right] \\
&- k\mathbb{E}\left[\sum_{\ell=1}^n \mathbb{1}_{\Omega_{\kappa,\ell-1}} \left\langle \nabla\Pi_h\boldsymbol{\eta}^\ell, \nabla\Delta_h\boldsymbol{\theta}^\ell\right\rangle\right] \\
&+ k\mathbb{E}\left[\sum_{\ell=1}^n \mathbb{1}_{\Omega_{\kappa,\ell-1}} \left\langle \nabla\Pi_h f_R(\boldsymbol{u}(t_\ell)) - \nabla\Pi_h f_R(\boldsymbol{u}_h^{\ell-1}), \nabla\Delta_h\boldsymbol{\theta}^\ell\right\rangle\right] \\
&+ \mathbb{E}\left[\sum_{\ell=1}^n \mathbb{1}_{\Omega_{\kappa,\ell-1}} \int_{t_{\ell-1}}^{t_\ell} \left\langle (\boldsymbol{\rho}^\ell + \boldsymbol{\theta}^\ell + \boldsymbol{u}(s) - \boldsymbol{u}(t_\ell)) \times \boldsymbol{H}(s), \Delta_h\boldsymbol{\theta}^\ell\right\rangle \mathrm{d}s\right] \\
&+ \mathbb{E}\left[\sum_{\ell=1}^n \mathbb{1}_{\Omega_{\kappa,\ell-1}} \int_{t_{\ell-1}}^{t_\ell} \left\langle \boldsymbol{u}_h^\ell \times (\boldsymbol{\eta}^\ell + \boldsymbol{\xi}^\ell + \boldsymbol{H}(s) - \boldsymbol{H}(t_\ell)), \Delta_h\boldsymbol{\theta}^\ell\right\rangle \mathrm{d}s\right] \\
&- \mathbb{E}\left[\sum_{\ell=1}^n \mathbb{1}_{\Omega_{\kappa,\ell-1}} \int_{t_{\ell-1}}^{t_\ell} \left\langle \boldsymbol{\nu}\cdot\nabla(\boldsymbol{\rho}^\ell + \boldsymbol{\theta}^\ell + \boldsymbol{u}(s) - \boldsymbol{u}(t_\ell)), \Delta_h\boldsymbol{\theta}^\ell\right\rangle \mathrm{d}s\right] \\
&- \mathbb{E}\left[\sum_{\ell=1}^n \mathbb{1}_{\Omega_{\kappa,\ell-1}} \int_{t_{\ell-1}}^{t_\ell} \left\langle (\boldsymbol{\rho}^\ell + \boldsymbol{\theta}^\ell + \boldsymbol{u}(s) - \boldsymbol{u}(t_\ell)) \times (\boldsymbol{\nu}\cdot\nabla)\boldsymbol{u}_h^\ell, \Delta_h\boldsymbol{\theta}^\ell\right\rangle \mathrm{d}s\right] \\
&- \mathbb{E}\left[\sum_{\ell=1}^n \mathbb{1}_{\Omega_{\kappa,\ell-1}} \int_{t_{\ell-1}}^{t_\ell} \left\langle \boldsymbol{u}(s) \times (\boldsymbol{\nu}\cdot\nabla)(\boldsymbol{\rho}^\ell + \boldsymbol{\theta}^\ell + \boldsymbol{u}(s) - \boldsymbol{u}(t_\ell)), \Delta_h\boldsymbol{\theta}^\ell\right\rangle \mathrm{d}s\right] \\
&- \mathbb{E}\left[\sum_{\ell=1}^n \mathbb{1}_{\Omega_{\kappa,\ell-1}} \int_{t_{\ell-1}}^{t_\ell} \left\langle \mathcal{M}_R(\boldsymbol{u}(s)) - \mathcal{M}_R(\boldsymbol{u}_h^{\ell-1}), \Delta_h\boldsymbol{\theta}^\ell\right\rangle \mathrm{d}s\right] \\
&- \mathbb{E}\left[\sum_{\ell=1}^n \mathbb{1}_{\Omega_{\kappa,\ell-1}} \int_{t_{\ell-1}}^{t_\ell} \left\langle G(\boldsymbol{u}(s)) - G(\boldsymbol{u}_h^{\ell-1}), \Delta_h\boldsymbol{\theta}^\ell\right\rangle \mathrm{d}W(s)\right] \\
&=: I_1 + I_2 + \cdots + I_{13}.
\end{aligned} \tag{3.44}
$$

We will estimate each term on the last line. In the following, whenever appropriate, we always use Proposition 2.2 and 3.5 without further mention. Let $\epsilon > 0$. For the first term, by (2.16) and Young's inequality we have

$$
|I_1| \le Ck\mathbb{E}\left[\sum_{\ell=1}^n \mathbb{1}_{\Omega_{\kappa,\ell-1}} \left\|\nabla\boldsymbol{\theta}^\ell\right\|_{\mathbb{L}^2}^2\right] + \epsilon k\mathbb{E}\left[\sum_{\ell=1}^n \mathbb{1}_{\Omega_{\kappa,\ell-1}} \left\|\nabla\Delta_h\boldsymbol{\theta}^\ell\right\|_{\mathbb{L}^2}^2\right]
$$

$$\leq Ce^{C\kappa^2}\left(h^2+k^{\frac{1}{2}-\delta}\right)+\epsilon k\mathbb{E}\left[\sum_{\ell=1}^{n}\mathbb{1}_{\Omega_{\kappa,\ell-1}}\left\|\nabla\Delta_h\boldsymbol{\theta}^\ell\right\|_{\mathbb{L}^2}^2\right].$$

For the second term, by Young's inequality and (2.13),

$$\begin{aligned}|I_2|&\leq Ck^{-1}\mathbb{E}\left[\sum_{\ell=1}^{n}\left\|\boldsymbol{\rho}^\ell-\boldsymbol{\rho}^{\ell-1}\right\|_{\mathbb{L}^2}^2\right]+\epsilon k\mathbb{E}\left[\sum_{\ell=1}^{n}\mathbb{1}_{\Omega_{\kappa,\ell-1}}\left\|\Delta_h\boldsymbol{\theta}^\ell\right\|_{\mathbb{L}^2}^2\right]\\&\leq Cnh^4k^{-1}+\epsilon k\mathbb{E}\left[\sum_{\ell=1}^{n}\mathbb{1}_{\Omega_{\kappa,\ell-1}}\left\|\Delta_h\boldsymbol{\theta}^\ell\right\|_{\mathbb{L}^2}^2\right].\end{aligned}$$

Similarly, noting Proposition 2.2 we have

$$\begin{aligned}|I_3|&\leq Ch^4+Ck\mathbb{E}\left[\sum_{\ell=1}^{n}\mathbb{1}_{\Omega_{\kappa,\ell-1}}\left\|\boldsymbol{\xi}^\ell\right\|_{\mathbb{L}^2}^2\right]+\epsilon k\mathbb{E}\left[\sum_{\ell=1}^{n}\mathbb{1}_{\Omega_{\kappa,\ell-1}}\left\|\Delta_h\boldsymbol{\theta}^\ell\right\|_{\mathbb{L}^2}^2\right]\\&\quad+C\mathbb{E}\left[\sum_{\ell=1}^{n}\mathbb{1}_{\Omega_{\kappa,\ell-1}}\int_{t_{\ell-1}}^{t_\ell}\|\boldsymbol{H}(s)-\boldsymbol{H}(t_\ell)\|_{\mathbb{L}^2}^2\,\mathrm{d}s\right]\\&\leq Ce^{C\kappa^2}\left(h^2+k^{\frac{1}{2}-\delta}\right)+\epsilon k\mathbb{E}\left[\sum_{\ell=1}^{n}\mathbb{1}_{\Omega_{\kappa,\ell-1}}\left\|\Delta_h\boldsymbol{\theta}^\ell\right\|_{\mathbb{L}^2}^2\right],\\|I_4|&\leq\epsilon k\mathbb{E}\left[\sum_{\ell=1}^{n}\mathbb{1}_{\Omega_{\kappa,\ell-1}}\left\|\nabla\Delta_h\boldsymbol{\theta}^\ell\right\|_{\mathbb{L}^2}^2\right]+C\mathbb{E}\left[\sum_{\ell=1}^{n}\mathbb{1}_{\Omega_{\kappa,\ell-1}}\int_{t_{\ell-1}}^{t_\ell}\|\nabla\boldsymbol{H}(s)-\nabla\boldsymbol{H}(t_\ell)\|_{\mathbb{L}^2}^2\,\mathrm{d}s\right]\\&\leq\epsilon k\mathbb{E}\left[\sum_{\ell=1}^{n}\mathbb{1}_{\Omega_{\kappa,\ell-1}}\left\|\nabla\Delta_h\boldsymbol{\theta}^\ell\right\|_{\mathbb{L}^2}^2\right]+Ck^{\frac{1}{2}-\delta}.\end{aligned}\tag{3.45}$$

For the term I_5, we use (2.10) and Young's inequality to deduce

$$|I_5|\leq Ch^2+\epsilon k\mathbb{E}\left[\sum_{\ell=1}^{n}\mathbb{1}_{\Omega_{\kappa,\ell-1}}\left\|\nabla\Delta_h\boldsymbol{\theta}^\ell\right\|_{\mathbb{L}^2}^2\right].$$

Next, for the term I_6 we apply Young's inequality, (2.10), (2.28), and (2.13). By Proposition 2.2 (Hölder continuity of $\boldsymbol{u}$) and the Sobolev embeddings, we obtain for any $\delta>0$,

$$\begin{aligned}|I_6|&\leq\epsilon k\mathbb{E}\left[\sum_{\ell=1}^{n}\mathbb{1}_{\Omega_{\kappa,\ell-1}}\left\|\nabla\Delta_h\boldsymbol{\theta}^\ell\right\|_{\mathbb{L}^2}^2\right]+Ck\mathbb{E}\left[\sum_{\ell=1}^{n}\mathbb{1}_{\Omega_{\kappa,\ell-1}}\left\|\nabla f_R(\boldsymbol{u}(t_\ell))-\nabla f_R(\boldsymbol{u}_h^{\ell-1})\right\|_{\mathbb{L}^2}^2\right]\\&\leq\epsilon k\mathbb{E}\left[\sum_{\ell=1}^{n}\mathbb{1}_{\Omega_{\kappa,\ell-1}}\left\|\nabla\Delta_h\boldsymbol{\theta}^\ell\right\|_{\mathbb{L}^2}^2\right]+Ck\mathbb{E}\left[\sum_{\ell=1}^{n}\mathbb{1}_{\Omega_{\kappa,\ell-1}}\left(1+\|\boldsymbol{u}(t_\ell)\|_{\mathbb{L}^\infty}^6\right)\left\|\nabla\boldsymbol{u}(t_\ell)-\nabla\boldsymbol{u}_h^{\ell-1}\right\|_{\mathbb{L}^2}^2\right]\\&\quad+Ck\mathbb{E}\left[\sum_{\ell=1}^{n}\mathbb{1}_{\Omega_{\kappa,\ell-1}}\left(1+\|\boldsymbol{u}(t_\ell)\|_{\mathbb{L}^\infty}^6\right)\|\nabla\boldsymbol{u}(t_\ell)\|_{\mathbb{L}^4}^2\left\|\boldsymbol{u}(t_\ell)-\boldsymbol{u}_h^{\ell-1}\right\|_{\mathbb{L}^4}^2\right]\\&\leq\epsilon k\mathbb{E}\left[\sum_{\ell=1}^{n}\mathbb{1}_{\Omega_{\kappa,\ell-1}}\left\|\nabla\Delta_h\boldsymbol{\theta}^\ell\right\|_{\mathbb{L}^2}^2\right]\\&\quad+C(1+\kappa^4)k\mathbb{E}\left[\sum_{\ell=1}^{n}\mathbb{1}_{\Omega_{\kappa,\ell-1}}\left\|\nabla\boldsymbol{u}(t_\ell)-\nabla\boldsymbol{u}(t_{\ell-1})+\nabla\boldsymbol{\theta}^{\ell-1}+\nabla\boldsymbol{\rho}^{\ell-1}\right\|_{\mathbb{L}^2}^2\right]\\&\quad+C(1+\kappa^4)k\mathbb{E}\left[\sum_{\ell=1}^{n}\mathbb{1}_{\Omega_{\kappa,\ell-1}}\left\|\boldsymbol{u}(t_\ell)-\boldsymbol{u}(t_{\ell-1})+\boldsymbol{\theta}^{\ell-1}+\boldsymbol{\rho}^{\ell-1}\right\|_{\mathbb{L}^4}^2\right]\end{aligned}$$

$$
\begin{aligned}
&\leq \epsilon k \mathbb{E}\left[\sum_{\ell=1}^n \mathbb{1}_{\Omega_{\kappa,\ell-1}} \left\|\nabla \Delta_h \boldsymbol{\theta}^\ell\right\|_{\mathbb{L}^2}^2\right] + C(1+\kappa^4)\left(k^{1-\delta}+h^2\right) \\
&\quad + C(1+\kappa^4) k \mathbb{E}\left[\sum_{\ell=1}^n \mathbb{1}_{\Omega_{\kappa,\ell-1}} \left\|\boldsymbol{\theta}^{\ell-1}\right\|_{\mathbb{H}^1}^2\right] \\
&\leq \epsilon k \mathbb{E}\left[\sum_{\ell=1}^n \mathbb{1}_{\Omega_{\kappa,\ell-1}} \left\|\nabla \Delta_h \boldsymbol{\theta}^\ell\right\|_{\mathbb{L}^2}^2\right] + C(1+\kappa^4) e^{C\kappa^2}\left(h^2 + k^{\frac{1}{2}-\delta}\right),
\end{aligned}
\tag{3.46}
$$

where in the last step we also used Proposition 3.5. For the term I_7, by Young's inequality and (2.13), we have

$$
\begin{aligned}
|I_7| &\leq \mathbb{E}\left[\sum_{\ell=1}^n \mathbb{1}_{\Omega_{\kappa,\ell-1}} \int_{t_{\ell-1}}^{t_\ell} \left\|\boldsymbol{\rho}^\ell + \boldsymbol{\theta}^\ell + \boldsymbol{u}(s) - \boldsymbol{u}(t_\ell)\right\|_{\mathbb{L}^4} \|\boldsymbol{H}(s)\|_{\mathbb{L}^2} \left\|\Delta_h \boldsymbol{\theta}^\ell\right\|_{\mathbb{L}^4} \mathrm{d}s\right] \\
&\leq \epsilon k \mathbb{E}\left[\sum_{\ell=1}^n \mathbb{1}_{\Omega_{\kappa,\ell-1}} \left\|\Delta_h \boldsymbol{\theta}^\ell\right\|_{\mathbb{L}^4}^2\right] + C h^4 \mathbb{E}\left[\|\boldsymbol{H}\|_{L^\infty_T(\mathbb{L}^2)}^2\right] \\
&\quad + C k \mathbb{E}\left[\sum_{\ell=1}^n \mathbb{1}_{\Omega_{\kappa,\ell-1}} \sup_{t\in[t_{\ell-1},t_\ell]} \|\boldsymbol{H}(t)\|_{\mathbb{L}^2}^2 \left\|\boldsymbol{\theta}^\ell\right\|_{\mathbb{L}^4}^2\right] \\
&\quad + C \mathbb{E}\left[\sum_{\ell=1}^n \mathbb{1}_{\Omega_{\kappa,\ell-1}} \int_{t_{\ell-1}}^{t_\ell} \|\boldsymbol{H}(s)\|_{\mathbb{L}^2}^2 \|\boldsymbol{u}(s)-\boldsymbol{u}(t_\ell)\|_{\mathbb{L}^4}^2 \mathrm{d}s\right] \\
&\leq \epsilon k \mathbb{E}\left[\sum_{\ell=1}^n \mathbb{1}_{\Omega_{\kappa,\ell-1}} \left\|\nabla \Delta_h \boldsymbol{\theta}^\ell\right\|_{\mathbb{L}^2}^2\right] + C(1+\kappa) e^{C\kappa^2}\left(h^2 + k^{\frac{1}{2}-\delta}\right),
\end{aligned}
$$

where in the last step we also used (2.16) and (2.22), Proposition 3.5, and the Hölder continuity of $\boldsymbol{u}$. Next, similarly we have

$$
\begin{aligned}
|I_8| &\leq \mathbb{E}\left[\sum_{\ell=1}^n \mathbb{1}_{\Omega_{\kappa,\ell-1}} \int_{t_{\ell-1}}^{t_\ell} \left\|\boldsymbol{u}_h^\ell\right\|_{\mathbb{L}^4} \left(\left\|\boldsymbol{\eta}^\ell\right\|_{\mathbb{L}^2} + \left\|\boldsymbol{\xi}^\ell\right\|_{\mathbb{L}^2} + \|\boldsymbol{H}(s)-\boldsymbol{H}(t_\ell)\|_{\mathbb{L}^2}\right) \left\|\Delta_h \boldsymbol{\theta}^\ell\right\|_{\mathbb{L}^4}\right] \mathrm{d}s \\
&\leq \epsilon k \mathbb{E}\left[\sum_{\ell=1}^n \mathbb{1}_{\Omega_{\kappa,\ell-1}} \left\|\Delta_h \boldsymbol{\theta}^\ell\right\|_{\mathbb{L}^4}^2\right] + C \mathbb{E}\left[\max_{j\leq n} \|\boldsymbol{\eta}^j\|_{\mathbb{L}^2}^4\right]^{\frac{1}{2}} \mathbb{E}\left[\left(k\sum_{\ell=1}^n \left\|\boldsymbol{u}_h^\ell\right\|_{\mathbb{H}^1}^2\right)^2\right]^{\frac{1}{2}} \\
&\quad + C\kappa k \mathbb{E}\left[\sum_{\ell=1}^n \mathbb{1}_{\Omega_{\kappa,\ell-1}} \left\|\boldsymbol{\xi}^\ell\right\|_{\mathbb{L}^2}^2\right] + C\kappa \mathbb{E}\left[\sum_{\ell=1}^n \mathbb{1}_{\Omega_{\kappa,\ell-1}} \int_{t_{\ell-1}}^{t_\ell} \|\boldsymbol{H}(s)-\boldsymbol{H}(t_\ell)\|_{\mathbb{L}^2}^2 \mathrm{d}s\right] \\
&\leq \epsilon k \mathbb{E}\left[\sum_{\ell=1}^n \mathbb{1}_{\Omega_{\kappa,\ell-1}} \|\nabla \Delta_h \boldsymbol{\theta}^n\|_{\mathbb{L}^2}^2\right] + C(1+\kappa) e^{C\kappa^2}\left(h^2 + k^{\frac{1}{2}-\delta}\right).
\end{aligned}
$$

For the terms I_{10} and I_{11}, we apply similar argument as above to infer that

$$
\begin{aligned}
|I_{10}| &\leq \mathbb{E}\left[\sum_{\ell=1}^n \mathbb{1}_{\Omega_{\kappa,\ell-1}} \int_{t_{\ell-1}}^{t_\ell} \left(\left\|\boldsymbol{\rho}^\ell\right\|_{\mathbb{L}^2} + \left\|\boldsymbol{\theta}^\ell\right\|_{\mathbb{L}^2} + \|\boldsymbol{u}(s)-\boldsymbol{u}(t_\ell)\|_{\mathbb{L}^2}\right) \left\|\nabla \boldsymbol{u}_h^\ell\right\|_{\mathbb{L}^2} \left\|\Delta_h \boldsymbol{\theta}^\ell\right\|_{\mathbb{L}^4}\right] \mathrm{d}s \\
&\leq \epsilon k \mathbb{E}\left[\sum_{\ell=1}^n \mathbb{1}_{\Omega_{\kappa,\ell-1}} \left\|\nabla \Delta_h \boldsymbol{\theta}^\ell\right\|_{\mathbb{L}^2}^2\right] + C(1+\kappa) e^{C\kappa^2}\left(h^2 + k^{\frac{1}{2}-\delta}\right), \\
|I_{11}| &\leq \mathbb{E}\left[\sum_{\ell=1}^n \mathbb{1}_{\Omega_{\kappa,\ell-1}} \int_{t_{\ell-1}}^{t_\ell} \|\boldsymbol{u}(s)\|_{\mathbb{L}^\infty} \left(\left\|\nabla \boldsymbol{\rho}^\ell\right\|_{\mathbb{L}^2} + \left\|\nabla \boldsymbol{\theta}^\ell\right\|_{\mathbb{L}^2} + \|\nabla \boldsymbol{u}(s)-\nabla \boldsymbol{u}(t_\ell)\|_{\mathbb{L}^2}\right) \left\|\Delta_h \boldsymbol{\theta}^\ell\right\|_{\mathbb{L}^2}\right] \mathrm{d}s
\end{aligned}
$$

$$\leq \epsilon k \mathbb{E}\left[\sum_{\ell=1}^{n} \mathbb{1}_{\Omega_{\kappa,\ell-1}} \left\|\Delta_h \boldsymbol{\theta}^\ell\right\|_{\mathbb{L}^2}^2\right] + C(1+\kappa)e^{C\kappa^2}\left(h^2 + k^{\frac{1}{2}-\delta}\right).$$

Next, by Young's inequality and (2.13), it is easy to see that

$$|I_9| + |I_{12}| \leq \epsilon k \mathbb{E}\left[\sum_{\ell=1}^{n} \mathbb{1}_{\Omega_{\kappa,\ell-1}} \left\|\nabla\Delta_h \boldsymbol{\theta}^\ell\right\|_{\mathbb{L}^2}^2\right] + \epsilon k \mathbb{E}\left[\sum_{\ell=1}^{n} \mathbb{1}_{\Omega_{\kappa,\ell-1}} \left\|\Delta_h \boldsymbol{\theta}^\ell\right\|_{\mathbb{L}^2}^2\right] + Ce^{C\kappa^2}\left(h^2 + k^{\frac{1}{2}-\delta}\right).$$

Finally, we split the stochastic integral in I_{13} as

$$\begin{aligned}
I_{13} &= \mathbb{E}\left[\max_{m\leq n} \sum_{\ell=1}^{m} \mathbb{1}_{\Omega_{\kappa,\ell-1}} \int_{t_{\ell-1}}^{t_\ell} \left\langle \nabla\Pi_h G(\boldsymbol{u}(s)) - \nabla\Pi_h G(\boldsymbol{u}_h^{\ell-1}), \nabla\boldsymbol{\theta}^{\ell-1}\right\rangle \mathrm{d}W(s)\right] \\
&\quad + \mathbb{E}\left[\max_{m\leq n} \sum_{\ell=1}^{m} \mathbb{1}_{\Omega_{\kappa,\ell-1}} \int_{t_{\ell-1}}^{t_\ell} \left\langle \nabla\Pi_h G(\boldsymbol{u}(s)) - \nabla\Pi_h G(\boldsymbol{u}_h^{\ell-1}), \nabla\boldsymbol{\theta}^{\ell} - \nabla\boldsymbol{\theta}^{\ell-1}\right\rangle \mathrm{d}W(s)\right] \\
&=: I_{13a} + I_{13b}.
\end{aligned}$$

For the first term above, noting the assumptions on G, the $\mathbb{H}^1$ stability of Π_h, and the Hölder continuity of $\boldsymbol{u}$, we apply the Burkholder–Davis–Gundy and the Young inequalities to obtain

$$\begin{aligned}
I_{13a} &\leq C\mathbb{E}\left[\left(\sum_{\ell=1}^{n} \mathbb{1}_{\Omega_{\kappa,\ell-1}} \int_{t_{\ell-1}}^{t_\ell} \left\|\nabla G(\boldsymbol{u}(s)) - \nabla G(\boldsymbol{u}_h^{\ell-1})\right\|_{\mathbb{L}^2}^2 \left\|\nabla\boldsymbol{\theta}^{\ell-1}\right\|_{\mathbb{L}^2}^2 \mathrm{d}s\right)^{\frac{1}{2}}\right] \\
&\leq C\mathbb{E}\left[\left(\max_{m\leq n} \mathbb{1}_{\Omega_{\kappa,m-1}} \left\|\nabla\boldsymbol{\theta}^{m-1}\right\|_{\mathbb{L}^2}\right)\left(\sum_{\ell=1}^{n} \mathbb{1}_{\Omega_{\kappa,\ell-1}} \int_{t_{\ell-1}}^{t_\ell} \left\|\nabla\boldsymbol{u}(s) - \nabla\boldsymbol{u}_h^{\ell-1}\right\|_{\mathbb{L}^2}^2\right)^{\frac{1}{2}}\right] \\
&\leq \epsilon\mathbb{E}\left[\max_{m\leq n} \mathbb{1}_{\Omega_{\kappa,m-1}} \left\|\nabla\boldsymbol{\theta}^{m}\right\|_{\mathbb{L}^2}^2\right] + Ce^{C\kappa^2}\left(h^2 + k^{\frac{1}{2}-\delta}\right).
\end{aligned}$$

For the term I_{13b}, we use Young's inequality, Itô's isometry, the assumptions on G, and the Hölder continuity of $\boldsymbol{u}$ to obtain

$$\begin{aligned}
I_{13b} &\leq C\mathbb{E}\left[\sum_{\ell=1}^{n} \left\|\mathbb{1}_{\Omega_{\kappa,\ell-1}} \int_{t_{\ell-1}}^{t_\ell} \left(\nabla\Pi_h G(\boldsymbol{u}(s)) - \nabla\Pi_h G(\boldsymbol{u}_h^{\ell-1})\right) \mathrm{d}W(s)\right\|_{\mathbb{L}^2}^2\right] \\
&\quad + \epsilon\mathbb{E}\left[\sum_{\ell=1}^{n} \mathbb{1}_{\Omega_{\kappa,\ell-1}} \left\|\nabla\boldsymbol{\theta}^\ell - \nabla\boldsymbol{\theta}^{\ell-1}\right\|_{\mathbb{L}^2}^2\right] \\
&= C\mathbb{E}\left[\sum_{\ell=1}^{n} \mathbb{1}_{\Omega_{\kappa,\ell-1}} \int_{t_{\ell-1}}^{t_\ell} \left\|\nabla\Pi_h G(\boldsymbol{u}(s)) - \nabla\Pi_h G(\boldsymbol{u}_h^{\ell-1})\right\|_{\mathbb{L}^2}^2 \mathrm{d}s\right] \\
&\quad + \epsilon\mathbb{E}\left[\sum_{\ell=1}^{n} \mathbb{1}_{\Omega_{\kappa,\ell-1}} \left\|\nabla\boldsymbol{\theta}^\ell - \nabla\boldsymbol{\theta}^{\ell-1}\right\|_{\mathbb{L}^2}^2\right] \\
&\leq Ce^{C\kappa^2}\left(h^2 + k^{\frac{1}{2}-\delta}\right) + \epsilon\mathbb{E}\left[\sum_{\ell=1}^{n} \mathbb{1}_{\Omega_{\kappa,\ell-1}} \left\|\nabla\boldsymbol{\theta}^\ell - \nabla\boldsymbol{\theta}^{\ell-1}\right\|_{\mathbb{L}^2}^2\right].
\end{aligned}$$

We now substitute all the above estimates into (3.44), set $\epsilon = 1/16$, and rearrange the terms. Altogether, continuing from (3.44) we deduce that

$$\mathbb{E}\left[\max_{m\leq n}\left(\mathbb{1}_{\Omega_{\kappa,m-1}} \left\|\nabla\boldsymbol{\theta}^m\right\|_{\mathbb{L}^2}^2\right)\right] + k\mathbb{E}\left[\sum_{\ell=1}^{n} \mathbb{1}_{\Omega_{\kappa,\ell-1}} \left\|\Delta_h\boldsymbol{\theta}^\ell\right\|_{\mathbb{L}^2}^2\right] + k\mathbb{E}\left[\sum_{\ell=1}^{n} \mathbb{1}_{\Omega_{\kappa,\ell-1}} \left\|\nabla\Delta_h\boldsymbol{\theta}^\ell\right\|_{\mathbb{L}^2}^2\right]$$

$$\leq \mathbb{E}\left[\left\|\nabla \boldsymbol{\theta}^0\right\|_{\mathbb{L}^2}^2\right] + C(1+\kappa^4)e^{C\kappa^2}\left(h^2 + k^{\frac{1}{2}-\delta}\right). \tag{3.47}$$

Furthermore, note that by setting $\boldsymbol{\phi}_h = -k\Delta_h \boldsymbol{\xi}^n$ in (3.35), multiplying by $\mathbb{1}_{\Omega_{\kappa,n-1}}$ and taking expectation, then applying (2.14) and (3.31), we obtain

$$\begin{aligned}
k\mathbb{E}\left[\mathbb{1}_{\Omega_{\kappa,n-1}} \|\nabla \boldsymbol{\xi}^n\|_{\mathbb{L}^2}^2\right] &= k\mathbb{E}\left[\mathbb{1}_{\Omega_{\kappa,n-1}} \langle \nabla \Pi_h \boldsymbol{\eta}^n, \nabla \boldsymbol{\xi}^n\rangle\right] + k\mathbb{E}\left[\mathbb{1}_{\Omega_{\kappa,n-1}} \langle \nabla \Delta_h \boldsymbol{\theta}^n, \nabla \boldsymbol{\xi}^n\rangle\right] \\
&\quad + k\mathbb{E}\left[\mathbb{1}_{\Omega_{\kappa,n-1}} \langle \nabla \Pi_h f_R(\boldsymbol{u}(t_n)) - \nabla \Pi_h f_R(\boldsymbol{u}_h^{n-1}), \nabla \boldsymbol{\xi}^n\rangle\right] \\
&\leq Ckh^2 + \epsilon k\mathbb{E}\left[\mathbb{1}_{\Omega_{\kappa,n-1}} \|\nabla \boldsymbol{\xi}^n\|_{\mathbb{L}^2}^2\right] + Ck\mathbb{E}\left[\mathbb{1}_{\Omega_{\kappa,n-1}} \|\nabla \Delta_h \boldsymbol{\theta}^n\|_{\mathbb{L}^2}^2\right] \\
&\quad + Ck\mathbb{E}\left[\mathbb{1}_{\Omega_{\kappa,n-1}} \left\|\nabla \Pi_h f_R(\boldsymbol{u}(t_n)) - \nabla \Pi_h f_R(\boldsymbol{u}_h^{n-1})\right\|_{\mathbb{L}^2}^2\right],
\end{aligned} \tag{3.48}$$

where in the last step we used Young's inequality and (2.13). The final term in (3.48) can be estimated using (2.28) as done in (3.46). Summing (3.48) over $\ell \in \{1, 2, \dots, n\}$ and applying (3.47), we obtain

$$k\mathbb{E}\left[\sum_{\ell=1}^n \mathbb{1}_{\Omega_{\kappa,\ell-1}} \left\|\nabla \boldsymbol{\xi}^\ell\right\|_{\mathbb{L}^2}^2\right] \leq \mathbb{E}\left[\left\|\nabla \boldsymbol{\theta}^0\right\|_{\mathbb{L}^2}^2\right] + C(1+\kappa^4)e^{C\kappa^2}\left(h^2 + k^{\frac{1}{2}-\delta}\right).$$

Choosing $\boldsymbol{u}_h^0$ such that $\mathbb{E}\left[\left\|\nabla \boldsymbol{\theta}^0\right\|_{\mathbb{L}^2}^2\right] \leq Ch^2$, say $\boldsymbol{u}_h^0 = \Pi_h \boldsymbol{u}_0$, we deduce the required result from the last estimate and (3.47). □

The following error estimate, which holds over a sample space with large probability, now follows from the above propositions. Indeed, note that by Chebyshev's inequality,

$$\mathbb{P}\left[\Omega_{\kappa,m}\right] \geq 1 - \frac{1}{\kappa}\left(\mathbb{E}\left[\max_{t\leq t_m\wedge T} \|\boldsymbol{u}(t)\|_{\mathbb{H}^2}^2\right] + \mathbb{E}\left[\max_{t\leq t_m\wedge T} \|\boldsymbol{H}(t)\|_{\mathbb{L}^2}^2\right] + \mathbb{E}\left[\max_{n\leq m} \|\boldsymbol{u}_h^n\|_{\mathbb{H}^1}^2\right]\right) \geq 1 - \frac{C_{R,T}}{\kappa}.$$

Here, $C_{R,T}$ is a constant depending on R, T, and the coefficients of the equation, which is conferred by Lemma 3.3 and Proposition 2.2. Therefore, $\mathbb{P}\left[\Omega_{\kappa,m}\right] \to 1$ as $\kappa \to \infty$.

Theorem 3.7. Assume that the hypotheses of Proposition 3.5 hold. Then for $n \in \{1, 2, \dots, N\}$, we have

$$\mathbb{E}\left[\max_{m\leq n}\left(\mathbb{1}_{\Omega_{\kappa,m-1}} \|\boldsymbol{u}(t_m) - \boldsymbol{u}_h^m\|_{\mathbb{H}^1}^2\right)\right] + k\sum_{\ell=1}^n \mathbb{E}\left[\mathbb{1}_{\Omega_{\kappa,\ell-1}} \left\|\boldsymbol{H}(t_\ell) - \boldsymbol{H}_h^\ell\right\|_{\mathbb{H}^1}^2\right] \leq \widetilde{C}e^{\widetilde{C}\kappa^2}\left(h^2 + k^{\frac{1}{2}-\delta}\right), \tag{3.49}$$

where $\widetilde{C}$ is a constant depending on R and T, but is independent of h and k.

Proof. This follows from Proposition 3.5, Proposition 3.6, equations (3.29) and (3.30), estimate (2.13), and the triangle inequality. □

We now define the following quantities:

$$A_n := \max_{m\leq n}\left(\mathbb{1}_{\Omega_{\kappa,m-1}} \|\boldsymbol{u}(t_m) - \boldsymbol{u}_h^m\|_{\mathbb{H}^1}^2\right) + k\sum_{\ell=1}^n \mathbb{1}_{\Omega_{\kappa,\ell-1}} \left\|\boldsymbol{H}(t_\ell) - \boldsymbol{H}_h^\ell\right\|_{\mathbb{H}^1}^2, \tag{3.50}$$

$$\widetilde{A_n} := \max_{m\leq n}\left(\mathbb{1}_{\Omega^{\complement}_{\kappa,m-1}} \|\boldsymbol{u}(t_m) - \boldsymbol{u}_h^m\|_{\mathbb{H}^1}^2\right) + k\sum_{\ell=1}^n \mathbb{1}_{\Omega^{\complement}_{\kappa,\ell-1}} \left\|\boldsymbol{H}(t_\ell) - \boldsymbol{H}_h^\ell\right\|_{\mathbb{H}^1}^2. \tag{3.51}$$

By choosing an appropriate value of κ, we will derive some convergence results. Firstly, we have the following theorem on convergence in probability with a rate.

Theorem 3.8. Under the hypothesis of Proposition 3.5, for $n \in \{1, 2, \dots, N\}$ we have

$$\lim_{h,k\to 0^+} \mathbb{P}\left[\max_{m\le n} \|\boldsymbol{u}(t_m) - \boldsymbol{u}_h^m\|_{\mathbb{H}^1}^2 + k\sum_{\ell=1}^n \left\|\boldsymbol{H}(t_\ell) - \boldsymbol{H}_h^\ell\right\|_{\mathbb{H}^1}^2 \ge \alpha\left(h^{2(1-2\delta)} + k^{\frac{1}{2}(1-8\delta)}\right)\right] = 0$$

for any $\alpha, \delta > 0$.

Proof. By Chebyshev's inequality and Theorem 3.7 with $\kappa^2 = O\left(\log(\log 1/h)\right)$, for any $\alpha, \delta > 0$ we have

$$\begin{aligned}
&\mathbb{P}\left[\max_{m\le n} \|\boldsymbol{u}(t_m) - \boldsymbol{u}_h^m\|_{\mathbb{H}^1}^2 + k\sum_{\ell=1}^n \left\|\boldsymbol{H}(t_\ell) - \boldsymbol{H}_h^\ell\right\|_{\mathbb{H}^1}^2 \ge \alpha(h^{2(1-2\delta)} + k^{\frac{1}{2}(1-8\delta)})\right] \\
&\le \alpha^{-1}\left(h^{2(1-2\delta)} + k^{\frac{1}{2}(1-8\delta)}\right)^{-1}\mathbb{E}\left[A_n\right] + \mathbb{P}\left[\Omega_{\kappa,n-1}^{\complement}\right] \\
&\le C\alpha^{-1}\left(h^{2(1-2\delta)} + k^{\frac{1}{2}(1-8\delta)}\right)^{-1}\left(h^{2(1-\delta)} + k^{\frac{1}{2}(1-4\delta)}\right) + C_{R,T}\big(\log(\log(1/h))\big)^{-\frac{1}{2}},
\end{aligned}$$

which tends to 0 as $h, k \to 0^+$. ∎

We now assume that $\beta_2 = 0$ to derive a strong order of convergence for the scheme.

Theorem 3.9. Suppose that $\beta_2 = 0$. Under the hypothesis of Proposition 3.5, for $n \in \{1, 2, \dots, N\}$ we have

$$\mathbb{E}\left[\max_{m\le n} \|\boldsymbol{u}(t_m) - \boldsymbol{u}_h^m\|_{\mathbb{H}^1}^2 + k\sum_{\ell=1}^n \left\|\boldsymbol{H}(t_\ell) - \boldsymbol{H}_h^\ell\right\|_{\mathbb{H}^1}^2\right] \le C_r\left|\log\left(h^2 + k^{\frac{1}{2}-\delta}\right)\right|^{-r}, \tag{3.52}$$

for any $r \ge 1$. In particular, the right-hand side of (3.52) tends to 0 as $h, k \to 0^+$.

Proof. Note that by Hölder's inequality with exponents 2^{q-1} and $p = 2^{q-1}/(2^{q-1}-1)$, where $q > 1$, we have

$$\mathbb{E}\left[\max_{m\le n} \mathbb{1}_{\Omega_{\kappa,m-1}^{\complement}} \|\boldsymbol{u}(t_m) - \boldsymbol{u}_h^m\|_{\mathbb{H}^1}^2\right] \le C\left[\mathbb{P}\left(\Omega_{\kappa,n-1}^{\complement}\right)\right]^{\frac{1}{p}}\left[\mathbb{E}\left(\max_{t\in[0,T]} \|\boldsymbol{u}(t)\|_{\mathbb{H}^1}^{2^q} + \max_{m\le n}\|\boldsymbol{u}_h^m\|_{\mathbb{H}^1}^{2^q}\right)\right]^{\frac{1}{2^{q-1}}}. \tag{3.53}$$

Similarly,

$$\mathbb{E}\left[k\sum_{\ell=1}^n \mathbb{1}_{\Omega_{\kappa,\ell-1}^{\complement}} \left\|\boldsymbol{H}(t_\ell) - \boldsymbol{H}_h^\ell\right\|_{\mathbb{H}^1}^2\right] \le C\left[\mathbb{P}\left(\Omega_{\kappa,n-1}^{\complement}\right)\right]^{\frac{1}{p}}\left[\mathbb{E}\left(k\sum_{\ell=1}^n \|\boldsymbol{H}(t_\ell)\|_{\mathbb{H}^1}^2 + \left\|\boldsymbol{H}_h^\ell\right\|_{\mathbb{H}^1}^2\right)^{2^{q-1}}\right]^{\frac{1}{2^{q-1}}}. \tag{3.54}$$

The last terms on the right-hand side of (3.53) and (3.54) are bounded due to the assumed regularity in Proposition 2.2 and the stability estimate (3.27). Therefore, it remains to establish a bound for the probability of the 'bad' set $\Omega_{\kappa,n-1}^{\complement}$. To this end, by Chebyshev's inequality and the definition of the set $\Omega_{\kappa,n-1}$,

$$\mathbb{P}\left(\Omega_{\kappa,n-1}^{\complement}\right) \le \kappa^{-2^{q-1}}\left[\mathbb{E}\left(\max_{t\in[0,T]} \|\boldsymbol{u}(t)\|_{\mathbb{H}^2}^{2^q} + \max_{t\in[0,T]} \|\boldsymbol{H}(t)\|_{\mathbb{L}^2}^{2^q} + \max_{m\le n}\|\boldsymbol{u}_h^m\|_{\mathbb{H}^1}^{2^q}\right)\right],$$

which implies by the definition (3.51),

$$\mathbb{E}\left[\widetilde{A_n}\right] \le C_q\kappa^{-2^{q-1}}. \tag{3.55}$$

For sufficiently small h and k, we now choose

$$\kappa^2 = \frac{1}{\widetilde{C}}\left(\left|\log\left(h^2 + k^{\frac{1}{2}-\delta}\right)\right| - (2^{q-1}-1)\log\left|\log\left(h^2 + k^{\frac{1}{2}-\delta}\right)\right|\right),$$

where $\widetilde{C}$ is the constant in (3.49). With this choice of κ, noting (3.55), we have by (3.49), (3.53), and (3.54),

$$\begin{aligned}\mathbb{E}\left[\max_{m\le n}\|\boldsymbol{u}(t_m)-\boldsymbol{u}_h^m\|_{\mathbb{H}^1}^2+k\sum_{\ell=1}^{n}\left\|\boldsymbol{H}(t_\ell)-\boldsymbol{H}_h^\ell\right\|_{\mathbb{H}^1}^2\right] &= \mathbb{E}\left[A_n\right]+\mathbb{E}\left[\widetilde{A_n}\right]\\ &\le \widetilde{C}e^{\widetilde{C}\kappa^2}\left(h^2+k^{\frac{1}{2}-\delta}\right)+C_q\kappa^{-2^{q-1}}\\ &\le C_r\left|\log\left(h^2+k^{\frac{1}{2}-\delta}\right)\right|^{-r},\end{aligned}$$

for any $r\ge 1$. This completes the proof of the theorem. □

Remark 3.10. If the initial data $\boldsymbol{u}_0$ is more regular, say $\boldsymbol{u}_0\in\mathbb{H}^3$, then by a similar argument as in [23], one can show that the pathwise solution $\boldsymbol{u}$ of (2.7) belongs to $L^p\big(\Omega;C^\alpha(0,T;\mathbb{H}^3)\big)$, where $\alpha\in(0,\frac{1}{2})$, for any $p\ge 1$. In that case, an $O(k^{1-\delta})$ bound can be obtained in (3.40) and (3.45), leading to an $O(k^{1-\delta})$ bound in Proposition 3.5, Proposition 3.6, and Theorem 3.7 (instead of $O(k^{\frac{1}{2}-\delta})$ as stated). Consequently, the right-hand side of (3.52) would read $C_r\left|\log\left(h^2+k^{1-\delta}\right)\right|^{-r}$ in this case.

Acknowledgements

The authors acknowledge financial support through the Australian Research Council's Discovery Projects funding scheme (project DP200101866).

Agus L. Soenjaya is supported by the Australian Government Research Training Program (RTP) Scholarship awarded at the University of New South Wales, Sydney.

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